International Journal of Pure and Applied Mathematics
Volume 53 No. 2 2009, 163-174

FORMULAE RELATIONSHIP TO THE DISTRIBUTIONAL PRODUCT OF $\delta^{(k-1)}(x).\delta^{(l-1)}(x)$ AND $\delta^{(k-1)}(x-a).\delta^{(l-1)}(x-b)$

Manuel A. Aguirre
Núcleo Consolidado Matemática Pura y Aplicada
Facultad de Ciencias Exactas
Universidad Nacional del Centro
Tandil, Provincia de Buenos Aires, ARGENTINA
e-mail: maguirre@exa.unicen.edu.ar

Abstract: One of the problem in distribution theory is the lack of definitions for products and power of distributions in general. In physics (c.f. [7], p. 141), oneself finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions. Chenkuan Li (see [11]) derives that $\delta^2(x) = 0$ on even-dimension space by applying the Laurent expansion of $r = 1$. Koh and Li in [10] give a sense to distribution δ^k and $(\delta^{`})^k$ for some k, using the concept of neutrix limit. Aguirre in [3], gives a sense to distributional product of $\delta^{(m)}(x).\delta^{(l)}(x)$, using the Hankel transform of generalized function of $\delta^{(m)}(x)$. In this paper using the Fourier transform of $\delta^{(k-1)}(x)$ we obtain formulae for the distributional product of $\delta^{(k-1)}(x).\delta^{(l-1)}(x)$ and $\delta^{(k-1)}(x-a).\delta^{(l-1)}(x-b)$, where k and $l = 1, 2, 3, ...$ As consequence we give a sense at the following product: $\delta^2(x), (\delta(x))^k, (\delta^{(k-1)}(x))^m, (\delta(x-a))^t$ and $(\delta^k(x-a))^m$. Finally, we write formulae relations with distributional products of $(\delta^k(P(x_1, ...x_n))^2$ and $(\delta^k(m^2 + P(x_1, ...x_n))^2$ where $P(x_1, ...x_n)$ is defined by (74).

AMS Subject Classification: 47Bxx, 45P05, 47G10, 32A25, 32M15
Key Words: distribution theory, Laurent expansion, Hankel transform

1. Introduction

One the problem in distributiontheory is the lack of definition for products and power of distributions in general. In physics (c.f. [7], p. 141), oneself

Received: March 31, 2009 © 2009 Academic Publications

finds the need to evaluate $t\delta^2$ when calculating the transition rates of certain particle interactions. Chenkuan Li (see [11]) derives that $\delta^2(x) = 0$ on even-dimension space by applying the Laurent expansion of $r = 1$. Koh and Li in [10] give a sense to distribution δ^k and $(\delta^{`})^k$ for some k, using the concept of neutrix limit. Aguirre in [3], gives a sense to distributional product of $\delta^{(m)}(x).\delta^{(l)}(x)$, using the Hankel transform of generalized function of $\delta^{(m)}(x)$. In this paper using the Fourier transform of $\delta^{(k-1)}(x)$ we obtain formulae for the distributional product of $\delta^{(k-1)}(x).\delta^{(l-1)}(x)$ and $\delta^{(k-1)}(x-a).\delta^{(l-1)}(x-b)$, where k and $l = 1, 2, 3, ...$ As consequence we give a sense at the following product: $\delta^2(x), (\delta(x))^k, (\delta^{(k-1)}(x))^m, (\delta(x-a))^t$ and $(\delta^k(x-a))^m$. Finally, we write formulae relations with distributional products of $(\delta^k(P(x_1, ...x_n))^2$ and $(\delta^k(m^2 + P(x_1, ...x_n))^2$ where $P(x_1, ...x_n)$ is defined by (74).

Let x_+^λ be the function equal to x^λ for $x > 0$, and zero for $x \leq 0$. The generalized functions corresponding to x_+^λ and x_-^λ are defined by

$$(x_+^\lambda, \varphi) = \int_0^\infty x^\lambda \varphi(x) dx \tag{1}$$

(see [8], p. 48) and

$$(x_+^\lambda, \varphi) = \int_0^\infty |x|^\lambda \varphi(x) dx \tag{2}$$

(see [8], p. 49).

The functions $x_\pm^\lambda$ have simple poles at $\lambda = -k, k = 1, 2, ...$ and the residues there are

$$\underset{\lambda=-k,k=1,2,..}{\mathrm{Res}\, x_+^\lambda} = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(x) \quad \text{(see [8], p. 49)} \tag{3}$$

and

$$\underset{\lambda=-k,k=1,2,..}{\mathrm{Res}\, x_-^\lambda} = \frac{1}{(k-1)!}\delta^{(k-1)}(x) \quad \text{(see [8], p. 50)}\,. \tag{4}$$

Let $\frac{x_+^\lambda}{\Gamma(\lambda+1)}$ be entire function of λ, where $\Gamma(\lambda + 1)$ is the gamma function defined by

$$\Gamma(\lambda + 1) = \int_0^\infty x^\lambda e^{-x} dx, \tag{5}$$

the values of this generalized function of the singular points of the numerator and denominator can be obtained by taking the ratios of the corresponding residues. From (3) and (4) and using the formula

$$\underset{z=-t,t=0,1,2,..}{\mathrm{Res}\, \Gamma(z)} = \frac{(-1)^t}{t!} \text{ for } t = 0, 1, 2, ..., \tag{6}$$

we arrive at the following formula

$$\lim_{\lambda\to -k, k=1,2,..} \frac{x_+^\lambda}{\Gamma(\lambda+1)} = \delta^{(k-1)}(x) \quad \text{(see [8], p. 57, formula (1))} \tag{7}$$

and

$$\lim_{\lambda\to -k, k=1,2,..} \frac{x_-^\lambda}{\Gamma(\lambda+1)} = (-1)^{k-1}\delta^{(k-1)}(x) \quad \text{(see [8], formula (2)).} \tag{8}$$

Let $(x \pm i0)^\lambda$ be the functions defined by

$$(x+i0)^\lambda = \lim_{\varepsilon\to 0^+}(x+i\varepsilon)^\lambda \quad \text{(see [8], p. 59)} \tag{9}$$

and

$$(x-i0)^\lambda = \lim_{\varepsilon\to 0^-}(x-i\varepsilon)^\lambda \quad \text{(see [8], p. 59)} \tag{10}$$

The generalized functions corresponding to these ordinary functions are denoted in the same form $(x \pm i0)^\lambda$ and the following properties are valid,

$$(x+i0)^\lambda = x_+^\lambda + e^{\lambda\pi i}x_-^\lambda \quad \text{(see [8], p. 60, formula (3))} \tag{11}$$

and

$$(x-i0)^\lambda = x_-^\lambda + e^{-\lambda\pi i}x_-^\lambda \text{((see [8], p. 60, formula (4))}\,. \tag{12}$$

We know from [8], p. 80 that the generalized functions $(x+i0)^\lambda$ and $(x-i0)^\lambda$ are homogeneous of degree λ and are generalized function entire of λ.

1.1. The Convolution Product of $(x+i0)^\lambda * (x-i0)^\lambda$

Let K be the space of complex test function infinitely differentiable with bounded support (c.f. [8], Chapter I, Sections 1-9) and the corresponding complex generalized function space $K^{‘}$.

As in [8]), we defined the Fourier transform of a function φ in K by

$$\psi(s) = \mathcal{F}\{\varphi\}(s) = \hat{\varphi}(s) = \int_{-\infty}^{+\infty} \varphi(x)e^{ixs}dx\,, \tag{13}$$

where $s = \sigma + i\tau$ is complex variable.

We know from [8], pp. 154-155 that $\psi(s)$ is an entire analytic function with the following property for $q = 0, 1, 2...$

$$|s^q\psi(s)| \le C_q e^{\alpha|\mathrm{Im}s|} \tag{14}$$

for the same constants C_q and α depending on $\psi(s)$.

The set of all entire analytic functions with property (14) is indeed the space

$$Z = F(K) = \{\psi : \exists\ \varphi\ \epsilon\ K \text{ and } \mathcal{F}\{\varphi\}=\psi\}\,. \tag{15}$$

The Fourier transform $\{f\}^{\Lambda}$ of a distribution f is an ultradistribution in $Z^{‘}$, i.e. a linear and continuous functional on Z. It is defined by Parseval's equation

$$< \{f\}^{\Lambda}, \overset{\Lambda}{\varphi} >= 2\pi < f, \varphi > \quad \text{(see [8], p. 167, formula (1))}\,. \tag{16}$$

We know from [8], p. 169-170 that the Fourier transform of f, $\{f\}^{\Lambda}$ can be extend from Z to S, where S is the Schwartz space (see [12], p. 233) and (see [8] p. 16) and from (see [8] p. 170), the generalized function $\mathcal{F}\{x_+^{\lambda}\}$, $\mathcal{F}\{x_-^{\lambda}\}$, $\mathcal{F}\{|x|^{\lambda}\}$ and $\mathcal{F}\{|x|^{\lambda}\, sqn(x)\}$ can be considered as functionals on S.

From [8], pp. 172-174 using (13) we have the Fourier transform of

$$\frac{x_+^{\lambda}}{\Gamma(\lambda+1)}, \frac{x_-^{\lambda}}{\Gamma(\lambda+1)}, (x+i0)^{\lambda} \text{ and } (x-i0)^{\lambda}.$$

$$\mathcal{F}\{\frac{x_+^{\lambda}}{\Gamma(\lambda+1)}\} = ie^{\frac{\lambda\pi i}{2}}(\sigma+i0)^{-\lambda-1} \quad \text{(see [8], p. 172, formula (3))}, \tag{17}$$

$$\mathcal{F}\{\frac{x_-^{\lambda}}{\Gamma(\lambda+1)}\} = -ie^{\frac{-\lambda\pi i}{2}}(\sigma-i0)^{-\lambda-1} \quad \text{(see [8], p. 172, formula (8))}, \tag{18}$$

$$\mathcal{F}\{(x+i0)^{\lambda}\} = \frac{2\pi e^{\frac{\lambda\pi i}{2}}}{\Gamma(-\lambda)}\sigma_-^{-\lambda-1} \quad \text{(see [8], p. 174, formula (22))} \tag{19}$$

and

$$\mathcal{F}\{(x-i0)^{\lambda}\} = \frac{2\pi e^{\frac{\lambda\pi i}{2}}}{\Gamma(-\lambda)}\sigma_+^{-\lambda-1} \quad \text{(see [8], p. 174, formula (23))}. \tag{20}$$

Lemma 1. *Let $(x\pm i0)^{\lambda}$ be the generalized functions defined by (9) and (10), then the following formula is valid*

$$(x+i0)^{\lambda} * (x-i0)^{\lambda} = C_{\lambda,\mu}(x\pm i0)^{\lambda+\mu-1} \text{ if } \lambda,\mu \text{ and } \lambda+\mu \neq 1,2,3,\ldots, \tag{21}$$

where the symbool $*$ we means convolution, λ and μ are complex numbers and

$$C_{\lambda,\mu} = \mp\frac{2\pi i\Gamma(1-\lambda-\mu)}{\Gamma(1-\lambda)\Gamma(1-\mu)}\,. \tag{22}$$

Proof. Considering that $(x+i0)^{\lambda}$ and $(x-i0)^{\lambda}$ are generalized function entire of λ, homogeneous of degree λ and taking into account that every homogeneous distribution is tempreated ([6], p. 154-155), using the classic theorem of L. Schwartz (see [12], p. 268, formula (IV, 8, 4)) we conclude the validy of the following formula:

$$\mathcal{F}\{(x+i0)^{\lambda-1} * (x+i0)^{\mu-1}\} = \mathcal{F}\{(x+i0)^{\lambda-1}\}.\mathcal{F}\{(x+i0)^{\mu-1}\} \tag{23}$$

and

$$\mathcal{F}\{(x-i0)^{\lambda-1}*(x-i0)^{\mu-1}\}=\mathcal{F}\{(x-i0)^{\lambda-1}\}.\mathcal{F}\{(x-i0)^{\mu-1}\}\,, \tag{24}$$

where $\mathcal{F}$ designates the Fourier transform defined by (13).

From (23) and considering (19) we have

$$\begin{aligned}\mathcal{F}\{(x+i0)^{\lambda-1}*(x+i0)^{\mu-1}\} &= \frac{2\pi.2\pi e^{(\lambda+\mu)\frac{\pi i}{2}}e^{-\frac{\pi i}{2}}e^{-\frac{\pi i}{2}}}{\Gamma(1-\lambda)\Gamma(1-\mu)}\sigma_-^{-\lambda-\mu}\\ &= -\frac{2\pi.i\Gamma(1-\lambda-\mu)}{\Gamma(1-\lambda)\Gamma(1-\mu)}\mathcal{F}\{(x+i0)^{\lambda+\mu-1}\}. \end{aligned} \tag{25}$$

From (25) we have,

$$(x+i0)^{\lambda-1}*(x+i0)^{\mu-1}=-\frac{2\pi i\Gamma(1-\lambda-\mu)}{\Gamma(1-\lambda)\Gamma(1-\mu)}(x+i0)^{\lambda+\mu-1} \tag{26}$$

if λ,μ, and $\lambda+\mu\neq 1,2,3,...$

Similarly from (24) and using (20) we have,

$$(x-i0)^{\lambda-1}*(x-i0)^{\mu-1}=\frac{2\pi i\Gamma(1-\lambda-\mu)}{\Gamma(1-\lambda)\Gamma(1-\mu)}(x-i0)^{\lambda+\mu-1} \tag{27}$$

if λ,μ, and $\lambda+\mu\neq 1,2,3,...$

From (26) and (27) we obtain the formula (21). The formula (21) appears in ([5], p. 601, formula (4)). □

1.2. The Distribution Product of $\delta^{(k-1)}(x)$ and $\delta^{(l-1)}(x)$

From (7) and (17) we have the Fourier transform of $\delta^{(k-1)}(x)$,

$$\mathcal{F}\{\delta^{(k-1)}(x)\}=\lim_{\lambda\to-k}\mathcal{F}\{\frac{x_+^\lambda}{\Gamma(\lambda+1)}\}=\lim_{\lambda\to-k}(ie^{\frac{\lambda\pi i}{2}}(\sigma+i0)^{-\lambda-1})\,. \tag{28}$$

From (28) we obtain the following formula

$$\mathcal{F}\{\delta^{(k-1)}(x)\}=ie^{\frac{k\pi i}{2}}(\sigma+i0)^{-k-1}. \tag{29}$$

Using (28) we have,

$$\begin{aligned}\mathcal{F}\{\delta^{(k-1)}(x)\}*\mathcal{F}\{\delta^{(l-1)}(x)\} &= \lim_{\lambda\to-k}(ie^{\frac{\lambda\pi i}{2}}(\sigma+i0)^{-\lambda-1})*\lim_{\mu\to-l}(ie^{\frac{\mu\pi i}{2}}(\sigma+i0)^{-\mu-1})\\ &= \lim_{\lambda\to-k}\lim_{\mu\to-l}ie^{\frac{\lambda\pi i}{2}}ie^{\frac{\mu\pi i}{2}}((\sigma+i0)^{-\lambda-1}*(\sigma+i0)^{-\mu-1}), \end{aligned} \tag{30}$$

using Lemma 1, fromula (21), we have,

$$\mathcal{F}\{\delta^{(k-1)}(x)\}*\mathcal{F}\{\delta^{(l-1)}(x)\}$$

$$= \lim_{\lambda\to -k}\lim_{\mu\to -l} i.i.e^{\frac{(\lambda+\mu)\pi i}{2}}(-\frac{2\pi i\Gamma(1-\lambda-\mu)}{\Gamma(1-\lambda)\Gamma(1-\mu)}(\sigma+i0)^{\lambda+\mu-1}), \quad (31)$$

if λ, μ and $\lambda+\mu \neq 1, 2, 3, ...$

Taking into account that the distribution $\delta^{(m-1)}(x)$ is homogeneous degree $-m$ (see [8], p. 80) and consecuently tempered ([6], p. 154-155), i.e.

$$\delta^{(k-1)}(x)\epsilon S', \quad (32)$$

where $S\prime$ is the dual of S and S is the Schwartz test space and considering the formula (31) we define the distributional product of $\delta^{(k-1)}(x)$ and $\delta^{(l-1)}(x)$ through the following form:

Definition 2. Let $k-1$ and $l-1$ be a non-negative integers and $\delta^{(m-1)}(x)$ the distributional defined by the equation (7) and taking into account the formula (31) we defined the multiplicative product of $\delta^{(k-1)}(x)$ and $\delta^{(l-1)}(x)$ by means of the following formula

$$\delta^{(k-1)}(x).\delta^{(l-1)}(x) = \mathcal{F}^{-1}\left\{\mathcal{F}\{\delta^{(k-1)}(x)\} * \mathcal{F}\{\delta^{(l-1)}(x)\}\right\}, \quad (33)$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform (see [8], p. 155).

Now using the definition (33) and considering (29) and (31) we arrive at the following formula

$$\delta^{(k-1)}(x).\delta^{(l-1)}(x) = \mathcal{F}^{-1}\left\{i.i.e^{\frac{(-k-l)\pi i}{2}}(-\frac{2\pi i\Gamma(k+l+1)}{\Gamma(1+k)\Gamma(1+l)}(\sigma+i0)^{-k-l-1})\right\}$$

$$= \mathcal{F}^{-1}\left\{i.i.(-\frac{2\pi i\Gamma(1+k+l)}{\Gamma(1+k)\Gamma(1+l)}\frac{e^{\frac{(-k-l)\pi i}{2}}}{ie^{\frac{(-k-l)\pi i}{2}}}\mathcal{F}\{\delta^{(k+l-1)}(x)\})\right\}$$

$$= \frac{2\pi(k+l)!}{k!l!}\mathcal{F}^{-1}\mathcal{F}\{\delta^{(k+l-1)}(x)\}$$

$$= \frac{2\pi(k+l)!}{k!l!}\delta^{(k+l-1)}(x) = 2\pi\binom{k+l}{l}\delta^{(k+l-1)}(x), \quad (34)$$

where

$$\binom{k+l}{l} = \frac{(k+l)!}{l!k!}. \quad (35)$$

Remark 3. In [3], appears the distribution product $\delta^{(k-1)}(x).\delta^{(l-1)}(x) = A_{k,l}\delta^{(k+l-1)}(x)$ using the Hankel transform. In this paper we obtain the distribution product $\delta^{(k-1)}(x).\delta^{(l-1)}(x) = C_{k,l}\delta^{(k+l-1)}(x)$ using the Fourier transform. The difference is in the value of constant: In [3] the constant is

$$A_{k,l} = \frac{(-1)^{\frac{2}{3}}(m!)^2(l!)^2}{(m+l+1)!}\frac{\Gamma(m+l+\frac{1}{2})}{\Gamma(m+\frac{1}{2})\Gamma(l+\frac{1}{2})} \quad (36)$$

and in this paper

$$C_{k,l} = \frac{2\pi(k+l)!}{k!l!} = 2\pi\binom{k+l}{l}. \tag{37}$$

On the other hand, by putting $k = l$ in (34) we have,

$$\delta^{(k-1)}(x).\delta^{(l-1)}(x) = \frac{2\pi(2k)!}{k!k!}\delta^{(2k-1)}(x) \tag{38}$$

and for $k = 1$ considering that

$$\delta^{(0)}(x) = \delta(x)\,, \tag{39}$$

we have,

$$\delta(x).\delta(x) = 4\pi\delta'(x). \tag{40}$$

Using the formula (40) in this paper we obtain a new formula for the square of Dirac's delta taking into account the definition (33).

Definition 4. Let $\delta(x)$ be the distribution of Dirac's delta, we means by definition the square of Dirac's delta

$$\delta^2(x) = (\delta(x))^2 = \delta(x).\delta(x)\,, \tag{41}$$

where $\delta(x).\delta(x)$ is defined by the formula (40).

Using the formula (40) we obtain the following formula

$$\delta^2(x) = (\delta(x))^2 = \delta(x).\delta(x) = 4\pi\delta'(x). \tag{42}$$

By iteration s times the formula (41) and using (34) we obtain the following formula

$$\delta^s(x) = (\delta(x))^s = \pi^{s-1}\delta^{s-1}s!\delta^{(s-1)}(x) \text{ for } s = 1, 2, 3, ... \tag{43}$$

From (39) and taking into account the definition

$$(\delta^{(k-1)}(x))^2 = \delta^{(k-1)}(x).\delta^{(k-1)}(x)\,, \tag{44}$$

we have

$$\left(\delta^{(k-1)}(x)\right)^2 = \frac{2\pi(2k)!}{k!k!}\delta^{(2k-1)}(x) = \frac{2\pi(2k)!}{(k!)^2}\delta^{(2k-1)}(x) \tag{45}$$

By iteration m time and using the formulae (44) and (54) we arrive at the following formula

$$\left(\delta^{(k-1)}(x)\right)^m = \frac{(2\pi)^{m-1}(mk)!}{(k!)^m}\delta^{(mk-1)}(x)\,. \tag{46}$$

It is clear that by putting $k = 1$ in (46) and using (39) we obtain the formula (43).

In this paragraph we give a sense to the distributional product of $\delta^{(k-1)}(x-a).\delta^{(l-1)}(x-b)$ using the formula (26), when a and b we real numbers.

In order to do it we need the Fourier transform of $\delta^{(k-1)}(x-a)$ and $\delta^{(l-1)}(x-b)$.

Consider te fucntion $(x-a)_+^\lambda$ equal $(x-a)^\lambda$ for $x-a>0$, and zero for $x-a\leq 0$. The generalized function corresponding to it is defined the following form

$$< (x-a)_+^\lambda, \varphi >= \int_a^\infty (x-a)^\lambda \varphi(x) dx \tag{47}$$

for all $\varphi =$ in K, where K is the set of functions continuos derivatives of all orders and with bounded supported (see [8], p. 2).

Acording to [8], Chapter I, Section 3 $(x-a)_+^\lambda$ is a generalization function which is analytic everywhere except at $\lambda = -k, k = 1, 2, ..$ at which points it has poles with residues

$$\underset{\lambda=-k, k=1,2,\ldots}{\mathrm{Re}s} (x-a)_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!}(x-a). \tag{48}$$

From (48) and considering (see [8]) we have

$$\lim_{\lambda\to -k} \frac{(x-a)_+^\lambda}{\Gamma(\lambda+1)} = \delta^{(k-1)}(x-a)\,. \tag{49}$$

Let us now find the Fourier transform of $(x-a)_+^\lambda$, for those λ for which the corresponding integrals converge, the Fourier transform can be obtained using the same method of Fourier transform of x_+^λ ([8], pp. 170-171).

From (13) we have

$$\mathcal{F}\{e^{-x\tau}(x-a)_+^\lambda\} = \int_{-\infty}^{+\infty} e^{-x\tau}(x-a)_+^\lambda e^{ix\sigma} dx = \int_a^\infty e^{-x\tau} e^{ix\tau} dx\,, \tag{50}$$

where $\tau = \mathrm{Im} s > 0$, if that $0 < \arg s < \pi$.

The integral (50) converges. As $\tau \to 0^+, e^{-x\tau}(x-a)_+^\lambda$ converge to $(x-a)_+^\lambda$ in the sense of generalized functions, so that its Fourier transform converges to the desired Fourier transform of $(x-a)_+^\lambda$.

We shall calculate the integral (49) by making the change of variables $y = x-a$ we have

$$\mathcal{F}\{e^{-x\tau}(x-a)_+^\lambda\} = e^{-a(\tau+i(-\sigma))} \int_0^\infty y^\lambda e^{-y(\tau+i(-\sigma))} dy \tag{51}$$

and using the formula

$$\begin{aligned}\int_0^\infty x^{v-1} e^{-x(p+iq)} dx &= \Gamma(v)(p^2+q^2)^{-\frac{v}{2}} e^{-iv \arctan\frac{q}{p}} \\ &= \Gamma(v)\Gamma(p+iq)^{-\frac{v}{2}} \quad \text{(see [9], p. 328, formula (5))}\,, \end{aligned} \tag{52}$$

and by putting $v = \lambda + 1$, $p = \tau, q = -\sigma$ we obtain

$$\mathcal{F}\{e^{-x\tau}(x-a)_+^{\lambda}\} = e^{-a(\tau+i(-\sigma))}\Gamma(\lambda+1)(\sigma+i\tau)^{-\lambda-1} \quad (53)$$

From (53) by passing to the limit $\tau \to 0^+$ we arrive at the formula ([8], p. 171)

$$\mathcal{F}\{\frac{(x-a)_+^{\lambda}}{\Gamma(\lambda+1)}\} = \lim_{\tau\to 0}\{e^{-x\tau}\frac{(x-a)_+^{\lambda}}{\Gamma(\lambda+1)}\} = e^{ai\sigma}e^{(\lambda+1)\frac{\pi i}{2}}(\sigma + i0)^{-\lambda-1}. \quad (54)$$

Now from (51) and using (49) we obtain the following formula

$$\mathcal{F}\{\delta^{(k-1)}(x-a)\} = e^{ai\sigma}e^{(1-k)\frac{\pi i}{2}}(\sigma+i0)^{k-1}. \quad (55)$$

It is clear that by putting $a = 0$ in (55) we obtain the formula (29).

Now using (29) we have

$$\begin{aligned}&\{\delta^{(k-1)}(x-a)\}^{\hat{}} * \{\delta^{(l-1)}(x-b)\}^{\hat{}}\\ &= e^{ai\sigma}.e^{bi\sigma}.\mathcal{F}\{\delta^{(k-1)}(x)\} * \mathcal{F}\{\delta^{(l-1)}(x)\} = e^{(a+b)i\sigma}.\mathcal{F}\{\delta^{(k-1)}(x)\} * \mathcal{F}\{\delta^{(l-1)}(x)\}\\ &= \frac{2\pi(k+l)!}{k!l!}e^{(a+b)i\sigma}.\mathcal{F}\{\delta^{(k+l-1)}(x)\}. \quad (56)\end{aligned}$$

Taking into account the definition (33) and and using the formula (56) we obtain the following formula

$$\begin{aligned}\delta^{(k-1)}(x-a).\delta^{(l-1)}(x-b) &= \mathcal{F}^{-1}\{\mathcal{F}\{\delta^{(k-1)}(x-a)\} * \mathcal{F}\{\delta^{(l-1)}(x-b)\}\}\\ &= \mathcal{F}^{-1}\{\frac{2\pi(k+l)!}{k!l!}e^{(a+b)i\sigma}.\mathcal{F}\{\delta^{(k+l-1)}(x)\}\}\\ &= \frac{2\pi(k+l)!}{k!l!}e^{(a+b)i\sigma}.\delta^{(k+l-1)}(x), \quad (57)\end{aligned}$$

for a and b real numbers.

In particular if $k = l$ in (57) we have

$$\delta^{(k-1)}(x-a).\delta^{(k-1)}(x-b) = \frac{2\pi(2k)!}{k!k!}e^{(a+b)i\sigma}.\delta^{(2k-1)}(x), \quad (58)$$

and by putting $k = 1$ in (58) we obtain the following formula

$$\delta(x-a).\delta(x-b) = 4\pi\delta'(x)e^{(a+ib)i\sigma} \quad (59)$$

for all a and b real numbers.

Taking into account the definition (see, formula (41)) and by putting $a = b$ in (59) we have

$$\delta^2(x-a) = (\delta(x-a))^2 = \delta(x-a).\delta(x-a) = e^{(a+b)i\sigma}4\pi\delta'(x). \quad (60)$$

By iteration s times the formula (60), using (57) we obtain the following formula

$$\delta^s(x-a) = (\delta(x-a))^s = (e^{2ai\sigma})^s\pi^{s-1}2^{s-1}s!\delta^{(s-1)}(x-a). \quad (61)$$

It is clear that by putting $a = 0$ in (61) we obtain the formula (43).

By putting $k = l$ and $a = b$ in (57) and using the definition

$$(\delta^{(k-1)}(x-a))^2 = (\delta^{(k-1)}(x-a))^2 \tag{62}$$

we have,

$$(\delta^{(k-1)}(x-a))^2 = \frac{2\pi(2k)!}{k!k!} e^{2ai\sigma}.\delta^{(2k-1)}(x). \tag{63}$$

By iteration m times and using the formulae (63) and (57) we arrive at the following formula

$$(\delta^{(k-1)}(x-a))^m = (e^{2ai\sigma})^m \tfrac{\pi^{m-1}2^{m-1}(mk)!}{(k!)^m} \delta^{(mk-1)}(x-a). \tag{64}$$

It is clear that by putting $k = 1$ in (64) and using that

$$\delta^{(0)}(x-a) = \delta(x-a) \tag{65}$$

we obtain the formula (61).

In the case of several variables the distributional product $\delta(x_1, ..., x_n)$ $\delta(x_1, ..., x_n)$ appear in [4],

$$\delta(x_1, ..., x_n).\delta(x_1, ..., x_n) = C_{\frac{n}{2}-1,\frac{n}{2}-1,p,q,n} L^{\frac{n}{2}} \{\delta(x_1...x_n)\} \tag{66}$$

if n is even.

Here

$$C_{\frac{n}{2}-1,\frac{n}{2}-1,p,q,n} = \frac{((\frac{n}{2}-1)!)^2}{2^{n+1}(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}(n-1)!(\frac{n}{2})!}.\frac{1}{[\Psi(\frac{p}{2}) - \Psi(\frac{n}{2})]^2}, \tag{67}$$

$$\Psi(k) = -\gamma + 1 + \frac{1}{2} + ... + \frac{1}{k-1}, \tag{68}$$

$$\Psi(k + \frac{1}{2}) = -\gamma + 2\ln(2) + 2(1 + \frac{1}{3} + ... + \frac{1}{2k-1}), \tag{69}$$

γ Euler's constant and

$$L^j = \{\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2}\}^j \tag{70}$$

$j = 1, 2, ...$ Therefore using the definition

$$\delta^2(x_1, ..., x_n) = (\delta(x_1, ..., x_n))^2 = \delta(x_1, ..., x_n).\delta(x_1, ..., x_n) \tag{71}$$

from (66) we obtain the formula

$$\delta^2(x_1, ..., x_n) = C_{\frac{n}{2}-1,\frac{n}{2}-1,p,q,n} L^{\frac{n}{2}} \{\delta(x_1, ..., x_n)\}, \tag{72}$$

where $C_{\frac{n}{2}-1,\frac{n}{2}-1,p,q,n}$ is defined by (67).

On the other hand we know that the distributional product of

$$\delta(P(x_1, ..., x_n)).\delta(P(x_1, ..., x_n)), \tag{73}$$

exists, where

$$P(x_1, ..., x_n) = x_1^2 + ... + x_\mu^2 - x_{\mu+1}^2 - ... - x_{\mu+v}^2, \tag{74}$$

$\mu + v = n$ dimension of space (see [1] and [2]) and the distributional product

$$\delta(m^2 + P(x_1, ..., x_n)).\delta(m^2 + P(x_1, ..., x_n)) \tag{75}$$

exists (see [2], p. 163).

Therefore from [2], p. 168, formula (66), we have,

$$\delta(P).\delta(P) = D_{1,1,n}\delta'(P) \tag{76}$$

if μ and v are both odd, where

$$D_{1,1,n} = \frac{1}{2}\frac{1}{\Psi(\frac{\mu}{2}) - \Psi(\frac{n}{2})}, \tag{77}$$

$\Psi(k)$ and $\Psi(k + \frac{1}{2})$ are defined by (68) and (69), respectively, and

$$\delta(m^2 + P).\delta(m^2 + P) = D_{1,1,n}\delta'(m^2 + P) \tag{78}$$

where $D_{1,1,n}$ is defined by (77).

From (76) and (78) and taking into account the definition (41) we have the following formulae

$$(\delta(P))^2 = \delta(P).\delta(P) = \frac{1}{2}\frac{1}{\Psi(\frac{\mu}{2}) - \Psi(\frac{n}{2})}\delta'(P)\,, \tag{79}$$

and

$$(\delta(m^2 + P))^2 = \delta(m^2 + P).\delta(m^2 + P) = \frac{1}{2}\frac{1}{\Psi(\frac{\mu}{2}) - \Psi(\frac{n}{2})}\delta'(m^2 + P)\,, \tag{80}$$

if μ and v are both odd and

$$(\delta(m^2 + P))^2 = \delta(m^2 + P).\delta(m^2 + P) = \frac{1}{2}\frac{1}{\Psi(\frac{\mu}{2}) - \Psi(\frac{n}{2})}\delta'(m^2 + P)\,. \tag{81}$$

Acknowledgements

This work was partially supported by Comisión de Investigaciones Científicas de la provincia de Buenso Aires (C.I.C.), Argentina.

References

[1] M.A. Aguirre, The distributional product of Dirac delta in hypercone, *Journal of Computational and Applied Mathematics*, **115** (2000), 13-21.

[2] M.A. Aguirre, A generalization of distributional product of Dirac's delta in a hypercone, *Far East Journal of Mathematical Sciences (FJMS)*, Special Volume (Functional Analysis and its Applications) (2001), 155-169.

[3] M.A. Aguirre, The distributional product of $\delta^{(m)}(x).\delta^{(l)}(x)$, *Pacific Journal of Mathematics*, **1**, No. 1 (2008), 105-114.

[4] M.A. Aguirre, The distributional product associed to Dirac's delta, To Appear

[5] A. Bredimas, Generalized convolution product and canonical product between distributions, *Lettere al Nuovo Cimento*, **73**, No. 16 (1975).

[6] W.R. Dunoghue, *Distribution and Fourier Transform*, Academic Press (1969).

[7] S. Gastorowicz, *Elementary Particle Physics*, J. Wiley and Sons, Inc., N.Y. (1966).

[8] I.M. Gelfand, G.E. Shilov, *Generalized Functions*, Volume 1, Academic Press, New York (1964).

[9] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press (1980).

[10] E.L. Koh, C.K. Li, On the distributions δ^k and $(\delta^{`})^k$, *Math. Nachr.*, **157**, No. 8 (1992), 243-248.

[11] C.K. Li, The product on the unit sphere even dimension spaces, *JMAA*.

[12] L. Schwartz, *Distributions Theory*, Hermann, Nouveau Tirage (1988).