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ACM REDUCIBLE CURVES WITH $h^1(X, \mathcal{O}_X(2)) = 0$

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we construct many arithmethically Cohen-Macaulay curves $X \subset \mathbb{P}^r$ such that $h^1(X, \mathcal{O}_X(2)) = 0$.

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1. Introduction

Here we construct many arithmetically Cohen-Macaulay curves $X \subset \mathbb{P}^r$ such that $h^1(X, \mathcal{O}_X(2)) = 0$. In a joint work we will consider (but in a much smaller range) the construction of smooth curves with the same properties. For all integers r, c such that $r \geq 3$ and $0 \leq c \leq r^2/2 - r/2 - 2$ define the integer $g_{r,c}$ in the following way. Set $g_{0,r} := 0$ and $g_{1,r} = r + 1$. If $r \geq 4$ and $c \leq r - 1$, then set $g_{r,c} := c + r$. If $r \geq 4$ and $c \geq r$, then set $g_{r,c} := g_{r-1,c-r} + 2r - 1$. In this way we define the integer $g_{r,c}$ for all integers $r \geq 3$ and $0 \leq c \leq r^2/2 - r/2 - 2$, because $c \leq r^2/2 - r/2 - 2$ if and only if $c - r \leq (r-1)^2/2 - (r-1)/2 - 2$.

Theorem 1. *Fix integers r, g, c, f such that $r \geq 3$, $f \geq 0$, $0 \leq c \leq r^2/2 - r/2 - 2$, and $g_{r,c} \leq g \leq r(r-1)/2 + 2c - 2f$. Then there is a nodal and connected curve $X \subset \mathbb{P}^r$ such that at least one of the irreducible components of X spans $\mathbb{P}^r$, $h^1(\mathbb{P}^r, \mathcal{I}_X(t)) = 0$ for all $t \geq 2$, $p_a(X) = g$, $h^0(X, \mathcal{O}_X(1)) = r+1+f$, $h^1(X, \mathcal{O}_X(1)) = c$ and $h^1(X, \mathcal{O}_X(2)) = 0$.*

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Take X as in the statement of Theorem 1. Riemann-Roch gives $\deg(X) = g+r-c+f$. Hence $\deg(\mathcal{O}_X(2)) = 2g+2r-2c+2f$. Since $h^1(X,\mathcal{O}_X(2)) = 0$, Riemann-Roch gives $h^0(X,\mathcal{O}_X(2)) = g+2r+1-2c+2f$. Since $h^0(\mathbb{P}^r,\mathcal{O}_{\mathbb{P}^r}(2)) = \binom{r+2}{2}$, the condition $g+2r+1-2c+2f \le \binom{r+2}{2}$ (i.e. $g \le r(r-1)/2+2c-2f$) is a necessary condition for the vanishing of $H^1(\mathbb{P}^r,\mathcal{I}_X(2))$.

2. Proof of Theorem 1

We need the following well-known lemma (the so-called Horace Lemma, see [4]).

Lemma 1. *Let $H \subset \mathbb{P}^r$ be a hyperplane. Fix any projective scheme $T \subset \mathbb{P}^r$. Let $\mathrm{Res}_H(T)$ be the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_T : \mathcal{I}_H$ as its ideal sheaf. Then*

$$h^i(\mathbb{P}^r,\mathcal{I}_T(t)) \le h^i(\mathcal{I}_{\mathrm{Res}_H(T)}(t-1)) + h^i(H,\mathcal{I}_{T\cap H,H}(t))$$

for all integers $i \ge 0$ and $t \ge 0$.

Proof. The definition of the residual scheme $\mathrm{Res}_H(T)$ gives the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(T)}(t-1) \to \mathcal{I}_T(t) \to \mathcal{I}_{T\cap H,H}(t) \to 0,$$

whose long cohomological exact sequence gives the lemma. □

Proposition 1. *Fix integers r, x, q, e such that $r \ge 4$, $0 \le e \le r$. Assume the existence of a reduced curve $Y \subset \mathbb{P}^{r-1}$ such that $\deg(Y) = x$, $p_a(Y) = q$, $h^1(\mathbb{P}^{r-1},\mathcal{I}_Y(t)) = 0$ for all $t \ge 2$, and $h^1(Y,\mathcal{O}_Y(2)) = 0$. Then there exists a reduced curve $X \subset \mathbb{P}^r$ such that $\deg(X) = x+r+e$, $p_a(Y) = q+e+r-1$, at least one of the irreducible components of X spans $\mathbb{P}^r$, $h^1(\mathbb{P}^r,\mathcal{I}_X(t)) = 0$ for all $t \ge 2$, $h^0(X,\mathcal{O}_X(1)) = r+1$, $h^1(\mathbb{P}^r,\mathcal{I}_X(t)) = 0$ for all $t \ge 2$ and $h^1(X,\mathcal{O}_X(2)) = 0$. If Y is nodal, then we may take X nodal. If $h^1(\mathbb{P}^{r-1},\mathcal{I}_Y(1)) = 0$, then we may take X such that $h^1(\mathbb{P}^r,\mathcal{I}_X(1)) = 0$. If Y has at most c irreducible components, then we may take X with at most $c+1$ irreducible components.*

Remark 1. Take Y, X as in Proposition 1. Y (resp. X) is arithmetically Cohen-Macaulay in $\mathbb{P}^{r-1}$ (resp. $\mathbb{P}^r$). In particular both Y and X are connected. Since $h^0(Y,\mathcal{O}_Y(1)) = r$, $h^1(Y,\mathcal{O}_Y(2)) = 0$ and $h^1(\mathbb{P}^{r-1},\mathcal{I}_Y(t)) = 0$, Riemann-Roch gives $2x+1-q \le \binom{r+1}{2}$. Since $h^0(X,\mathcal{O}_X(1)) = r+1$, $h^1(X,\mathcal{O}_X(2)) = 0$ and $h^1(\mathbb{P}^{r-1},\mathcal{I}_Y(2)) = 0$, Riemann-Roch gives $2(x+r+e)-q-e-r \le \binom{r+2}{2}$.

Proof of Proposition 1. See $\mathbb{P}^{r-1}$ as a hyperplane H of $\mathbb{P}^r$. Take $Y \subset H$ as above. Hence $\langle Y\rangle = H$ and H is linearly normal and arithemetically Cohen-

Macaulay in H.

First Claim. *There is $S\subset Y_{reg}$ such that $\sharp(S)=r+e$ and any r of the points of S are linearly independent.*

Proof of the First Claim. By assumption there is an irreducible component T of Y spanning H. Take as S a general subset of T with cardinality $r+e$. □

Fix S as in the First Claim. Since $e\le r$, there is a smooth and connected curve $D\subset\mathbb{P}^r$ such that $p_a(D)=e$, $\deg(D)=x+e$, D is linearly normal, $h^1(D,\mathcal{O}_D(1))=0$, and $D\cap H=S$. We have $h^1(\mathbb{P}^r,\mathcal{I}_D(t))=0$ for all $t\ge 0$. Set $X:=Y\cup D$. Hence $\deg(X)=\deg(Y)+\deg(D)$. Since D is non-degenerate, at least one of the irreducible components of X is non-degenerate. Thus X spans $\mathbb{P}^r$. Since $S\subset Y_{reg}$, $p_a(X)=p_a(Y)+p_a(D)+\sharp(S)-1$ and $\mathrm{Sing}(X)=\mathrm{Sing}(Y)\cup S$. If Y has c irreducible components, then X has $c+1$ irreducible components. If Y is nodal, then X is nodal. Since $S\neq\emptyset$ and Y is connected, X is connected, i.e. $h^1(\mathbb{P}^r,\mathcal{I}_X)=0$.

Second Claim. $h^i(\mathbb{P}^r,\mathcal{I}_X(1))=0$, $i=0,1$, *and* $h^1(X,\mathcal{O}_X(2))=0$.

Proof of the Second Claim. Since $D\cap H=S$ (scheme-theoretically), for any integer t we have the following Mayer-Vietoris exact sequence

$$0\to\mathcal{O}_X(t)\to\mathcal{O}_Y(t)\oplus\mathcal{O}_D(t)\to\mathcal{O}_S(t)\to 0\,. \tag{1}$$

Since $H\cap D=S$, Y is linearly normal in H and D is linearly normal, the case $t=1$ of (1) gives $h^0(X,\mathcal{O}_X(1))=r+1$. Since X spans $\mathbb{P}^r$, we get $h^i(\mathbb{P}^r,\mathcal{I}_X(1))=0$, $i=0,1$. We assumed $h^1(Y,\mathcal{O}_Y(2))=0$. Since $2\cdot\deg(D)=2r+2e>2(r+e-1)=2p_a(D)-2$, we get $h^1(D,\mathcal{O}_D(2))=0$ (here we use the irreducibility of D). Since $S=H\cap D$ and $h^1(D,\mathcal{O}_D(1))=0$, the restriction map $H^0(D,\mathcal{O}_D(2))\to H^0(S,\mathcal{O}_S(2))$ is surjective. Hence (1) gives $h^1(X,\mathcal{O}_X(2))=0$, concluding the proof of the Second Claim. □

Third Claim. $h^1(\mathbb{P}^r,\mathcal{I}_{Y\cup D}(t))=0$ *for all* $t\ge 2$.

Proof of Third Claim. We have $\mathrm{Res}_H(Y\cup D)=D$, $h^1(\mathbb{P}^r,\mathcal{I}_D(t-1))=0$ for all $t\ge 1$ and $(Y\cup D)\cap H=Y$ (scheme-theoretically). Apply Lemma 1. □

The proof of the Third Claim shows that if $h^1(\mathbb{P}^{r-1},\mathcal{I}_Y(t))=0$ (i.e. Y is linearly normal), then $h^1(\mathbb{P}^r,\mathcal{I}_{Y\cup D}(1))=0$ (i.e. $Y\cup D$ is linearly normal). □

Lemma 2. *Let $A\subset\mathbb{P}^r$, $r\ge 3$, be a reduced curve such that $\langle A\rangle=\mathbb{P}^r$ and A has at least one irreducible component, T, such that $\dim(\langle T\rangle)\ge r-1$. Assume $f:=h^0(\mathbb{P}^r,\mathcal{I}_A(2))>0$ and fix an integer z such that $1\le z\le f$. Let U be the union of A and z general secant lines to T. Then $\deg(U)=\deg(A)+z$, $p_a(U)=p_a(A)+z$ and $h^0(\mathbb{P}^r,\mathcal{I}_U(2))=f-z$. Moreover, $h^0(U,\mathcal{O}_U(t))=h^0(A,\mathcal{O}_A(t))+t-1$, $h^1(U,\mathcal{O}_U(t))=h^1(A,\mathcal{O}_A(t))$, $h^0(\mathbb{P}^r,\mathcal{I}_A(t))=h^1(\mathbb{P}^r,\mathcal{I}_U(t))+$*

$t-1$ and $h^1(\mathbb{P}^r,\mathcal{I}_A(t)) = h^1(\mathbb{P}^r,\mathcal{I}_U(t))$ for all integers $t \ge 1$.

Proof. By induction on z we reduce to the case $z = 1$. Since $f > 0$, there is a quadric hypersurface Q of $\mathbb{P}^r$ such that $A \subset Q$. Since $\langle A\rangle = \mathbb{P}^r$, Q is not a double hyperplane, i.e. the singular locus of Q is a linear space of dimension at most $r-2$. Hence the singular locus of Q does not contain T. Fix a general $P \in T$. Since the singular locus of any quadric hypersurface is a linear space and T spans $\mathbb{P}^r$, Q is smooth at P, i.e. it is not a cone with vertex containing P. Since T is non-degenerate, the general secant line D to T passing through P is not contained in Q. Hence $h^0(\mathbb{P}^r,\mathcal{I}_U(2)) \le f-1$, where $U := A\cup D$. Since any quadric hypersurface containing 3 points of D contains D and $\sharp(D\cap A) \ge 2$, we have $h^0(\mathbb{P}^r,\mathcal{I}_U(2)) \ge f-1$. Hence $h^0(\mathbb{P}^r,\mathcal{I}_U(2)) = f-1$. Since A spans $\mathbb{P}^r$, U spans $\mathbb{P}^r$. Hence U is linearly normal if and only if $h^0(U,\mathcal{O}_U(1))$. Fix any integer $t \ge 1$. Look at the Mayer-Vietoris exact sequence (1) with $S := A\cap D$. Since $S = A\cap D$ is scheme-theoretically the union of two points, the restriction map $H^0(D,\mathcal{O}_D(t)) \to H^0(A\cap D,\mathcal{O}_{A\cap D}(t))$ is surjective and its kernel has dimension $t-1$. Thus (1) gives $h^0(U,\mathcal{O}_U(t)) = h^0(A,\mathcal{O}_A(t)) + t - 1$ and $h^1(U,\mathcal{O}_U(t)) = h^1(A,\mathcal{O}_A(t))$ for all integers $t \ge 1$. From the case $t = 2$ just proved and using the union of Q and $t-2$ hyperplanes we get that D for every integer $t \ge 3$ the line D imposes exactly $t-1$ linearly independent conditions to the linear system $|\mathcal{I}_A(t)|$, i.e. $h^0(\mathbb{P}^r,\mathcal{I}_A(t)) = h^1(\mathbb{P}^r,\mathcal{I}_U(t)) + t - 1$. Since $h^0(U,\mathcal{O}_U(t)) = h^0(A,\mathcal{O}_A(t)) + t - 1$, we obtain $h^1(\mathbb{P}^r,\mathcal{I}_A(t)) = h^1(\mathbb{P}^r,\mathcal{I}_U(t))$ for all $t \ge 3$. The case $t = 1$ is obvious, because A is non-denerate, $A \subset U$, and we proved that $h^0(U,\mathcal{O}_U(1)) = h^0(A,\mathcal{O}_A(1))$. □

Lemma 3. *Fix integers r, x, q, a such that $r \ge 4$, $0 \le a \le r-2$. Assume the existence of a reduced curve $Y \subset \mathbb{P}^{r-1}$ such that $\deg(Y) = x$, for all $t \ge 0$, $h^0(Y,\mathcal{O}_Y(1)) = r$, and $h^1(Y,\mathcal{O}_Y(2)) = 0$. Then there exists a reduced curve $X \subset \mathbb{P}^r$ such that $\deg(X) = x+a+1$, $p_a(Y) = q+a$, $h^1(\mathbb{P}^r,\mathcal{I}_X(t)) = 0$ for all $t \ge 0$, $h^0(X,\mathcal{O}_X(1)) = r+1$, and $h^1(X,\mathcal{O}_X(2)) = 0$. If Y is nodal, then we may take X nodal. If Y has at most c irreducible components, then we may take X with at most $c+1$ irreducible components. If Y has an irreducible component spanning a linear subspace of dimension $\ge m$, then we may find X with the additional property of having an irreducible component spanning a linear space of dimension $\ge \max\{m, a\}$.*

Proof. Take a hyperplane $H \subset \mathbb{P}^r$ and $Y \subset H$ as in the statement. Fix an irreducible component T of Y spanning H and a general $S \subset T$ such that $\sharp(S) = a+1$. Hence $S \subset T\cap Y_{reg}$ and $\dim(\langle S\rangle) = a$. Let M be any $(a+1)$-dimensional linear subspace of $\mathbb{P}^r$ such that $M\cap H = \langle S\rangle$. Let D be any rational normal curve of M such that $S \subset D$. Set $X := Y \cup D$. Adapt the proof of Proposition 1 to show that X has all the properties listed in the statement of Lemma 3. □

Notice that in the statement of Lemma 3 we do not claim that at least one of the irreducible components of X spans $\mathbb{P}^r$.

Lemma 4. *Let $A \subset \mathbb{P}^r$, $r \geq 3$, be a reduced curve such that $\langle A \rangle = \mathbb{P}^r$ and A has at least one irreducible component, T, such that $\dim(\langle T \rangle) \geq r-1$.Assume $h^0(\mathbb{P}^r, \mathcal{I}_A(2)) \geq 2$. Fix a general $P \in T$ and a general line $D \subset \mathbb{P}^r$, such that $P \in D$. Set $U := A \cup D$. Then $\deg(U) = \deg(A)+1$, $p_a(U) = p_a(A)$ and $h^0(\mathbb{P}^r, \mathcal{I}_U(2)) = h^0(\mathbb{P}^r, \mathcal{I}_A(2)) - 2$. We have $h^0(U, \mathcal{O}_U(t)) = h^0(A, \mathcal{O}_A(t)) + t$, $h^1(U, \mathcal{O}_U(t)) = h^1(A, \mathcal{O}_A(t))$, $h^0(\mathbb{P}^r, \mathcal{I}_A(t)) = h^1(\mathbb{P}^r, \mathcal{I}_U(t)) + t$ and $h^1(\mathbb{P}^r, \mathcal{I}_A(t)) = h^1(\mathbb{P}^r, \mathcal{I}_U(t))$ for all integers $t \geq 1$.*

Proof. Take two generals $Q_1, Q_2 \in |\mathcal{I}_A(2)|$. Since $h^0(\mathbb{P}^r, \mathcal{I}_A(2)) \geq 2$, Q_1 and Q_2 are linearly independent. As in the proof of Lemma 2 we see that it is sufficient to prove $h^0(\mathbb{P}^r, \mathcal{I}_U(2)) = h^0(\mathbb{P}^r, \mathcal{I}_A(2)) - 2$. By semicontinuity it is sufficient to prove the existence of a line $R \subset \mathbb{P}^r$ such that $P \in R$ and $h^0(\mathbb{P}^r, \mathcal{I}_{A \cup R}(2)) = h^0(\mathbb{P}^r, \mathcal{I}_A(2)) - 2$. Since the singular locus of a quadric hypersurface is a linear space, T is non-degenerate and P is general in T, $P \in (Q_1)_{reg} \cap (Q_2)_{reg}$. Since Q_1 is not a cone with vertex P, a general line through P is not contained in Q_1. Hence $h^0(\mathbb{P}^r, \mathcal{I}_U(2)) \leq h^0(\mathbb{P}^r, \mathcal{I}_A(2)) - 1$ and $h^0(\mathbb{P}^r, \mathcal{I}_U(2)) = h^0(\mathbb{P}^r, \mathcal{I}_A(2)) - 1$ if and only if for a general $P \in T$ the set $E_1(P)$ of all lines passing through P is contained in Q_1 is equal to the set $E_2(P)$ of all lines passing through P is contained in Q_2. Assume that this is the case for all sufficiently general $P \in T$. Notice the cone $T_P Q_i \cap Q_i$ is the union of all lines of $E_i(P)$ and that, varying P in a non-empty open subset V of T the set $\cup_{P \in V}(T_P Q_i \cap Q_i)$ is dense in Q_i. Hence $E_1(P) = E_2(P)$ for all general $P \in T$ implies $Q_1 = Q_2$, contradiction. □

Proof of Theorem 1. Set $c_3 := 1$. For all integers $r \geq 4$ define inductively the integer c_r by the formula $c_r := c_{r-1} + r$. Hence $c_r = r^2/2 - r/2 - 2$ for all $r \geq 3$. The integer c_r is the upper bound for the integers c with respect to $\mathbb{P}^r$ in the statement of Theorem 1. Since the case $c = 0$ is true ([1] or [3]) (and we may even take as X a smooth and connected curve), from now on we assume $c > 0$. In steps (a), (b), and (c) we assume $f = 0$.

(a) Assume $r = 3$. By [2] everything is true (for $c = 1$ and $g = 4$ we take a canonically embedded curve, for $c = 1$ and $g = 5$ we may use [2], theorem 1).

From now on we assume $r \geq 4$ and that the case $f = 0$ of Theorem 1 is true in $\mathbb{P}^{r-1}$. Fix a hyperplane H of $\mathbb{P}^r$.

(b) Here we assume $c \geq r$. Set $c' := c - r$ and $g' := g - 2r + 1$. Notice that $0 \leq c' \leq c_{r-1}$. Hence we may apply the inductive assumption with respect to the datum $(r', g', c', 0)$. We get a reduced and curve $Y \subset H$ such that at least one of the irreducible components of Y spans H, $p_a(Y) = g'$, $h^1(Y, \mathcal{O}_Y(1)) = c'$, $h^1(Y, \mathcal{O}_Y(2)) = 0$ $h^1(\mathbb{P}^r, \mathcal{I}_U(t))$ for all integers $t \geq 0$.

(c) Here we assume $1 \leq c \leq r - 1$. By assumption we have $g \geq c + r - 1$. Set $c' := 0$ and $g' := g - c - r + 1$. Take $Y \subset H$ satisfying the thesis of Theorem 1 for the datum $(r - 1, c', g', f')$ with $f' = 0$. Then apply Proposition 1 with $q = g - c - r + 1$, $x = q + r - 1$ and $e = c$.

(d) Now assume $f > 0$. Take a solution Y for the datum $(r', g', c', f') = (r, g, c, 0)$ and then apply f times Lemma 4. □

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