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A COMPARISON BETWEEN THE SELBERG AND THE BRUGGEMAN-KUTZNETSOV TRACE FORMULAS III

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Abstract: In this paper we elucidate and make somewhat transparent the clever technique of first introducing and then removing weights (Fourier coefficients of eigenfunctions) when employing the Bruggeman-Kuznetsov trace formula to obtain information on the distribution of the eigenvalues of the hyperbolic Laplacian for the modular group.

Frequently, this technique yields improvement of results obtained by the Selberg trace formula. This gain is realized because the sums on the geometric side of the Bruggeman-Kuznetsov trace formula involve sums and integrals, which apparently package certain cancellations in a more efficient way than do the sums involving class numbers, which appear naturally on the geometric side of the Selberg trace formula.

We do this by elaborating and significantly modifying the argument outlined in a letter from Sarnak to Rudnick, and, in the process, we improve one of the results obtained there. The limit of the construction is also discussed.

To the memory of my friend Paul Cohen.

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1. Introduction

We consider $\Gamma = \mathrm{PSL}(2, Z)$.

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Hence we have that

$$\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

are the eigenvalues (i.e. the point spectrum) of the hyperbolic Laplacian associated with Γ and

$$\lambda_j = s_j(1-s_j) \quad \text{with} \quad s_j = \frac{1}{2} + it_j\,,$$

so

$$\lambda_j = \frac{1}{4} + t_j^2 \quad \text{with} \quad t_j > 0 \quad j = 1, 2, 3, \dots\ .$$

The Weyl-Selberg formula states that for $\Gamma = \mathrm{PSL}(2, Z)$

$$M(T) = \sum_{0<t_j \le T} m(t_j) = \frac{1}{12}T^2 + c_1 T \log T + O(T),$$

where $m(t_j)$ denotes the multiplicity of λ_j, which is equal to the dimension of the eigenspace of λ_j (cf [11]). Let $m(T) = \sum\limits_{t_j = T} 1$. Let $A > 0$. Let $g \in C_c^\infty(1,3)$, $g(x) \ge 0$, and $\int\limits_{-\infty}^{\infty} g(x)dx = 1$.

Fix h an even test function in $\mathscr{S}(R)$, the space of rapidly decreasing functions with $\hat{h} \in C_c^\infty(-1,1)$ and $\int\limits_{-\infty}^{\infty} h(x)dx = 1$ and $\hat{h}(t) = \int\limits_{-\infty}^{\infty} h(x)e^{-2\pi ixt}dx$.

For T large and $1 \ll L \ll T^{1-\epsilon}$ define

$$H_T(t) \circeq h(L(t-T)) + h(L(-t-T)), \tag{1.1}$$

$$N_h(T,L) \circeq \sum_{0<t_j} h(L(t_j - T)), \tag{1.2}$$

$$\bar{N}_h(T,L) \circeq \frac{2\mathrm{vol}\,\Gamma}{4\pi}\left(\int\limits_{-\infty}^{\infty} h(x)dx\right)\frac{T}{L} = \frac{2\mathrm{vol}\,\Gamma}{4\pi}\frac{TI}{L}. \tag{1.3}$$

If h were allowed to be the characteristic function of $[-1,1]$, then $N_h(T,L)$ would count the numbers of t_j within $\frac{1}{L}$ of T. Hence, the larger we can take L the more information about the local distribution of the spectrum we can determine.

Using the Selberg trace formula Rudnick (cf. [10]) established in a straightforward way

Theorem 1.1. *If* $1 \le L \le \frac{\log T}{\pi}$, *then*

$$N_h(T,L) \sim \bar{N}_h(T,L) \quad \text{as} \quad T \to \infty.$$

Using the weighted Bruggeman-Kuznetsov trace formula, Sarnak (cf. [9]) has shown

Theorem 1.2. $1 \leq L \leq \frac{2\log T}{\pi}$, *then*

$$N_h(T,L) \sim \bar{N}_h(T,L) \quad \text{as} \quad T \to \infty.$$

Corollary 1.1.

$$\overline{\text{Limit}}_{T\to\infty} \frac{m(T)\log T}{T} \leq \frac{\pi}{12}.$$

This corollary gives the best known upper bound for the multiplicity $m(T)$. It is half of what one gets using the trace formula (i.e. from Theorem 1.1). It is also no doubt very far from the truth as one expects that $m(T)$ is at most 1!

Let h be defined as above except that for each $A > 0$, $\hat{h} \in C_c^\infty\left(-\frac{1}{A}, \frac{1}{A}\right)$. Then in this paper we modify significantly the proof of Theorem 1.2 to obtain

Theorem 1.3. *If* $1 \leq L \leq (2-\varepsilon_0)A\,\frac{\log T}{\pi}$ *for any* $\varepsilon_0 > 0$ *for any fixed* $A > 0$, *then there exists an* h *such that*

$$N_h(T,L) \sim \bar{N}_h(T,L) \quad \text{as} \quad T \to \infty.$$

2. The Weighted Bruggeman-Kuznetsov Trace Formula

Theorem 2.1. (Bruggeman-Kuznetsov) *Let h satisfy the Selberg trace formula conditions* (cf. [2]). *Then*

$$\sum_{1\leq j} h(t_j)\bar{\mathscr{V}_j}(m)\mathscr{V}_j(n) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\bar{\mathscr{N}}(m,t)\mathscr{N}(n,t)dt$$

$$= \delta_{mn}\frac{1}{\pi}\int_{-\infty}^{\infty} t\tanh(\pi t)h(t)dt + \sum_{c=1}^{\infty}\frac{s(m,n,c)}{c}h^+\left(\frac{4\pi\sqrt{|mn|}}{c}\right),$$

where $h^+(x) = 2i\int_{-\infty}^{\infty} J_{2it}(x)\frac{h(t)t}{\cosh \pi t}dt$ *and where* $\mathscr{V}_j(n)$, $\mathscr{V}_j(m)$, $\mathscr{N}(n,t)$, $\mathscr{N}(m,t)$ *are Fourier coefficients of any arithmetical system of Maass forms, and the eigenpacket of Eisenstein series.*

For a nice proof of this theorem, see [2].

Now choose the unique Hecke basis $\{\phi_j\}$. Then

$$\mathscr{V}_j(n) = \lambda_j(n)\gamma_{\phi_j}$$

and

$$\mathscr{V}_j(m) = \lambda_j(m)\gamma_{\phi_j}\ ,$$

where

$$\gamma_{\phi_j} = \left(\frac{L(1, \mathrm{sym}^2\,\phi_j)}{2\pi}\right)^{1/2},$$

and

$$\lambda_j(1) = 1.$$

Letting $n = m$ we get the so-called weighted Bruggeman-Kuznetsov trace formula.

Theorem 2.2.

$$\sum_{1\le j} h(t_j)|\lambda_j(n)|^2\gamma_{\phi_j}^2 + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)|\mathscr{N}(n,t)|^2 dt$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} t\,\tanh(\pi t)h(t)dt + \sum_{c=1}^{\infty}\frac{s(n,n,c)}{c}h^+\left(\frac{4\pi n}{c}\right).$$

The techniques used in this paper, as described in the abstract, were first used by Iwaniec in [3].

3. Lemmata

Lemma 3.0.

$$h(z) \textit{ is even}\,,\quad h(z) \textit{ is entire}\,,\quad h(z) \ll_k \frac{e^{\frac{2\pi}{A}|y|}}{(1+|z|)^k}.$$

Proof. The proof follows from partial summation and the elementary properties of the Fourier transform. □

Lemma 3.1. $t_j^{-\epsilon} \ll_\epsilon |\gamma_{\phi_j}|^2 \ll \log t_j$.

Proof. This is established in [6]. □

Lemma 3.2. $\displaystyle\sum_{0<t_j\le\frac{5}{2}T} |H_T(t_j)| = O(T)$.

Proof. The proof follows by partial summation applied to the expansion

$$S = \frac{1}{L^{2k}} \sum_{0<t_j<T-1} (t_j - T)^{-2k} + \sum_{T-1 \leq t_j \leq T+1} 1$$
$$+ \frac{1}{L^{2k}} \sum_{T+1<t_j \leq \frac{5}{2}T} (t_j + T)^{-2k}. \quad \square$$

Let

$$S(\phi_j) = \sum_{n=1}^{\infty} \gamma_{\phi_j}^2 |\lambda_j(n)|^2 g\left(\frac{n}{M}\right), \tag{3.1}$$

where $M = T^{\Delta}$ for $0 < \Delta < \frac{1}{100}$.

Lemma 3.3. *$S(\phi_j) = E_1(t_j)$ where $E_1(t_j) = O(M^{\frac{5}{4}}|t_j^{\epsilon}|)$*

Proof. This is immediate from the estimate due to Kim and Sarnak (cf. [2])

$$|\lambda_j(n)| \ll n^{\frac{7}{64}+\epsilon}$$

and Lemma 3.1. $\square$

In Lemma 1 in [6] let $\sigma = 1 - \delta$, where $\delta > 0$ is chosen so that $(\delta b + \epsilon) < \frac{1}{10}$. Consider $0 < t_j \leq (5/2)T$. Define $R = \{\rho = \beta + i\gamma \mid 0 < \delta \leq 1/2, (1-\delta) \leq \beta < 1, |\gamma| \leq \log^3 \frac{5}{2} T\}$. Define $B = \{t_j \mid 0 < t_j \leq \frac{5}{2} T$ and $L(\rho, \mathrm{sym}^2 \phi_j) = 0$ if $\rho \in R\}$. By Lemma 1 in [6] we have

$$\sum_{t_j \in B} m(t_j) \ll T^{\frac{1}{10}}. \tag{3.2}$$

Define $G = \{t_j | 0 < t_j \leq \frac{5}{2} T\} - B$. Clearly, if $t_j \in G$, then $L(s, \mathrm{sym}^2 \phi_j)$ has no zero in the domain $(1-\delta) < \sigma < 1$, $|t| \leq \log^3 T$. Hence by Lemma 2 in [6] in the domain $(1-\delta/2) < \sigma < 1$, $|t| \leq \log^2 T$ we have

$$L(s, \mathrm{sym}^2 \phi_j) \ll_{\epsilon} T^{\epsilon} \quad \text{for any } \epsilon > 0. \tag{3.3}$$

Lemma 3.4. *If $t_j \in G$, $\sum\limits_{n=1}^{\infty} |\lambda_j(n)|^2 g\left(\frac{n}{M}\right) = \frac{ML(1,\mathrm{sym}^2\phi_j)}{\zeta(2s)} + O\left(M^{1-\delta}T^{\epsilon}\right)$.*

Proof.

$$S = \sum_{n=1}^{\infty} |\lambda_j(n)|^2 g\left(\frac{n}{M}\right) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=2} \frac{L(s, \mathrm{sym}^2 \phi_j)}{\zeta(s)} \tilde{g}(s) M^s ds\,,$$

where $\tilde{g}(s) = \int_0^{\infty} g(x) x^{s-1} dx$. By moving the line of integration to $(1-\delta)$ we get

$$S = \frac{ML(1, \mathrm{sym}^2 \phi_j)}{\zeta(2)} + \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=1-\delta} \frac{L(s, \mathrm{sym}^2 \phi_j)}{\zeta(2s)} \tilde{g}(s) M^s ds,$$

and the result follows from (3.3). $\square$

Lemma 3.5. *If $t_j \in G$, then*

$$S(\phi_j) = \frac{12}{\pi} M + E_2(T)\,, \quad \text{where} \quad E_2(T) = O\left(M^{1-\delta} T^{\epsilon}\right).$$

Proof. This follows immediately from Lemma 3.4 and the fact that $\zeta(2) = \frac{\pi^2}{6}$, $\gamma_{\phi_j}^2 = \frac{2\pi}{L(1,\mathrm{sym}^2\phi_j)}$ and, by Lemma 3.1, $\gamma_{\phi_j}^2 \ll t_j^{\epsilon}$. □

Lemma 3.6. $\sum\limits_{\frac{5}{2}T<t_j} |H_T(t_j)| = O(T^{-\theta})$ *for any $\theta > 0$ where the implied constant only depends on θ.*

Proof. The proof follows from (1.1) and partial summation. □

Lemma 3.7. $\sum\limits_{\frac{5}{2}T<t_j} S(\phi_j) H_T(t_j) = O(1)$.

Proof. The proof follows from (1.1), Lemma 3.3 and partial summation. □

Lemma 3.8. $\sum\limits_{n=1}^{\infty} g\left(\frac{n}{M}\right) = M + O(1)$.

Proof. Immediate by partial summation, and the trivial observation $\sum\limits_{n=1}^{x} 1 = x + O(1)$. □

Lemma 3.9. $\sum\limits_{n=1}^{\infty} g\left(\frac{n}{M}\right)\left(\frac{1}{\pi}\int\limits_{-\infty}^{\infty} t\,\tanh(\pi t) H_T(t)dt\right) = \frac{2MTI}{\pi L} + O\left(\frac{T}{L}\right)$.

Proof. The proof follows in a straightforward manner from the easily established decomposition

$$\frac{2}{\pi}\int\limits_0^{\infty} t\, h(L(t-T))dt + \frac{2}{\pi}\int\limits_0^{\infty} t\, h(L(-t-T)dt + O(1)$$

and Lemma 3.8. □

Lemma 3.10. *Assume $0 < x$ and $0 < t$. Let $z = (\frac{x^2}{4} + t^2)^{1/2}$. Then*

$$J_{2it}(x) = \pi^{-1/2} z^{-1/2} e^{\frac{-\pi i}{4}} e^{\pi t} e\left(\frac{z}{\pi} - \frac{t}{\pi}\log\left(\frac{2(z+t)}{x}\right)\right) \cdot \left\{1 + O\left(\frac{1}{t}\right)\right\}$$

Proof. This is established in [1]. □

Clearly, $h_T^+(x) = 2i\int\limits_{-\infty}^{\infty} J_{2it}(x)\left(\frac{h(L(t-T))+h(L(t+T))}{\cosh\,\pi t}\right) t\,dt$. Using the fact that

$J_{2it}(x) = \overline{J_{-2it}(x)}$ we have

$$h_T^+(x) = 4i^2 \mathrm{Im} \int\limits_{-\infty}^{\infty} J_{2it}(x) \frac{h(L(t-T))}{\cosh \pi t} t \, dt.$$

Lemma 3.11. *For every $0 < \varepsilon < \frac{1}{100}$ there exists $0 < \Delta(\varepsilon) < \frac{1}{100}$ such that if $0 < \Delta < \Delta(\varepsilon)$*

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \sum_{c=1}^{T^{1-\varepsilon}} \frac{S(n,n,c)}{c} h_T^+(x) \ll T.$$

Proof.

$$h_T^+(x) = 4i^2 \,\mathrm{Im} \int\limits_0^B J_{2it}(x) \frac{h(L(t-T))}{\cosh \pi t} t \, dt - 4i^2 \,\mathrm{Im} \int\limits_0^A \bar{J}_{2it}(x) \frac{h(L(t+T))}{\cosh \pi t} t \, dt$$

$$\ll \int\limits_0^B |J_{2it}(x)| \frac{|h(L(t-T))|}{\cosh \pi t} t \, dt + \int\limits_0^A |J_{2it}(x)| \frac{|h(L(t+T)|}{\cosh \pi t} t \, dt = I_1 + I_2.$$

We first consider I_1.

$$I_1 \ll \int\limits_0^1 \frac{|J_{2it}(x)| t \, dt}{\cosh \pi t} + \int\limits_1^B \frac{|J_{2it}(x)| \, |h(L(t-T))| t \, dt}{\cosh \pi t} = I_a + I_b.$$

We first consider I_a.

$$I_a \ll \int\limits_0^1 \frac{\left(\frac{x^2}{4} + t^2\right)^{-\frac{1}{4}} e^{\pi t} t}{\cosh \pi t} dt + \int\limits_0^1 \frac{\left(\frac{x^2}{4} + t^2\right)^{-\frac{1}{4}} e^{\pi t}}{\cosh \pi t} dt.$$

But it is easy to see that

$$\left(\frac{x^2}{4} + t^2\right)^{-1/4} \ll T^{1/2 - \frac{\varepsilon}{2}}.$$

Hence $I_a \ll T^{1/2-\frac{\varepsilon}{2}}$.

We now consider I_b.

$$I_b \ll \int\limits_1^B \frac{t^{1/2} e^{\pi t} |h(L(t-T)| dt}{\left(1 + \frac{x^2}{4t^2}\right)^{1/4} \cosh \pi t} + \int\limits_1^B \frac{t^{-1/2} e^{\pi t} |h(L(t-T)| dt}{\left(1 + \frac{x^2}{4t^2}\right)^{1/4} \cosh \pi t}$$

$$\ll \int_1^B t^{1/2}|h(L(t-T)|dt,$$

so that by a simple change of variable we have

$$\ll \frac{T^{1/2}}{L}\int_{L(1-T)}^{L(B-T)}\left(1+\frac{\theta}{LT}\right)^{1/2} h(\theta)d\theta$$
$$\ll \frac{T^{1/2}}{L}\int_{L(1-T)}^{L(B-T)}(1+|\theta|)^{1/2}h(\theta)d\theta \ll \frac{T^{1/2}}{L}.$$

Hence

$$I_1 \ll \frac{T^{1/2}}{L}.$$

Similarly

$$I_2 \ll \frac{T^{1/2}}{L}$$

so that

$$h_T^+(x) \ll \frac{T^{1/2}}{L}.$$

The proof is completed by a straightforward application of the Weil bound for Kloosterman sums, namely

$$S(n,n,c) \ll n^{1/2}c^{1/2+\varepsilon_2}. \quad \square \tag{3.4}$$

The Taylor series for $J_{2it}(x)$ is

$$J_{2it}(x) = \left(\frac{x}{2}\right)^{2it}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+1+2it)}\left(\frac{x}{2}\right)^{k}$$
$$= \frac{\left(\frac{x}{2}\right)^{2it}}{\Gamma(1+2it)} + \left(\frac{x}{2}\right)^{2it}\sum_{k=1}^{\infty}\frac{1}{k!\Gamma(k+1+2it)}\left(\frac{x}{2}\right)^{2k}$$
$$= J_{2it}^a(x) + J_{2it}^b(x).$$

And if $0<x<1$ we have

$$J_{2it}^b(x) \ll \frac{x^2}{|\Gamma(1+2it)|},$$

since $|\Gamma(1+v)| \le |\Gamma(k+1+v)|$.

Lemma 3.12. *For every $0<\varepsilon_0<\frac{1}{100}$ there exists $0<\varepsilon_0'<\frac{1}{100}$ and $0<\Delta(\varepsilon_0)<\frac{1}{100}$ such that for all $0<\Delta\le\Delta(\varepsilon_0)$ for all $0<\varepsilon\le\varepsilon_0'/2$ (where ε_0*

is as specified in Theorem 1.3)

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \sum_{c=T^{1-\varepsilon}}^{\infty} \frac{S(n,n,c)}{c} \left(4i^2 \mathrm{Im} \int_{-\infty}^{\infty} J_{2it}^{a}(x) \frac{h(L(t-T))t\,dt}{\cosh \pi t}\right) \ll T.$$

Proof. Consider

$$I_a = \int_{-\infty}^{\infty} \frac{\left(\frac{x}{2}\right)^{2it} h(L(t-T))t}{\Gamma(1+2it)}\,dt\,.$$

It is immediate from Stirling's formula that

$$\text{For} \quad 0 < t_0 \le t \tag{3.5}$$
$$\frac{1}{\Gamma(\sigma+2it)} \ll_\sigma e^{\pi t} t^{1/2-\sigma}.$$

In I_a we move the line of integration to $-i\theta$ where $\theta = \left(\frac{1}{4}+\varepsilon_2\right)$ where ε_2 is that of (3.4) obtaining

$$I_1 = \left(\frac{x}{2}\right)^{2\theta} \int_{-A}^{B} \frac{\left(\frac{x}{2}\right)^{2it} h(L((t-T)-i\theta)tdt}{\Gamma(1+2\theta+2it)\cosh(\pi(t-i\theta))}$$
$$-\, i\theta \left(\frac{x}{2}\right)^{2\theta} \int_{-A}^{B} \frac{\left(\frac{x}{2}\right)^{2it} h(L((t-T)-i\theta)dt}{\Gamma(1+2\theta+2it)\cosh(\pi(t-i\theta)}.$$

By a simple change of variable, using the fact that $\frac{1}{\Gamma(z)}$ is entire and $\Gamma(z) = \overline{\Gamma(\bar{z})}$ together with (3.5) we obtain

$$I_1 \ll \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{B} |h(L(t-T)-iL\theta)| t^{(1/2-2\theta}dt$$
$$+ \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{A} |h(L(t+T)+iL\theta)| t^{(1/2-2\theta)}dt$$
$$+ \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{B} |h(L(t-T)-iL\theta)| t^{(-1/2-2\theta)}dt$$
$$+ \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{A} |h(L(t+T)+iL\theta)| t^{(-1/2-2\theta)}dt + O\left(\left(\frac{x}{2}\right)^{2\theta}\right).$$

Then by Lemma 3.0 we have

$$I_1 \ll \left(\frac{x}{2}\right)^{2\theta} T^{2\theta(2-\varepsilon_0)} \int_{t_0}^{B} \frac{t^{(1/2-2\theta)}dt}{(1+L|(t-T)-i\theta|)^k}$$

$$+ \left(\frac{x}{2}\right)^{2\theta} T^{2\theta(2-\varepsilon_0)} \int_{t_0}^{A} \frac{t^{(1/2-2\theta)}dt}{(1+L|(t+T)+i\theta|)^k}$$

$$+ \left(\frac{x}{2}\right)^{2\theta} T^{2\theta(2-\varepsilon_0)} \int_{t_0}^{B} \frac{t^{(-1/2-2\theta)}dt}{(1+L|(t-T)-i\theta|)^k}$$

$$+ \left(\frac{x}{2}\right)^{2\theta} T^{2\theta(2-\varepsilon_0)} \int_{t_0}^{A} \frac{t^{(-1/2-2\theta)}dt}{(1+L|(t+T)+i\theta|)^k} + O\left(\frac{x}{2}\right)^{2\theta}.$$

So that

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{2\theta(2-\varepsilon_0)} \int_{t_0}^{A} \frac{dt}{(1+|(t-T)|)^k}$$

$$+ \left(\frac{x}{2}\right)^{2\theta} T^{2\theta(2-\varepsilon_0)} \int_{t_0}^{A} \frac{dt}{(1+|(t+T)|)^k} + O\left(\frac{x}{2}\right)^{2\theta},$$

and by a simple change of variable

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{2\theta(2-\varepsilon_0)} \int_{t_0-T}^{B-T} \frac{dt}{(1+|x|)^k}$$

$$+ \left(\frac{x}{2}\right)^{2\theta(2-\varepsilon_0)} \int_{t_0+T}^{B+T} \frac{dt}{(1+|x|)^k} + \left(\frac{x}{2}\right)^{2\theta}.$$

Hence

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{2\theta(2-\varepsilon_0)}.$$

The above moving of the line of integration is valid if

$$I_B = \int_0^{\theta} J^a_{2iz}(x) \frac{h(L(z-T))zdz}{\cosh \pi z} \to 0 \quad \text{for each } x \text{ and } T$$

as $B \to \infty$ where $z(t) = B - it$ for $0 \le t \le \theta$, and

$$I_A = \int_0^\theta J^a_{2iz}(x) \frac{h(L(z-T))zdz}{\cosh \pi z} \to 0 \quad \text{for each } x \text{ and } T$$

as $A \to \infty$ where $z(t) = -A - it$ for $0 \le t \le \theta$.

By direct substitution and using the fact that

$$\frac{1}{|\Gamma(\sigma + it)|} \le e^{\frac{\pi t}{2}} (ct)^{1/2-\sigma} \quad \text{we have}$$

$$I_B \ll \int_0^\theta \left(\frac{x}{2}\right)^{2t} |h(L(B-T) - iLt)| B (cB)^{-1/2-2t} dt$$
$$+ \int_0^\theta \left(\frac{x}{2}\right)^{2t} |h(L(B-T) - iLt)| t (cB)^{-1/2-2t} dt$$

so that by Lemma 3.0 we have

$$I_B \ll \int_0^\theta \frac{\left(\frac{x}{2}\right)^{2t} e^{2\pi t L} B dt}{(1 + |B-T|)^k} + \int_0^\theta \frac{\left(\frac{x}{2}\right)^{2t} e^{2\pi t L} t\, dt}{(1 + |B-T|)^k}$$

so that $I_B \to 0$ for each x, T as $B \to \infty$. In a similar way we show that $I_A \to 0$ for each x, T as $A \to \infty$. The proof of the lemma is completed by straightforward application of the Weil bound for Kloosterman sums. □

Lemma 3.13. *For every $0 < \varepsilon_0 < \frac{1}{100}$ there exists $0 < \varepsilon_0' < \frac{1}{100}$ and $0 < \Delta(\varepsilon_0) < \frac{1}{100}$ such that for all $0 < \Delta \le \Delta(\varepsilon_0)$ for all $0 < \varepsilon < \varepsilon_0'/2$ (where ε_0 is as specified in Theorem 1.3)*

$$\sum_{n=1}^\infty g\left(\frac{n}{M}\right) \sum_{c=T^{1-\varepsilon}}^\infty \frac{S(n,n,c)}{c} \left(4i^2 \mathrm{Im} \int_{-\infty}^\infty J^b_{2it}(x) \frac{h(L(t-T))tdt}{\cosh \pi t} \right) \ll T.$$

Proof. Consider

$$I_b = \int_{-\infty}^\infty J^b_{2it}(x) \frac{h(L(t-T))tdt}{\cosh \pi t}.$$

By a simple change of variable using the fact that $\frac{1}{\Gamma(z)}$ is entire and $\Gamma(z) =$

$\overline{\Gamma(\bar{z})}$ together with (3.5) we obtain

$$I_b \ll x^2 \int_{t_0}^{B} |h(L(t-T))| t^{1/2} dt + x^2 \int_{t_0}^{A} |h(L(t+T))| t^{1/2} dt + O(x^2).$$

Then by Lemma 3.0 we have

$$I_b \ll x^2 \int_{t_0}^{B} \frac{t^{1/2}}{(1+|t-T|)^k} \, dt + x^2 \int_{t_0}^{A} \frac{t^{1/2}}{(1+|t+T|)^k} + O(x^2).$$

By a simple change of variable we have

$$I_b \ll x^2 \int_{t_0-T}^{B-T} \frac{(t+T)^{1/2}}{(1+|t|)^k} \, dt + x^2 \int_{t_0+T}^{B+T} \frac{(t-T)^{1/2}}{(1+|t|)^k} \, dt + O(x^2).$$

Hence we have

$$I_b \ll x^2 T^{1/2}.$$

The proof of the lemma is completed by straightforward application of the Weil bound for Kloosterman sums. □

Lemma 3.14.

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} H_T(t) |\mathcal{N}(n,t)|^2 dt\right) = O(M^{1+\epsilon}).$$

Proof. The result follows by straightforward calculation from the well-known facts and definitions:

$$\mathcal{N}(n,t) \stackrel{\circ}{=} \left(\frac{4\pi|n|}{\cosh \pi t}\right)^{1/2} \varphi(n, 1/2 + it),$$

where

$$\varphi(n,s) = \pi^s \Gamma^{-1}(s) \zeta(2s)^{-1} |n|^{-1/2} \sum_{ab=|n|} \left(\frac{a}{b}\right)^{s-1/2},$$

$$\frac{1}{\zeta(s)} = O(\log^{\Delta}(t)),$$

uniformly for $1 \leq \sigma$, and Sterling's asymptotic formula in the form

$$\Gamma(\sigma + at) = (2\pi)^{1/2} t^{\sigma - 1/2} e^{\frac{-\pi t}{2}} \left(\frac{t}{e}\right)^{it} (1 + O(t^{-1}))$$

if $t > 0$, where the implied constant depends on σ. □

4. Proofs

We first consider Theorem 1.2.

Let $S = \sum_{0<t_j} S(\phi_j)H_T(t_j)$.

$$S = \sum_{0<t_j \le \frac{5}{2}T} S(\phi_j)H_T(t_j) + \sum_{\frac{5}{2}T<t_j} S(\phi_j)H_T(t_j).$$

By Lemma 3.5 we have

$$S = \frac{12M}{\pi} \sum_{\substack{0<t_j \le \frac{5}{2}T \\ t_j\in G}} H_T(t_j) + \sum_{\substack{0<t_j \le \frac{5}{2}T \\ t_j\in G}} E_2(T)H_T(t_j) + \sum_{\substack{0<t_j \le \frac{5}{2}T \\ t_j\in B}} S(\phi_j)H_T(t_j)$$
$$+ \sum_{\frac{5}{2}T<t_j} S(\phi_j)H_T(t_j) \pm \frac{12M}{\pi} \sum_{\substack{0<t_j \le \frac{5}{2}T \\ t_j\in B}} H_T(t_j) \pm \frac{12M}{\pi} \sum_{\frac{5}{2}T \le t_j} H_T(t_j).$$

Hence

$$S = \frac{12M}{\pi} \sum_{0<t_j} H_T(t_j) + \sum_{\substack{0<t_j \le \frac{5}{2}T \\ t_j\in G}} E_2(T)H_T(t_j) + \sum_{\substack{0<t_j \le \frac{5}{2}T \\ t_j\in B}} S(\phi_j)H_T(t_j)$$
$$+ \sum_{\frac{5}{2}T<t_j} S(\phi_j)H_T(t_j) - \frac{12M}{\pi} \sum_{\substack{0<t_j \le \frac{5}{2}T \\ t_j\in B}} H_T(t_j) - \frac{12M}{\pi} \sum_{\frac{5}{2}T<t_j} H_T(t_j).$$

Hence by Lemma 3.2, Lemma 3.3 Lemma 3.5, Lemma 3.6 and Lemma 3.7 we have

$$S = \frac{12M}{\pi} \sum_{0<t_j} H_T(t_j) + O(M^{1-\delta}T^{1+\epsilon}) + O\left(M^{5/4}T^{\frac{1}{10}+\epsilon}\right)$$
$$+ O(1) + O(MT^{1/10}) + O(1);\ \text{so that}$$
$$S = \frac{12M}{\pi} \sum_{0<t_j} H_T(t_j) + O(M^{1-\delta}T^{1+\epsilon}) + O\left(M^{5/4}T^{\frac{1}{10}+\epsilon}\right).$$

But by interchanging the order of summation we have

$$S = \sum_{n=1}^{\infty} g\left(\frac{n}{M}\right)\left(\sum_{0<t_j} H_T(t_j)|\lambda_j(n)|^2\gamma_{\phi_j}^2\right).$$

Then Theorem 1.2 follows from Theorem 2.2, Lemma 3.9, Lemma 3.11, Lemma 3.12, Lemma 3.13, Lemma 3.14 and the trivial observation

$$N_h(T,L) = \sum_{0<t_j} H_T(t_j) + O(1).$$

We now consider Corollary 1.1.

Let h be as above but choose it such that $h(x) \geq 0$, and recall that $\hat{h}(0) = 1$.

$$h(0)M(T) \leq \sum_{0<t_j} h(L(t_j - T) \sim \frac{T}{6L}\hat{h}(0).$$

Taking $L = (2-\varepsilon)\frac{A \log T}{\pi}$ for $\epsilon > 0$

$$\overline{\underset{T\to\infty}{\text{Limit}}} \frac{m(T)\log T}{T} \leq \frac{\pi}{6(2-\varepsilon)A} \frac{\hat{h}(0)}{h(0)}(1+o(1)).$$

But it is shown in [4] that

$$\min_{\substack{h \geq 0 \\ \operatorname{supp} \hat{h} \subset [-\frac{1}{A}, \frac{1}{A}]}} \frac{\hat{h}(0)}{h(0)} = A.$$

Hence

$$\overline{\underset{T\to\infty}{\text{Limit}}} \frac{m(T)\log T}{T} \leq \frac{\pi}{12}.$$

Remark 1. If we could find a test function with the properties:

1. $h(z)$ is even,
2. $h(z)$ is entire,
3. $h(z) \ll_k \frac{|y|^k}{(1+|z|)^{2+\epsilon}}$ for some fixed k,

then one could establish, by the same proof, $L \ll T^\epsilon$ which would be best possible by the method of proof used to establish Theorem 1.2.

Unfortunately, by elementary entire function theory, it can be shown that $h(z)$ would have to be a polynomial and hence, by condition (3), identically equal to zero.

Of course, it is possible that one might be able to improve Lemma 3.12 by a different method of proof.

Remark 2. In this area of research there are two fundamental problems.

Problem 1. Define a test function, so that for an admissible L, $N_h(T,L) \sim \bar{N}_h(T,L)$ as $T \to \infty$, where the larger L can be taken the better the solution is.

Problem 2. Define a test function so that $\overline{\underset{T\to\infty}{\text{Limit}}} \frac{m(T)\log T}{T} \leq \alpha$, where the smaller α is the better the solution is.

In [8] Rudnick (our Theorem 1.1) provides a solution of Problem 1 by defining a test function such that $L \leq \log T$ is admissible, but he does not address Problem 2.

In [7] Sarnak (our Theorem 1.2 and Corollary 1.1) provides a solution to both Problem 1 and Problem 2 by defining a test function such that $1 \leq L \leq \frac{2 \log T}{\pi}$ and $\alpha = \frac{\pi}{12}$.

In this paper we (our Theorem 1.3) provide a solution to Problem 1 by exhibiting a test function so that $1 \leq L \leq (1-\varepsilon)A\,\frac{\log T}{\pi}$ for any $\varepsilon > 0$ is admissible, but since the support of our $\hat{h}$ is included in $\left(-\frac{1}{A}, \frac{1}{A}\right)$ for each $A > 0$, we do not get an improvement to Sarnak's solution to Problem 2.

Our result is somewhat unexpected (and hence of particular interest), for prior to our result it was generally assumed that any improvement in Problem 1 would ultimately yield a corresponding improvement in Problem 2.

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