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ON THE ITERATED MULTIPLICATION MAP FOR LINE BUNDLES ON CURVES

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Abstract: Let X be an integral curve. Here we study the multiplication map $H^0(X, L_1) \otimes \cdots \otimes H^0(X, L_s) \to H^0(X, L_1 \otimes \cdots \otimes L_s)$ of global sections of $s \geq 3$ line bundles, mainly when $h^0(X, L_i) = 2$ for all i.

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Let X be an integral projective curve defined over an algebraically closed field $\mathbb{K}$. Set $g := p_a(X)$. Let $S(X)$ denote the set of all spanned $L \in \mathrm{Pic}^{g+1}(X)$ such that $h^1(X, L) = 0$. The set $S(X)$ is an integral quasi-projective variety of dimension g. For any finite $S \subset S(X)$ such that $s := \sharp(S) \geq 2$ and any ordering η of S, say $\Sigma := (S, \eta) = (L_1, \dots, L_s)$ let $\mu_\Sigma : \otimes_{i=1}^s H^0(X, L_i) \to H^0(X, \otimes_{i=1}^s L_i)$ denote the multiplication map. Notice the target of μ_Σ is a vector space of dimension $s(g+1)+1-g = (s-1)g+s+1$, while its domain is a vector space of dimension 2^s. The rank of the linear map μ_Σ does not depend from the choice of the ordering η. Hence we set $\rho_S := \mathrm{rank}(\mu_\Sigma)$. We prove the following results.

Theorem 1. *Fix an integer $s \geq 2$. Let $S \subset S(X)$ be a general subset such that $\sharp(S) = s$. Then $\rho_S = \min\{2^s, (s-1)g+s+1\}$.*

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Theorem 2. *Assume $g \geq 5$. There is $S \subset S(X)$ such that $\sharp(S) = 3$ and $\rho_S \leq 6$ if and only if g is even and there are a degree $(g+1)/2$ finite covering $u : X \to C$ with C a smooth elliptic curve and $A \subset S(C)$ such that $\sharp(A) = 3$ and $u^*(A) = S$.*

It is obvious that $\rho_S = \min\{4, g+3\}$ for any $S \subset S(X)$ such that $\sharp(S) = 2$.

We raised the following questions.

Question 1. Fix an even integer $g \geq 5$ and a general $X \in \mathcal{M}_g$. Compute ρ_S for all $S \subseteq W^1_{g/2+1}(X)$. Is it possible to do the same in the odd genus case for all finite $S \subset W^1_{(g+3)/2}(X)$?

Question 2. Fix integers $g \geq 2k \geq 8$ such that g is even. Let X be a general k-gonal curve of genus g. Let $W^1_{g/2+1}(X)'$ denote the set of all spanned and complete linear systems of degree $g/2+1$. Is it possible to compute ρ_S for all $S \subseteq W^1_{g/2+1}(X)'$?

For the finiteness of $W^1_{g/2+1}(X)'$ and the nonemptiness of $W^1_{g/2+1}(X)'$, see [1], Theorem 1. It is easy to construct curves X and large sets $S \subset \mathrm{Pic}(X)$ such that every element of S is spanned, $h^0(X, L) = 2$ for all $L \in S$, $s := \sharp(S) \geq 3$ and $\rho_S = \binom{s+2}{2} - s$ (take as X a smooth plane curve of degree $d \geq s+1$, $A \subset X$ such that $\sharp(A) = s$ and set $S := \{\mathcal{O}_X(1)(-P)\}_{P \in A}$).

Lemma 1. *Fix any $L \in \mathrm{Pic}(X)$, any integer $t > 0$, and any linear subspace $V \subseteq H^0(X, L)$. Let $D \subset X$ be a general subset such that $\sharp(D) = t$. Use D to see $L(-D)$ as a subsheaf of L. Then $\dim(V \cap H^0(X, L(-D)) = \max\{0, \dim(V) - t\}$.*

Proof. If $t = 1$ we just use that if $V \neq \{0\}$, then the general point of X is not a base point of the linear system associated to V. If $t \geq 2$ use induction on t and apply the first part of the proof to the line bundle $L' := L(-A)$ and the linear space $V' := H^0(X, L') \cap V$, where A is a general subset of X such that $\sharp(A) = t - 1$. □

Proof of Theorem 1. Fix a general $A \subset S(X)$ such that sharp $A) = s - 1$ and any ordering β of it, say $A = (A_1, \dots, A - s - 1)$. Set $\Phi = (A, \beta)$. Set $L := \otimes_{i=1}^{s-1} A_i$. If $s = 2$, then set $V := H^0(X, L) = H^0(X, A_1)$. If $s \geq 3$, then set $V := \mathrm{Im}(\mu_\Phi)$. Fix a general $D \subset X$ such that $\sharp(D) = g + 1$. Notice that $\mathcal{O}_X(D) \in S(X)$. Set $S := A \cup \{\mathcal{O}_X(D)\}$ and apply Lemma 1. □

Proof of Theorem 2. Since the "if" part is easy, we only prove the "only if" part. Assume the existence of a smooth curve C of genus $q < g$, a finite covering $u : X \to C$ and a set A of 3 spanned line bundles such that $S = u^*(A)$. Every element of A has degree $(g+1)/\deg(u)$. Since $\sharp(S) > 1$, $q \neq 0$. Since each element of A is spanned and $\sharp(A) \geq 2$, either $q = 1$ or $(g+1)/\deg(u) \geq 3$. Let $\wp_i$, $i = 1,2,3$, be the i-th factor of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \wp_i$ denote the projection. For all $i < j$ set $\wp_{i,j} := \wp_i \times \wp_j$. Let $\pi_{i,j} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \wp_{i,j}$ denote the projection. Fix $S \subset S(X)$ such that $\sharp(S) = 3$ and $\rho_S \leq 6$. Fix an ordering η of S, say $S = L_1, L_2, L_3$ and set $\Sigma = (S, \eta)$. Set $W := \mathrm{Im}(\mu_\Sigma)$. The linear space W spans $L_1 \otimes L_2 \otimes L_3$. Since each L_i is spanned and $h^0(X, L_i) = 2$ for all i, Σ induces a morphism $\phi : X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. See $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as a degree 9 linearly normal subvariety of $\mathbb{P}^7$ (the Segre embedding). It is easy to check that $\rho_{\{R_1,R_2\}} = 4$ for any two distinct base point free and complete pencils R_1, R_2 on an integral curve T (use the base point free pencil trick and that $h^0(T, R_1 \otimes R_2^*) = 0$). Hence $\pi_{i,j}(\phi(X))$ spans $\wp_{i,j}$ for all $1 \leq i < j \leq 3$. Since $\rho_S = 6$, the linear space $J := \langle \phi(X) \rangle$ has dimension 5. Since each $\pi_{i,j}(\phi(X))$ spans $\wp_{i,j}$, we see that the irreducible component of $J \cap \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ containing $\phi(u)$ is a curve. Hence $\deg(\phi(u)) \leq 9$. Since $J \cong \mathbb{P}^5$, the genus bound for integral non-degenerate curve gives $p_a(\phi(u)) \leq 4$. Since $g \geq 5$, ϕ is not birational onto its image. Let $v : Y \phi(X)$ be the normalization of $\phi(X)$. The universal property of the normalzization map shows the existence of a morphism $u : X \to Y$ such that $\phi = v \circ u$. If $p_a(Y) = 1$, then we are done (and we even see that v is an isomorphism). Assume $p_a(Y) > 1$. We have $2 \leq p_a(Y) \leq 4$ and $\pi_i \circ v : Y \to \mathbb{P}^1$ are 3 different morphisms of the same degree associated to non-special line bundles. Hence they have degree $\geq p_a(Y) + 1$. Since the sum of their degrees is at most 9, we get $p_a(Y) = 2$. In this range we may use the surjectivity of a suitable multiplication map and gent $\rho_S = 9 + 1 - p_a(Y) > 6$, contradiction. □

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