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ON THE NUMERICAL SOLUTION OF AN OSCILLATING MAGNETOHYDRODYNAMIC FLOW PAST A FIXED WALL

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Abstract: In this paper, the numerical solutions of an oscillatory, viscous, electrically conducting, incompressible fluid stream past a fixed plane wall, are presented and compared with the analytical solutions studied by Poria et al [10]. The solutions of the problem, obtained by invoking an invariance principle, that satisfy the respective sets of boundary and initial conditions computed numerically by a finite difference method for different values of a magnetic parameter, are compared graphically with the analytical solutions and discussed.

AMS Subject Classification: 65C20
Key Words: MHD fluid flow, oscillating rigid plane wall, Stokes problem, Crank-Nicholson implicit scheme

1. Introduction

Stokes [8] was the first who studied the problem of an incompressible viscous fluid flow, produced by the oscillation of a plane solid wall. The problem is also known as Stokes second problem. Panton [5] obtained the transient solution for the flow produced by the oscillating plate. Ghosh [3] studied the velocity distribution of the flow near a vertically wall oscillating harmonically. Erdogan [2] derived the analytic solutions of the flow of a viscous fluid past the harmonically oscillating wall for small and large times by the Laplace transformed method. Poria, Mamaloukas et al [6] have determined some effects of a trans-

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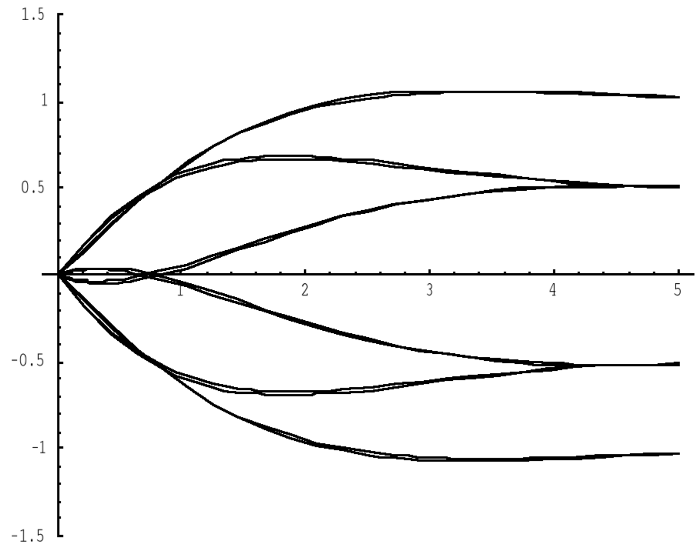

Figure 1: Variation of the velocity profiles due to cosine oscillation with time increments and $M = 0$, obtained by analytical solution and numerical solution

verse magnetic field on the flow of a viscous conducting fluid, produced by the harmonically oscillating plane wall. Recently, Poria et al [10] have studied the case when the rigid plane wall is fixed and the incompressible viscous conducting stream past it is oscillating and subjected to a transverse magnetic field. Such motions are not only of fundamental theoretical interest but they also occur in many applied problems such as acoustic streaming around an oscillating body and in many engineering applications such as flows in vibrating media.

2. Formulation of the Problem

In the second Stokes problem, the flow is produced by the oscillating rigid plane wall since such a velocity field satisfies the boundary and initial conditions $u\,(0,t) = U_0 \cos\,(\omega t)\,,\; u\,(\infty,t) = 0$ or $u\,(0,t) = U_0 \sin\,(\omega t)\,,\; u\,(\infty,t) = 0$. The momentum equation is written as (see [9])

$$\frac{\partial u}{\partial t} = v \frac{\partial^2 u}{\partial y^2}\,. \tag{1}$$

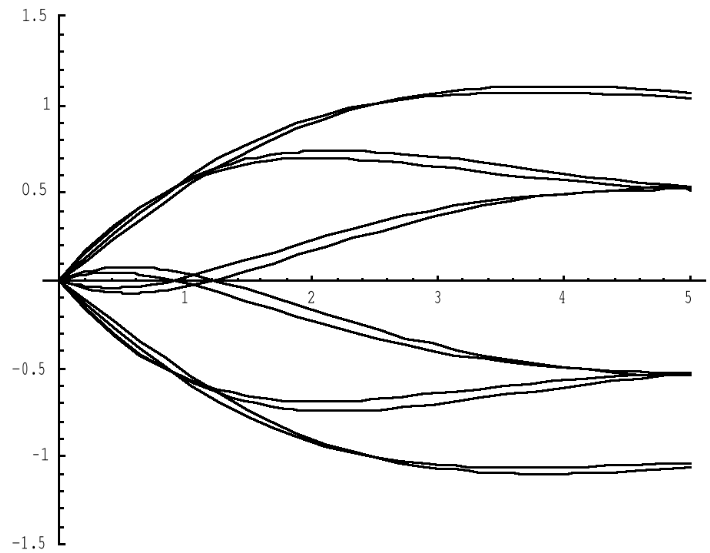

Figure 2: Variation of the velocity profiles due to cosine oscillation with time increments and $M = 0.2$, obtained by analytical solution and numerical solution

We nondimensionalize the coordinate y, time t and velocity u as

$$\overline{u} = \frac{u}{U_0}, \quad \tau = \omega t, \quad \eta = y\left(\frac{\omega}{v}\right)^{\frac{1}{2}} . \tag{2}$$

In view of (2) equation (1) reduces to

$$\frac{\partial \overline{u}}{\partial \tau} = \frac{\partial^2 \overline{u}}{\partial \eta^2} . \tag{3}$$

Now, the flow produced by the oscillating rigid plane wall is subjected to a uniform transverse magnetic field, so equation (3) is modified to

$$\frac{\partial \overline{u}}{\partial \tau} = \frac{\partial^2 \overline{u}}{\partial \eta^2} - M\overline{u} , \tag{4}$$

where $M = \frac{\sigma B_0^2}{\rho \omega}$ is the magnetic parameter, σ is the electrical conductivity and B_0 is the strength of the applied magnetic field.

Since, the oscillatory motion of the incompressible viscous conducting fluid is produced by a pressure force, the momentum equation can be written, in

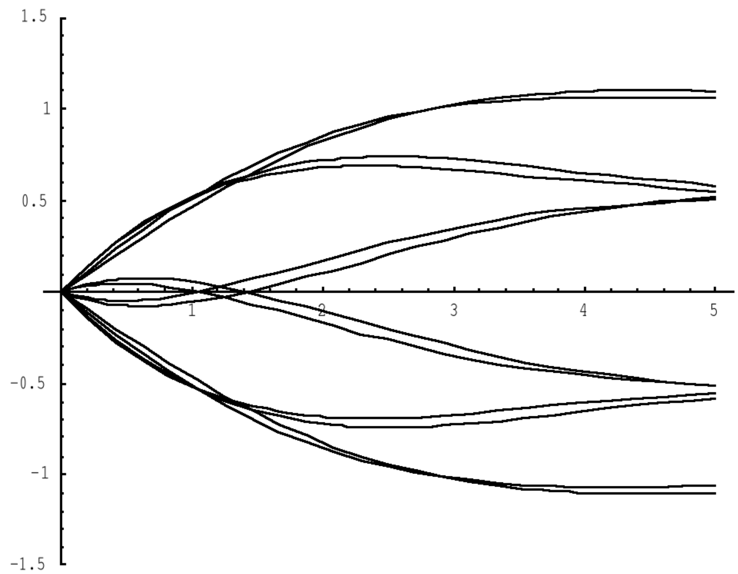

Figure 3: Variation of the velocity profiles due to cosine oscillation with time increments and $M = 0.5$, obtained by analytical solution and numerical solution

view of the relations (2) and $\overline{p} = \frac{p}{\rho u_0 \omega L}$, $\overline{x} = \frac{x}{L}$, as

$$\frac{\partial \overline{u}}{\partial \tau} = -\frac{d\overline{p}}{d\overline{x}} + \frac{\partial^2 \overline{u}}{\partial \eta^2} - M\overline{u}\,, \tag{5}$$

where L is a characteristic length.

Considering the fluid is oscillating with a velocity $\overline{u}\,(\eta \to \infty, \tau) = \cos\tau$, and the plane wall is stationary, $\overline{u}\,(\eta = 0, \tau) = 0$, the velocity of the $\widehat{x}_i$ system is also equal to $\cos\tau$, so any fluid velocity $\widehat{\overline{u}}$ in the new coordinate system can be written as

$$\widehat{\overline{u}} = \overline{u} - \cos\tau\,. \tag{6}$$

Keeping in mind that the pressure gradient acts through the layer to drive the flow and additionally, be affected by the imposed magnetic field, we choose

$$-\frac{d\overline{p}}{d\overline{x}} = -\sin\tau + M\cos\tau\,. \tag{7}$$

This form of the pressure gradient is suitable to convert the case under consideration to the corresponding second Stokes problem. Substituting (7) in (5),

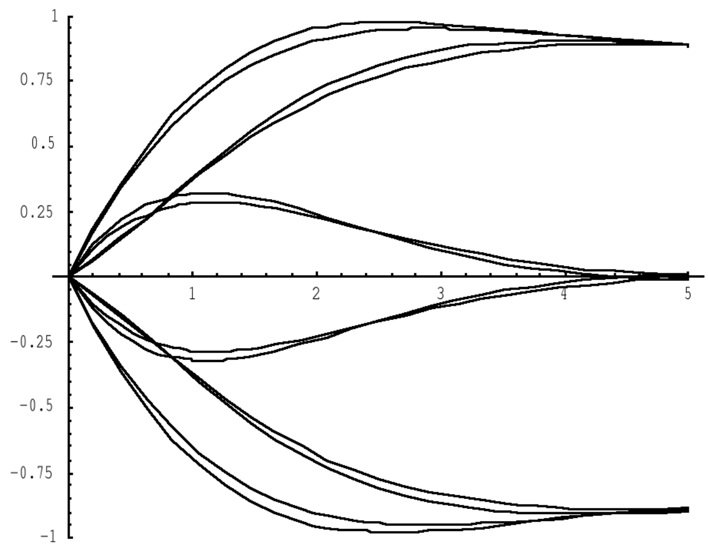

Figure 4: Variation of the velocity profiles due to sine oscillation with time increments and $M = 0$, obtained by analytical solution and numerical solution

we obtain

$$\frac{\partial \overline{u}}{\partial \tau} = -\sin\tau + M\cos\tau + \frac{\partial^2 \overline{u}}{\partial \eta^2} - M\overline{u}\,. \tag{8}$$

Taking (6) into account in (8), the momentum equation in the new coordinate system $\widehat{x}_i$, is obtained as

$$\frac{\partial \widehat{\overline{u}}}{\partial \tau} = \frac{\partial^2 \widehat{\overline{u}}}{\partial \eta^2} - M\widehat{\overline{u}}\,. \tag{9}$$

Clearly, the boundary and initial conditions in the new coordinate system are obtained with the help of (6) as

$$\widehat{\overline{u}}\,(\eta \to \infty, \tau) = 0, \quad \widehat{\overline{u}}\,(\eta = 0, \tau) = -\cos\tau\,. \tag{10}$$

The problem reduces to the corresponding Stokes problem in the new coordinate system, simply with a wall motion $-\cos\tau$. The solution of equation (9) using boundary and initial conditions (10) obtained by Poria et al [10] as

$$\overline{u}_1 = -e^{-\left[\frac{-M+\left(1+M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\eta} \cos\left(\tau - \left[\frac{-M+\left(1+M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\eta\right) + \cos\tau\,. \tag{11}$$

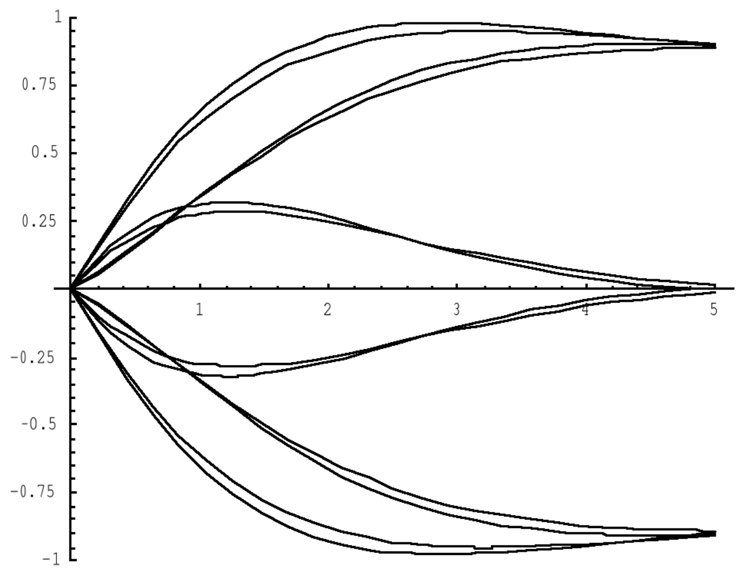

Figure 5: Variation of the velocity profiles due to sine oscillation with time increments and $M = 0.2$, obtained by analytical solution and numerical solution

Considering now, the fluid is oscillating with a velocity $\overline{u}\,(\eta \to \infty, \tau) = \sin\tau$, as before, the problem reduces again to the corresponding Stokes problem in the new coordinate system, simply with a wall motion $-\sin\tau$. The boundary and initial conditions are transformed to

$$\widehat{\overline{u}}\,(\eta \to \infty, \tau) = 0, \quad \widehat{\overline{u}}\,(\eta = 0, \tau) = -\sin\tau\,. \tag{12}$$

The solution to the original problem in this case is obtained by Poria et al [10] as

$$\overline{u}_1 = -e^{-\left[\frac{-M+\left(1+M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\eta} \sin\left(\tau - \left[\frac{-M+\left(1+M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\eta\right) + \sin\tau\,. \tag{13}$$

We now proceed to the numerical solution.

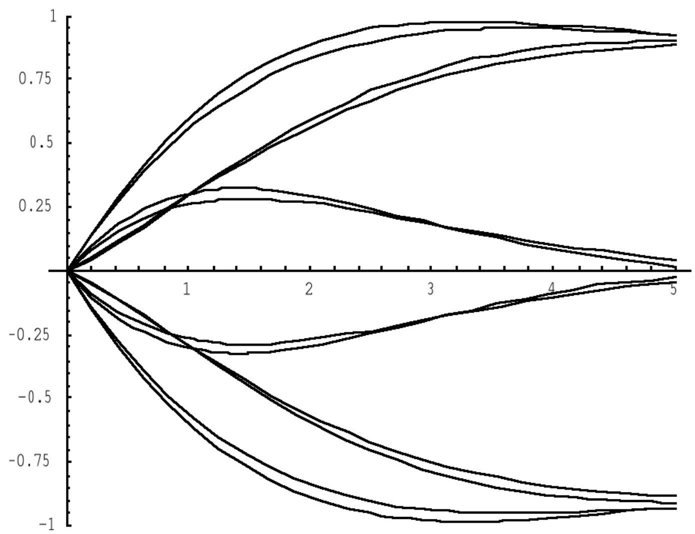

Figure 6: Variation of the velocity profiles due to sine oscillation with time increments and $M = 0.5$, obtained by analytical solution and numerical solution

3. Numerical Computations

In order to obtain the numerical solution of equation (9), the Crank-Nicholson's finite difference method is used separately with the two boundary and initial conditions, given by (10) and (12). The use of forward – differences for the time derivative term $\frac{\partial \bar{u}}{\partial \tau}$ gives

$$\frac{\partial \bar{u}}{\partial \tau} = \bar{u}_\tau\big|_{i,j} = \frac{\bar{u}_{i,j+1} - \bar{u}_{i,j}}{k} + 0(R) \tag{14}$$

and the average central differences at the present and new time step [4] for the space double derivative gives

$$\begin{aligned}\frac{\partial^2 \bar{u}}{\partial \eta^2} &= \bar{u}_{\eta\eta}\big|_{i,j} \\ &= \frac{1}{2}\left[\frac{\bar{u}_{i+1,j+1} - 2\bar{u}_{i,j+1} + \bar{u}_{i-1,j+1}}{h^2} + \frac{\bar{u}_{i+1,j} - 2\bar{u}_{i,j} + \bar{u}_{i-1,j}}{h^2}\right] + 0^2(R)\,. \end{aligned} \tag{15}$$

Substituting (14) and (15) into (9) and then performing all necessary calculations it can be obtained:

$$- r\bar{u}_{i-1,j+1} + (1 + 2r)\,\bar{u}_{i,j+1} - r\bar{u}_{i+1,j+1}$$

$$= r\left(\bar{u}_{i-1,j} + \bar{u}_{i+1,j}\right) + \left(1 - 2r - Mk\right)\bar{u}_{i,j}\,, \quad (16)$$

where $r = \frac{k}{2h^2}$.

The above form (16) derives in a tri-diagonal system of algebraic equations which is solved by Thomas algorithm for each time level. For the numerical computations in this problem grid-points net has been considered where is obtained at each grid-point at each time level.

4. Results and Discussions

Velocity profiles $\overline{u}\,(\eta, \tau)$ as obtained by analytical solution (11) and (13) and by solving the finite-difference equation (16) with the specified sets of initial and boundary conditions (10) and (12) for various values of times τ and various values of the magnetic parameter M are represented by graphs over a full cycle ($-\pi$ to π) respectively in Figures 1, 2, 3 and 4, 5, 6.

Figures 1, 2, 3 exhibits the time development of the velocity profiles due to cosine oscillation (equation (11)) respectively for values $M = 0$, 0.2 and 0.5. It is clear from these figures that the velocity field is accelerated or suppressed with the imposition of magnetic field. Such acceleration or suppression increases with the increase of the value of M. These phenomena are caused by the presence of the first term on the right-hand side of equation (11).

Similar features are noticed in Figures 4, 5, 6 for the velocity profiles for the case of sine oscillation (equation (12)) respectively for values of the magnetic parameter $M = 0$, 0.2 and 0.5. It is found that there is a significant change in the velocity profiles with sine oscillation compared to the case of cosine oscillation. The phenomena of acceleration or suppression of the velocity field by the magnetic field, however, remains qualitatively similar to the case of cosine oscillation.

Comparing the velocity profiles obtained by the analytical and the numerical solution we see that the more magnetic parameter M increases the less the matching results are satisfactory in the near field while the matching results in the far field are satisfactory in any case. Moreover, overshoot of velocity is found to occur in the case of cosine oscillation at 0 and π and in the case of sine oscillation at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. It is worth mentioning that in both cases the velocity field far away from the wall is governed mainly by the pressure forces. It is to be mentioned that in the present analysis the values of the magnetic parameter are chosen to be small enough so that the induced magnetic field may be negligible.

The results obtained in the present problem have relevance to some pathophysiological condition, for example, the magneto-therapeutic treatment of arterial blood flow.

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