International Journal of Pure and Applied Mathematics
Volume 53 No. 2 2009, 237-245

P-ADIC INTERTWINING OPERATORS AND LOCAL L-FACTORS FOR SU(2, 1)

Andrea Miller
Department of Mathematics
Harvard University
One Oxford Street, Cambridge, MA 02138, USA
e-mail: millerae@math.harvard.edu

Abstract: We prove that non-archimedean intertwining operators between induced principal series representations of certain unitary groups are quotients of local L-factors.

AMS Subject Classification: 11F70, 22D30, 33D80
Key Words: P-adic induced principal series representations, intertwining operators, non-archimedean special functions

1. Introduction

Classical hypergeometric functions at an archimedean prime of a number field play a crucial role in many parts of mathematics. For non-archimedean primes, analogues of Gamma functions and Beta functions have been studied by Gelfand and Graev in [2]. In this paper we show that, using these non-archimedean special functions, local non-archimedean intertwining operators for principal series representations of certain unitary groups are given by quotients of local L-factors. This generalizes the well known situation for $SL(2)$. As a consequence we show in [4] (among other things) that the reduction to the special fibre of the above situation (Iwahori case) leads to Gauß (and Jacobi-) sums.

In Section 2 we fix notation and state some facts about the group SU(2, 1). In Section 3 we prove a lemma concerning the non-archimedean Gamma and Beta functions, which will be needed later in Section 4. In Section 4 we compute

Received: April 24, 2009

the local intertwining operators making use of a computation of D. Keys in [3], and finally prove the above mentioned Theorem 4.

2. The Underlying Group

2.1. The Group $SU(2,1)$

Let $E/\mathbb{Q}$ be an imaginary quadratic number field, $\mathcal{O}(E)$ the ring of integers in E, and V a 3-dimensional vector space over E. Let f be the following non-degenerate hermitian form of signature $(2,1)$ on $V \times V$:

$$f(x,y) = x_1\overline{y_3} - x_2\overline{y_2} + x_3\overline{y_1}\,,$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in V$. Here $x \mapsto \overline{x}$ denotes the action of the non-trivial element in $\mathrm{Gal}(E/\mathbb{Q})$.

A matrix representative of f is given by $J = \left(\begin{smallmatrix} & & 1 \\ & -1 & \\ 1 & & \end{smallmatrix}\right)$. We set

$$G(\mathbb{Q}) = \mathrm{SU}(2,1)(\mathbb{Q}) = \{g \in \mathrm{SL}_3(E) \mid g = J[(\overline{g})^t]^{-1} \cdot J^{-1}\}$$

and accordingly $G(R) = \{g \in \mathrm{SL}_3(E \otimes_{\mathbb{Q}} R) \mid g = J[(\overline{g})^t]^{-1}J^{-1}\}$ for any $\mathbb{Q}$-algebra R. A maximal torus T (non-split over $\mathbb{Q}$) is given by

$$T(\mathbb{Q}) = \left\{ \begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & 1/\bar{t} \end{pmatrix}, t \in E^* \right\}.$$

The maximal $\mathbb{Q}$-split torus T' in T is given by $T'(\mathbb{Q}) = \left\{\left(\begin{smallmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{smallmatrix}\right)\right\}$. The group G is quasi-split with Borel subgroup $B = T \ltimes N$, where

$$N(\mathbb{Q}) = \left\{ n(x,y) = \begin{pmatrix} 1 & x & y \\ & 1 & \overline{x} \\ & & 1 \end{pmatrix} \;\middle|\; y + \overline{y} - x\overline{x} = 0 \text{ for } x, y \in E^* \right\}.$$

Note that we have a filtration $0 \to [N,N] \to N \to N/[N,N] \to 0$, where

$$[N,N](\mathbb{Q}) \cong \{y \in E \mid \mathrm{Tr}_{E/\mathbb{Q}}(y) = 0\} \cong \mathbb{G}_a(\mathbb{Q}) \cong \mathbb{Q}$$

and

$$(N/[N,N])(\mathbb{Q}) \cong R_{E/\mathbb{Q}}\mathbb{G}_a(\mathbb{Q}) \cong E$$

via $n(x,y) \longmapsto x$. Here $\mathrm{Tr}_{E/\mathbb{Q}}$ denotes the trace and $R_{E/Q}$ the restriction of scalars.

The character module $X^*(T)$ is generated by the two simple positive roots

α and β, where

$$\alpha\left(\begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & 1/\bar{t}\end{pmatrix}\right) = \frac{t^2}{\bar{t}}, \qquad \beta\left(\begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & 1/\bar{t}\end{pmatrix}\right) = \frac{\bar{t}^2}{t}.$$

Let $W \cong S_3$ denote the absolute Weyl group and $W' \cong \mathbb{Z}/2\mathbb{Z}$ the relative Weyl group. A matrix representative for the longest element $w = w_\ell$ of W is given by $w = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$.

2.2.

Let $p \neq 2$ be a non-archimedean prime of $\mathbb{Q}$. Let $E_{\mathfrak{P}} = E \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and let $\mathcal{O}_{\mathfrak{P}}$ be the ring of integers in $E_{\mathfrak{P}}$. We have

$$p \cdot \mathcal{O}_{\mathfrak{P}} = \begin{cases} \mathfrak{P} & \text{if } p \text{ is unramified and non-split in } E, \\ \mathfrak{P}^2 & \text{if } p \text{ is ramified in } E \end{cases}$$

(we ignore the split case, since this can be reduced to the case of SL_2).

Let $N_{E_{\mathfrak{P}}/\mathbb{Q}_p} : E_{\mathfrak{P}} \to \mathbb{Q}_p$, $x \mapsto x\overline{x}$ denote the norm map, and let π denote a local uniformizing element. We write multiplicative characters as $\chi = \chi_0 \cdot |\cdot|^s$, where $\chi_o : E_{\mathfrak{P}}{}^* \to \mathbb{C}^*$ is a unitary character with $\chi_0(\pi) = 1$; also $|\cdot| = |\cdot|_{\mathfrak{P}}$ is the $\mathfrak{P}$-adic valuation and $s \in \mathbb{C}$. All additive characters are translates of a non-trivial base character ϕ_{base} which satisfies $\phi_{\text{base}}|_{\mathfrak{P}^{-1}\cdot\mathcal{D}^{-1}_{E_{\mathfrak{P}}/\mathbb{Q}_p}} \neq \text{id}$ and which factorizes over the trace $\mathrm{Tr}_{E_{\mathfrak{P}}/\mathbb{Q}_p}$. Here $\mathcal{D}_{E_{\mathfrak{P}}/\mathbb{Q}_p}$ denotes the local differential.

A character of $T(\mathbb{Q}_p)$ is given simply by a character $\chi : E_{\mathfrak{P}} \to \mathbb{C}^*$. Using the above choices of simple positive roots, we find that the modulus character $\delta = \left|\frac{1}{2}(\alpha + \beta + (\alpha + \beta))\right|_{\mathfrak{P}}$ is given by

$$\delta\left(\begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & 1/\bar{t}\end{pmatrix}\right) = \left|\frac{t^2}{\bar{t}} \cdot \frac{\bar{t}^2}{t}\right|_{\mathfrak{P}} = |t|^2_{E_{\mathfrak{P}}}.$$

3. Generalized Gamma and Beta Functions

Let ϕ be given by $\phi(x) = \phi_{\text{base}}(\pi^{-d}x)$, where d is given by $\mathcal{D}_{E_{\mathfrak{P}}/\mathbb{Q}_p} = \mathfrak{P}^d$. For $f \in L^1(E_{\mathfrak{P}})$, the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(u) = \int_{E_{\mathfrak{P}}} \phi(-ux) f(x) dx.$$

For $\chi = \chi_0 \cdot |\cdot|^s$ as above set $\Gamma(\chi) = \int_{E_{\mathfrak{P}}} \phi(x)\chi_0(x)|x|^{s-1}dx$.

Proposition 1. *(1) The function $\Gamma(\chi) = \Gamma(\chi_0, s)$ converges and is holomorphic for $\mathrm{Re}(x) = \mathrm{Re}(s) > 0$. It posesses a meromorphic continuation to $\mathbb{C}$.*

(2) If $\chi = |\cdot|^s$, $s \neq 0$, then $\Gamma(\chi) = \Gamma(s) = \frac{1-q^{s-1}}{1-q^{-s}}$. The only zero is given by $\chi = |\cdot|$, and the only pole by $\chi = \mathrm{id}$, at which the residue is $\frac{1-q^{-1}}{\ln q}$.

(3) If $\chi = \chi_0 \cdot |\cdot|^s$ is ramified of ramification degree $\mathfrak{f}$, then $\Gamma(\chi) = W_\chi \cdot q^{\mathfrak{f}(s-\frac{1}{2})}$ and $|W_\chi| = 1$.

Proof. See [5] and [2]. □

Comparing $\Gamma(\chi)$ with the local factors $\zeta(f,\chi)$ of Tate's thesis [6] (here f is a suitably chosen function), we find

$$\Gamma(\chi_0, s) \cdot \Gamma(\chi_0^{-1}, 1-s) = \chi(-1). \tag{3.1}$$

We now set $L_{\mathfrak{P}}(\chi) = \zeta(f_{\mathfrak{f}}, \chi)$, where $\mathfrak{f}$ is the ramfication index of χ and

$$f_{\mathfrak{f}}(x) = \begin{cases} \phi(x) & \text{if } \chi \in \mathfrak{P}^{-\mathfrak{f}}, \\ 0 & \text{otherwise.} \end{cases}$$

For two multiplicative characters χ_1, χ_2 and $|\cdot| = |\cdot|_{\mathfrak{P}}$ as before define

$$B(\chi_1, \chi_2) = \int_{E_{\mathfrak{P}}} \chi_1(x)|x|^{-1} \cdot \chi_2(1-x)|1-x|^{-1}dx.$$

The convergence of this integral can be shown by splitting it up into integrals over $\{x \mid |x| \leq 1\}$ and $\{x \mid |x| > 1\}$, proving convergence on these domains for (χ_1, χ_2) in suitable halfspaces, and then analytically continuing both integrals. It is not too hard to show:

Lemma 2.

$$B(\chi_1, \chi_2) = \frac{\Gamma(\chi_1) \cdot \Gamma(\chi_2)}{\Gamma(\chi_1\chi_2)}.$$

Proof. See [4]. □

Now choose τ such that $E_{\mathfrak{P}} = \mathbb{Q}_p(\tau)$ with $\overline{\tau} = -\tau$ and $\mathcal{O}_p = \mathbb{Z}_p + \tau\mathbb{Z}_p$. Normalise $\mathrm{vol}(\mathcal{O}_{\mathfrak{P}}) = 1$.

Lemma 3. *The integral $I(1) = \int_{\mathbb{Q}_p} \chi(1-y\tau) \cdot |1-y\tau|_{E_{\mathfrak{P}}}^{-1}\, dy$ converges on a halfplane and is given by*

$$I(1) = \chi(-2) \cdot d_{\mathfrak{P}}^{-(s-\frac{1}{2})} \cdot \frac{L_{E_{\mathfrak{P}}}(\chi)}{L_{E_{\mathfrak{P}}}(\hat{\chi})} \cdot \frac{L_{\mathbb{Q}_p}(\widehat{\chi_{\mathbb{Q}_p}})}{L_{\mathbb{Q}_p}(\chi_{\mathbb{Q}_p})}.$$

Here $d_{\mathfrak{P}}$ is the $\mathfrak{P}$ part of the discriminant $d_{E/\mathbb{Q}}$ and $\hat{\chi} = \chi^{-1} \cdot |\cdot|$.

Proof. Convergence follows from the convergence on a halfplane of $B(\mathrm{id},\chi)$ and a non-archimedean Fubini Theorem.

For $u \in \mathbb{Q}_p$ set

$$\begin{aligned} I(u) &:= \int_{\mathbb{Q}_p} \chi(u-y\tau)\cdot|u-y\tau|_{E_\mathfrak{P}}^{-1}\,dy \\ &= \int_{\mathbb{Q}_p} \chi(u-uy\tau)\cdot|u-uy\tau|_{E_\mathfrak{P}}^{-1}\cdot|u|_{\mathbb{Q}_p}\,dy \\ &= \chi|_{\mathbb{Q}_p}(u)\cdot|u|_{\mathbb{Q}_p}^{-1}\underbrace{\int_{\mathbb{Q}_p}\chi(1-y\tau)\cdot|1-y\tau|_{E_\mathfrak{P}}^{-1}\,dy}_{=I(1)}. \end{aligned}$$

We obtain

$$I(u) = (\chi|_{Q_p}\cdot|\cdot|^{-1})(u)\cdot I(1)$$

and therefore

$$\hat{I}(1) = \Gamma_{\mathbb{Q}_p}(\chi|_{\mathbb{Q}_p})\cdot I(1).$$

On the other hand we have

$$\begin{aligned} \hat{I}(1) &= \int_{\mathbb{Q}_p}\phi_{\mathbb{Q}_p}(x)\int_{\mathbb{Q}_p}\chi(x-y\tau)|x-y\tau|_{E_\mathfrak{P}}^{-1}\,dydx \\ &= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\phi_{\mathbb{Q}_p}(2x)\cdot\chi(2x-2y\tau)|2x-xy\tau|_{E_\mathfrak{P}}^{-1}|2|_{\mathbb{Q}_p}^2\,dydx \\ &= \chi(2)\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\phi_{\mathbb{Q}_p}(Tr_{E_\mathfrak{P}/\mathbb{Q}_p}(x+y\tau))\chi(x+y\tau)\cdot|x+y\tau|_{E_\mathfrak{P}}^{-1}\,dydx \\ &= \chi(2)\int_{E_\mathfrak{P}}\phi_{E_\mathfrak{P}}(z)\cdot\chi(z)|z|_{E_\mathfrak{P}}^{-1}\,dx \\ &= \chi(2)\cdot(\widehat{\chi\cdot|\cdot|^{-1}})(1) = \chi(2)\cdot\Gamma_{E_\mathfrak{P}}(\chi). \end{aligned}$$

Putting these calculations together gives

$$I(1) = \chi(2)\cdot\frac{\Gamma_{E_\mathfrak{P}}(\chi)}{\Gamma_{\mathbb{Q}_p}(\chi_{\mathbb{Q}_p})}.$$

Using the functional equation (3.1), we obtain

$$\begin{aligned} I(1) &= \chi(2)\cdot\Gamma_{E_\mathfrak{P}}(\chi)\cdot\chi|_{\mathbb{Q}_p}(-1)\cdot\Gamma_{\mathbb{Q}_p}(\chi^{-1}|_{\mathbb{Q}_p}\cdot|\cdot|_{\mathbb{Q}_p}) \\ &= \chi(-1)\chi(2)\cdot\Gamma_{E_\mathfrak{P}}(\chi)\cdot\Gamma_{\mathbb{Q}_p}(\chi^{-1}|_{\mathbb{Q}_p}\cdot|\cdot|_{\mathbb{Q}_p}) \\ &= \chi(-2)\cdot d_\mathfrak{P}^{-(s-\frac{1}{2})}\cdot\frac{L_{E_\mathfrak{P}}(\chi)}{L_{E_\mathfrak{P}}(\hat{\chi})}\cdot\frac{L_{\mathbb{Q}_p}(\widehat{\chi|_{\mathbb{Q}_p}})}{L_{\mathbb{Q}_p}(\chi|_{\mathbb{Q}_p})}. \end{aligned}$$

□

4. Local Intertwining Operators

Local intertwining operators for $SU(3)$ have been computed by Keys in [3]. We will now prove that they are equal to quotients of local L-factors.

Let $p \neq 2$. Let

$$I_\chi = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} = \{f : G(\mathbb{Q}_p) \to \mathbb{C} \mid f(b \cdot g) = \chi(b) \cdot \delta^{\frac{1}{2}}(b) f(g)\},$$

where $b \in B(\mathbb{Q}_p)$, $g \in G(\mathbb{Q}_p)$, and f is locally constant. Now let χ be a regular character, i.e. $w\chi \neq \chi$, where $w = w_\ell \in W$. An intertwining operator $T_{w,\chi} : I_\chi \to I_{w\chi}$ is given by

$$T_{w,\chi}(f)(g) = \int_{N(\mathbb{Q}_p)} f(w \cdot u \cdot g) du, \qquad f \in I_\chi,\ g \in G(\mathbb{Q}_p).$$

$T_{w,\chi}$ converges for $\mathrm{Re}(\chi) >> 0$ and can be meromorphically continued to all χ (see [1]). Recall the Bruhat decomposition $G = Bw \cup BN^{-1}$, where $N^{-1} = wNw^{-1}$ (for convenience we write $G = G(\mathbb{Q}_p)$, $B = B(\mathbb{Q}_p)$, etc.). Let ν denote the $\mathfrak{P}$-adic order, i.e. $\nu(\pi^n) = n$. We now define

$$\theta_\chi(g) = \begin{cases} 0 & \text{if } g \in Bw, \\ 0 & \text{if } g = b \cdot u^- \in B \cdot N^- \text{ and } u^- \notin k_\mathfrak{f}, \\ \chi(b) \cdot \delta^{\frac{1}{2}} \cdot \phi_{-2\mathfrak{f}}(u^-) & \text{if } g = b \cdot u^- \in B \cdot N^- \text{ and } u^- \in k_\mathfrak{f}. \end{cases}$$

Here $k_f = \{g \in G(\mathcal{O}_\mathfrak{P}) \mid g \equiv \mathrm{Id}(\bmod \mathfrak{P}^\mathfrak{f})\}$ and $\phi_{-2\mathfrak{f}}$ denotes the following character of N^-:

$$\phi_{-2\mathfrak{f}}\left(\begin{pmatrix} 1 & & \\ x & 1 & \\ y & \overline{x} & 1 \end{pmatrix}\right) = \phi(\pi^{-2\mathfrak{f}} \cdot x),$$

where $\mathfrak{f}$ is the ramification index of χ. The function θ_χ extends to all of G. We are going to compute $T_{w,\chi}$ for $\theta_\chi \in I\chi$.

Theorem 4. *Let $p \neq 2$. Then $T_{w,\chi}(\theta_\chi) = c_{w,\chi} \cdot \theta_{w\chi}$, where*

$$c_{w,\chi} = C \cdot \frac{L_{E_\mathfrak{P}}(\chi \circ N_{E_\mathfrak{P}/\mathbb{Q}_p})}{L_{E_\mathfrak{P}}(\hat{\chi} \circ N_{E_\mathfrak{P}/\mathbb{Q}_p})} \cdot \frac{L_{E_\mathfrak{P}}(\chi)}{L_{E_\mathfrak{P}}(\hat{\chi})} \cdot \frac{L_{\mathbb{Q}_p}(\widehat{\chi|_{\mathbb{Q}_p}})}{L_{\mathbb{Q}_p}(\chi|_{\mathbb{Q}_p})}$$

and $C = \chi(-1) \cdot |\pi|_{E_\mathfrak{P}}^{4\mathfrak{f}s} \cdot d_\mathfrak{P}^{-(2s-1)}$.

Proof. From results of Casselman [1] and the choice of our function θ_χ, we know that $(T_{w^{-1},w\chi} \circ T_{w,\chi})(\theta_\chi)$ is a homothety. It is therefore sufficient to evaluate $T_{w,\chi}(\theta_\chi)$ at 1.

Recall the notation $n(x,y) = \begin{pmatrix} 1 & x & y \\ & 1 & \overline{x} \\ & & 1 \end{pmatrix}$, $w = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$. We write $x = x_1 + \tau x_2$, $y = y_1 + \tau y_2$, where $x_i, y_i \in \mathbb{Q}_p$ and $\overline{\tau} = -\tau$. We then have

$$\int_N \theta_\chi(wn)dn = \int_N \theta_\chi\left(\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & x & \frac{1}{2}x\overline{x} + \tau y_2 \\ & 1 & \overline{x} \\ & & 1 \end{pmatrix}\right) dxdy_2$$

$$= \int_N \theta_\chi\left(\begin{pmatrix} 1/\overline{y} & & \\ & \overline{y}/y & \\ & & y \end{pmatrix}\begin{pmatrix} 1 & -\frac{x\overline{y}}{y} & \overline{y} \\ & 1 & -\frac{\overline{x}y}{\overline{y}} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ \frac{\overline{x}}{\overline{y}} & 1 & \\ \frac{1}{y} & \frac{x}{y} & 1 \end{pmatrix}\right) dxdy_2$$

$$= \int_{\nu(\frac{1}{y}),\nu(\frac{x}{y})\geq \mathfrak{f}} \chi\delta^{1/2}\left(\frac{1}{\overline{y}}\right)\cdot \phi_{-2\mathfrak{f}}\left(\frac{\overline{x}}{\overline{y}}\right) dxdy_2$$

$$= \int_{\nu(x),\nu(x\overline{x}(1-\tau y_2))\geq \mathfrak{f}} \chi\delta^{1/2}\left(\frac{1}{2}x\overline{x}(1-y_2\tau)\right)\phi_{-2\mathfrak{f}}(x)\cdot|1-y_2\tau|_{E_\mathfrak{P}}^{-2}\cdot|x|_{E_\mathfrak{P}}^{-2}\frac{dx}{|x|E_\mathfrak{P}}dy_2$$

$$= \chi^{-1}(2)\sum_{i=\mathfrak{f}}^{\infty}\int_{\nu(x)=i}\chi(x\overline{x})\phi_{-2\mathfrak{f}}(x)\cdot|x|^2\cdot|x|^{-2}$$

$$\frac{dx}{|x|E_\mathfrak{P}}\int_{\nu(\tau y_2)\geq \mathfrak{f}-2i}\chi\delta^{1/2}(1-y_2\tau)\cdot|1-y_2\tau|^{-2}dy_2$$

$$= \chi^{-1}(2)\sum_{i=\mathfrak{f}}^{\infty}\int_{\nu(x)=i}\chi(x\overline{x})\phi_{-2\mathfrak{f}}(x)\frac{dx}{|x|E_\mathfrak{P}}\int_{\nu(\tau y_2)\geq \mathfrak{f}-2i}\chi(1-y_2\tau)\cdot|1-y_2\tau|^{-1}dy_2$$

$$= \chi^{-1}(2)\sum_{i=\mathfrak{f}}^{\infty}\int_{\nu(x)=i}\chi(x\overline{x})\phi_{-2\mathfrak{f}}(x)\frac{dx}{|x|E_\mathfrak{P}}\left[\int_{\mathbb{Q}_p}\chi(1-y_2\tau)\cdot|1-y_2\tau|^{-1}dy_2\right.$$

$$\underbrace{\left.-\int_{\nu(\tau y_2)<\mathfrak{f}-2i}\chi(1-y_2\tau)\cdot|1-y_2\tau|_{E_\mathfrak{P}}^{-1}dy_2\right]}_{=:A_i}$$

$$= \chi^{-1}(2)\chi(\pi^{2\mathfrak{f}}\cdot\overline{\pi}^{-2\mathfrak{f}})\sum_{i=\mathfrak{f}}^{\infty}\int_{\nu(x)=i-2\mathfrak{f}}\chi(x\overline{x})\phi(x)\frac{dx}{|x|E_\mathfrak{P}}$$

$$\left[\chi(-2)\cdot d_\mathfrak{P}^{-(s-\frac{1}{2})}\cdot\frac{L_{E_\mathfrak{P}}(\chi)}{L_{E_\mathfrak{P}}(\hat{\chi})}\cdot\frac{L_{\mathbb{Q}_p}(\widehat{\chi|\mathbb{Q}_p})}{L_{\mathbb{Q}_p}(\chi|\mathbb{Q}_p)} - A_i\right]$$

$$= \underbrace{\chi(-1)\cdot|\pi|^{4\mathfrak{f}s}\cdot d_{E_\mathfrak{P}}^{-2(s-\frac{1}{2})}}_{C}\cdot\frac{L_{E_\mathfrak{P}}(\chi\circ N_{E_\mathfrak{P}/\mathbb{Q}_p})}{L_{E_\mathfrak{P}}(\hat{\chi}\circ N_{E_\mathfrak{P}/\mathbb{Q}_p})}\cdot\frac{L_{E_\mathfrak{P}}(\chi)}{L_{E_\mathfrak{P}}(\hat{\chi})}\cdot\frac{L_{\mathbb{Q}_p}(\widehat{\chi|\mathbb{Q}_p})}{L_{\mathbb{Q}_p}(\chi|\mathbb{Q}_p)}$$

$$- \chi^{-1}(2) \cdot \underbrace{\sum_{i=\mathfrak{f}}^{\infty} A_i \cdot \int_{\nu(x)=i} \chi(x\overline{x}) \cdot \phi_{-2\mathfrak{f}}(x) \frac{dx}{|x|E_{\mathfrak{P}}}}_{=:A}$$

which is our claim if we can show that $A = 0$.

If $\nu(\tau y_2) < -\mathfrak{f}$ we have $\chi(1 - \tau y_2) = \chi(-\tau y_2(1 - \frac{1}{\tau y_2})) = \chi(-\tau y_2)$ and $|1 - \tau y_2| = |-\tau y_2(1 = \frac{1}{\tau y_2})| = |\tau y_2|$. So using the substitution $y_2 \mapsto \tilde{y}_2 = x\overline{x}y_2$ we obtain

$$A = \sum_{i=\mathfrak{f}}^{\infty} \int_{\nu(\tau y_2) < \mathfrak{f} - 2i} \chi(1 - y_2\tau) \cdot |1 - y_2\tau|_{E_{\mathfrak{P}}}^{-1} dy_2 \cdot \int_{\nu(x)=i} \chi(x\overline{x}) \cdot \phi_{-2\mathfrak{f}}(x) \frac{dx}{|x|E_{\mathfrak{P}}}$$

$$= \sum_{i=\mathfrak{f}}^{\infty} \int_{\nu(\tau y_2) < \mathfrak{f} - 2_i, \nu(x)=i} \chi(-\tau y_2 x\overline{x}) \cdot |\tau y_2 \cdot x\overline{x}|_{E_{\mathfrak{P}}}^{-1} \cdot |x\overline{x}|_{E_{\mathfrak{P}}} \cdot \phi_{-2\mathfrak{f}}(x) \cdot |x\overline{x}|_{\mathbb{Q}_p}^{-1} \frac{dx}{|x|E_{\mathfrak{P}}} d\tilde{y}_2$$

$$= \underbrace{\int_{\nu(\tau\tilde{y}_2) < \mathfrak{f}} \chi(-\tau\tilde{y}_2) \cdot |\tau\tilde{y}_2|_{E_{\mathfrak{P}}}^{-1} d\tilde{y}_2}_{=:D} \cdot \sum_{i=\mathfrak{f}}^{\infty} \int_{\nu(x)=i} \phi_{-2\mathfrak{f}}(x) dx$$

$$= D \cdot \int_{\nu(x) \geq -\mathfrak{f}} \phi(x) dx = 0.$$

This achieves our goal. □

Acknowledgements

This is the analytic part of the authors Bonn Diplom thesis, which she now decided to publish long after its appearance. I wish to thank my advisor at the time, Günter Harder, for the very interesting topic and many helpful discussions.

References

[1] W. Casselman, Some general results in the theory of admissible representations of p-adic groups, Unpublished Manuscript.

[2] I. Gelfand, M. Graev, Representations of a group of matrices of the second order with elements from a locally compact field, *Russian Math. Surveys*, **18** (1963), 29-100.

[3] D. Keys, Principal series representations of special unitary groups over local fields, *Comp. Math.*, **51** (1984), 115-130.

[4] A. Miller, Lokale Berechnung von unitären Hecke-Moduln in der Kohomologie des Randes von Picardschen Modulflächen, Preprint.

[5] P. Sally, H. Taibleson, Special functions on locally compact fields, *Acta. Math.*, **116** (1966), 279-309.

[6] J. Tate, *Fourier Analysis in Number Fields and Hecke's Zeta Functions*, Thesis in Algebraic Number Theory by Cassels and Fröhlich, Academic Press (1967).