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SOME NEW HARDY-HILBERT TYPE INEQUALITIES AND APPLICATIONS

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Abstract: In this paper it is shown that some new Hardy-Hilbert type inequalities for double series can be established by introducing a logarithm function of the form $\ln(n/m)$ and a parameter s. And the weight function is estimated by means of the Euler-Maclaurin summation formula. At the same time the constant factor is proved to be the best possible. In particular, for case $p = 2$, some new Hilbert type inequalities are built. As applications, some equivalent inequalities are studied.

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1. Introduction

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \le \frac{\pi}{\sin\frac{\pi}{p}}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

where the coefficient $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible. This is the famous Hardy-Hilbert's theorem for double series. In the papers [3], [4], the following inequality of the form

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$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\ln\frac{m}{n}\right)a_m b_n}{m-n}\le\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}, \qquad (1.2)$$

was established, and the coefficient $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2$ is also the best possible.

Owing to the importance of the Hardy-Hilbert inequality and the Hardy-Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) and (1.2) appear in a great deal of papers (see [9], [5], [6], [1], [2], etc.). Specially, Gao and Hsu enumerated more than 40 the research articles in the paper [9]. The aim of the present paper is to build some new Hardy-Hilbert type inequalities for double series by introducing a logarithm function and a parameter s and by applying the Euler-Maclaurin summation formula to estimate the weight function. And then the constant factor is proved to be the best possible. At last some equivalent forms of them are studied.

In the sake of convenience, we introduce some notations and define some functions.

Throughout this paper we stipulate that $\left(\ln\frac{n}{m}\right)^0=1$ when $m=n$.

Let $0<\alpha<1$ and s be a nonnegative integer. Define a function ζ^* by

$$\zeta^*(s,\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(\alpha+k)^s}. \qquad (1.3)$$

Let $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. And further define the function ζ_p by

$$\zeta_p=(2s)!\left(\zeta^*(2s+1,\frac{1}{p})+\zeta^*(2s+1,1-\frac{1}{p})\right), \qquad (1.4)$$

It is obvious that $\zeta_p=\zeta_q$.

Let $z>1$. Then the Riemann zeta function is defined by $\zeta(z,\alpha)=\sum\limits_{k=0}^{\infty}\frac{1}{(\alpha+k)^z}$ (see [11]). When $s>0$, we can write (1.3) in the form:

$$\begin{aligned}\zeta^*(2s+1,\alpha)&=\frac{1}{2^{2s+1}}\left(\sum_{m=0}^{\infty}\frac{1}{(\frac{\alpha}{2}+m)^{2s+1}}-\sum_{m=0}^{\infty}\frac{1}{(\frac{\alpha+1}{2}+m)^{2s+1}}\right)\\&=\frac{1}{2^{2s+1}}\left\{\zeta\left(2s+1,\frac{\alpha}{2}\right)-\zeta\left(2s+1,\frac{\alpha+1}{2}\right)\right\}.\end{aligned}$$

Consequently, we can write (1.4) in the following form:

$$\zeta_p=\frac{(2s)!}{2^{2s+1}}\left\{\zeta(2s+1,\frac{1}{2p})-\zeta(2s+1,\frac{1}{2p}+\frac{1}{2})+\zeta(2s+1,\frac{1}{2q})\right.$$

$$-\zeta(2s+1,\frac{1}{2q}+\frac{1}{2})\Big\}. \quad (1.5)$$

It shows that ζ_p can be expressed by the Riemann zeta function.

2. Some Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *Let s be a nonnegative integer and $r>1$. Define a function by*

$$\varphi(x)=\begin{cases}\varphi_1(1) & \text{if} \quad x=1\,,\\ \varphi_1(x)-\varphi_2(x)-\varphi_3(x) & \text{if} \quad x>1\,,\end{cases} \quad (2.1)$$

where $\varphi_1(x)=\int\limits_0^{1/x}\frac{\left(\ln\frac{1}{u}\right)^{2s}}{(1+u)}\left(\frac{1}{u}\right)^{\frac{1}{r}}du$, $\varphi_2(x)=\frac{x^{\frac{1}{r}}(\ln x)^{2s}}{2(x+1)}$,

$$\varphi_3(x)=\frac{x^{\frac{1}{r}}(\ln x)^{2s-1}\theta}{12}\left(\frac{\ln x}{(x+1)^2}+\frac{2s}{x+1}+\frac{\ln x}{r(x+1)}\right) \quad (0<\theta<1)\,.$$

Then $\varphi(x)>0$ and when $x>1$, $\varphi(x)\downarrow 0$.

Proof. When $x=1$, it is obvious that $\varphi(1)=\int\limits_0^1\frac{\left(\ln\frac{1}{u}\right)^{2s}}{(1+u)}\left(\frac{1}{u}\right)^{\frac{1}{r}}du>0$.

When $x>1$, we firstly consider the derivative of $\varphi_3(x)$ and write it in the following form:

$$\varphi_3'(x)=u(x)+v(x)\,,$$

where $u(x)$ is the sum of all the positive terms of $\varphi_3'(x)$ and $v(x)$ is the sum of all the negative terms of $\varphi_3'(x)$. So we have

$$\varphi_3'(x)\geq v(x)=-\frac{x^{\frac{1}{r}}(\ln x)^{2s}\theta}{x+1}\left(\frac{1}{6(x+1)^2}+\frac{s}{6(x+1)\ln x}+\frac{1}{12r(x+1)}\right).$$

Next we consider the derivative of $\varphi(x)$ and notice that $0<\theta<1$, so we have

$$\begin{aligned}\varphi'(x)&=\left(\varphi_1'(x)-\varphi_2'(x)\right)-\varphi_3'(x)\leq\left(\varphi_1'(x)-\varphi_2'(x)\right)-v(x)\\ &=-\frac{x^{\frac{1}{r}}(\ln x)^{2s}}{x+1}\left(\frac{(r+1)x+(2r+1)}{2rx(x+1)}+\frac{s}{x\ln x}\right)-v(x)\\ &=-\frac{x^{\frac{1}{r}}(\ln x)^{2s}}{x+1}\Big\{\left(\frac{(r+1)x+(2r+1)}{2rx(x+1)}+\frac{s}{x\ln x}\right)\end{aligned}$$

$$
\begin{aligned}
&-\theta\left(\frac{1}{6(x+1)^2}+\frac{s}{6(x+1)\ln x}+\frac{1}{12r(x+1)}\right)\Bigg\}\\
&<-\frac{x^{\frac{1}{r}}(\ln x)^{2s}}{x+1}\left\{\left(\frac{(r+1)x+(2r+1)}{2rx(x+1)}+\frac{s}{x\ln x}\right)\right.\\
&\left.-\left(\frac{1}{6(x+1)^2}+\frac{s}{6(x+1)\ln x}+\frac{1}{12r(x+1)}\right)\right\}\\
&=-\frac{x^{\frac{1}{r}}(\ln x)^{2s}}{x+1}\left\{\left(\frac{(r+1)x+(2r+1)}{2rx(x+1)}\right)+\left(\frac{s}{x\ln x}-\frac{s}{6(x+1)\ln x}\right)\right.\\
&\left.-\left(\frac{1}{6(x+1)^2}+\frac{1}{12r(x+1)}\right)\right\}\\
&<-\frac{x^{\frac{1}{r}}(\ln x)^{2s}}{x+1}\left\{\frac{(r+1)x+(2r+1)}{2rx(x+1)}-\left(\frac{1}{6(x+1)^2}+\frac{1}{12r(x+1)}\right)\right\}\\
&=-\frac{x^{\frac{1}{r}}(\ln x)^{2s}}{2(x+1)^2}\left\{\frac{r+1}{r}+\frac{2r+1}{rx}-\frac{1}{3(x+1)}-\frac{1}{6r}\right\}\\
&<-\frac{x^{\frac{1}{r}}(\ln x)^{2s}}{2(x+1)^2}\left\{\frac{r+1}{r}-\frac{1}{3(x+1)}-\frac{1}{6r}\right\}<0\,.
\end{aligned}
$$

Hence $\varphi(x)$ is monotonously decreasing in (I, ∞). Notice that $\lim_{x\to\infty}\varphi(x)=0$, Hereby we have $\varphi(x)\downarrow 0$. The lemma is proved. □

Lemma 2.2. *Let $0<\alpha<1$ and s be a nonnegative integer. Then*

$$
\int_0^1 u^{\alpha-1}\left(\ln\frac{1}{u}\right)^{2s}\frac{1}{1+u}du=(2s)!\zeta^*(2s+1,\alpha)\,, \tag{2.2}
$$

where ζ^* is defined by (1.3).

This result has been given in the paper [10]. Hence its proof is omitted here.

Lemma 2.3. *With the assumptions as in Lemma 2.2, then*

$$
\int_0^\infty u^{\alpha-1}\left(\ln\frac{1}{u}\right)^{2s}\frac{1}{1+u}du=(2s)!\left\{\zeta^*\left(2s+1,\alpha\right)+\zeta^*\left(2s+1,1-\alpha\right)\right\}, \tag{2.3}
$$

where ζ^* is defined by (1.3).

Proof. It is easy to deduce that

$$
\int_0^\infty u^{\alpha-1}\left(\ln\frac{1}{u}\right)^{2s}\frac{1}{1+u}du
$$

$$= \int_0^1 u^{\alpha-1}\left(\ln\frac{1}{u}\right)^{2s}\frac{1}{1+u}du + \int_1^\infty u^{\alpha-1}\left(\ln\frac{1}{u}\right)^{2s}\frac{1}{1+u}du$$

$$= \int_0^1 u^{\alpha-1}\left(\ln\frac{1}{u}\right)^{2s}\frac{1}{1+u}du + \int_0^1 v^{-\alpha}\left(\ln v\right)^{2s}\frac{1}{1+v}dv$$

$$= \int_0^1 u^{\alpha-1}\left(\ln\frac{1}{u}\right)^{2s}\frac{1}{1+u}du + \int_0^1 v^{(1-\alpha)-1}\left(\ln\frac{1}{v}\right)^{2s}\frac{1}{1+v}dv.$$

By using Lemma 2.2, the relation (2.3) follows at once. □

3. Main Results

We are ready now to formulate our main results.

Theorem 3.1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$, and s be a nonnegative integer. If $\sum_{n=1}^{\infty} a_n^p < +\infty$ and $\sum_{n=1}^{\infty} b_n^q < +\infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m b_n}{m+n} \le \zeta_p\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}, \tag{3.1}$$

where ζ_p is defined by (1.4), and the coefficient ζ_p is the best possible. In particular, when $s>0$, ζ_p can be expressed by the Riemann zeta function.

Proof. We may apply the Hölder inequality to estimate the left-hand side of (3.1) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m b_n}{m+n}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\left(\frac{\left(\ln\frac{n}{m}\right)^{2s}}{m+n}\right)^{\frac{1}{p}}\left(\frac{m}{n}\right)^{\frac{1}{pq}}a_m\right\}\left\{\left(\frac{\left(\ln\frac{n}{m}\right)^{2s}}{m+n}\right)^{\frac{1}{q}}\left(\frac{n}{m}\right)^{\frac{1}{pq}}b_n\right\}$$

$$\le \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{p}}b_n^q\right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=1}^{\infty} \omega_q(n) a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \omega_p(n) b_n^q \right\}^q, \quad (3.2)$$

where

$$\omega_r(n) = \sum_{m=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^{2s}}{m+n} \left(\frac{n}{m}\right)^{\frac{1}{r}}, \quad r = p, q\,. \quad (3.3)$$

Let $f(x) = \frac{\left(\ln \frac{n}{x}\right)^{2s}}{n+x} \left(\frac{n}{x}\right)^{\frac{1}{r}}$. Then it is easy to deduce that $f(\infty) = f'(\infty) = 0$. Applying the Euler-Maclaurin summation formula to $\omega_r(n)$, we have

$$\omega_r(n) = \int_1^{\infty} f(x)dx + \frac{1}{2} f(1) - \frac{\theta}{12} f'(1)$$

$$= \int_1^{\infty} \frac{\left(\ln \frac{n}{x}\right)^{2s}}{n+x} \left(\frac{n}{x}\right)^{\frac{1}{r}} dx + \frac{n^{\frac{1}{r}} (\ln n)^{2s}}{2(n+1)} + \frac{n^{\frac{1}{r}} (\ln n)^{2s-1} \theta}{12(n+1)} \left(\frac{\ln n}{n+1} + 2s + \frac{\ln n}{r}\right)$$

$$= \int_0^{\infty} \frac{\left(\ln \frac{1}{u}\right)^{2s}}{(1+u)} \left(\frac{1}{u}\right)^{\frac{1}{r}} du - \varphi(n), \quad (3.4)$$

where $\varphi(x)$ is defined by (2.1).

By using (2.3) and (1.4), we have

$$\omega_r(n) = \zeta_p - \varphi(n) \quad (r = p, q)\,. \quad (3.5)$$

It is known from Lemma 2.1 that $\varphi(n) \geq 0$. Hence we obtain

$$\omega_r(n) \leq \zeta_p \quad (r = p, q)\,. \quad (3.6)$$

It follows from (3.2) and (3.6) that the inequality (3.1) is valid.

It remains to need only to show that ζ_p in (3.1) is the best possible. $\forall \varepsilon > 0$. Define two sequences by $\tilde{a}_m = m^{-\frac{1+\varepsilon}{p}}$ and $\tilde{b}_n = n^{-\frac{1+\varepsilon}{q}}$ $(m, n = 1, 2, 3, \cdots)$. Since the sequence $\{\tilde{a}_m\}$ is monotonously decreasing in $[1, \infty)$, we have

$$\frac{1}{\varepsilon} = \int_1^{\infty} x^{-1-\varepsilon} dx < \sum_{m=1}^{\infty} m^{-1-\varepsilon} = \sum_{m=1}^{\infty} \tilde{a}_m^p = \tilde{a}_1^p + \sum_{m=2}^{\infty} \tilde{a}_m^p < 1 + \int_1^{\infty} x^{-1-\varepsilon} dx$$

$$= 1 + \frac{1}{\varepsilon}.$$

Hence we obtain $\sum\limits_{m=1}^{\infty} \tilde{a}_m^p = \frac{1}{\varepsilon} + o(1)$ $(\varepsilon \to 0)$.

Similarly, we obtain $\sum\limits_{n=1}^{\infty} \tilde{b}_n^q = \frac{1}{\varepsilon} + o(1)$ $(\varepsilon \to 0)$.

If the constant factor ζ_p in (3.1) is not the best possible, then there exists a constant $C > 0$ such that $C < \zeta_p$ and

$$S(\tilde{a}, \tilde{b}) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^{2s} \tilde{a}_m \tilde{b}_n}{m+n} \leq C \left(\sum_{m=1}^{\infty} \tilde{a}_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \tilde{b}_n^q \right)^{\frac{1}{q}} = \frac{C}{\varepsilon} \{1 + \circ(1)\} \quad (\varepsilon \to 0). \quad (3.7)$$

On the other hand, we have

$$S(\tilde{a}, \tilde{b}) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^{2s} m^{-\frac{1+\varepsilon}{p}} n^{-\frac{1+\varepsilon}{q}}}{m+n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^{-\frac{1+\varepsilon}{p}} \left(\left(\ln \frac{n}{m}\right)^{2s} n^{-\frac{1+\varepsilon}{q}} \right)}{m+n}$$
$$= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^{2s}}{m+n} \left(\frac{n}{m} \right)^{-\frac{1+\varepsilon}{q}} \right) m^{-1-\varepsilon}. \quad (3.8)$$

When ε is small enough, it is known from (3.3) and (3.5) that

$$\sum_{n=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^{2s}}{m+n} \left(\frac{n}{m} \right)^{-\frac{1+\varepsilon}{q}} = \sum_{n=1}^{\infty} \frac{\left(\ln \frac{m}{n}\right)^{2s}}{m+n} \left(\frac{m}{n} \right)^{\frac{1+\varepsilon}{q}} = \omega_q(m) + \tilde{\circ}(1)$$
$$= (\zeta_p - \varphi(m)) + \tilde{\circ}(1) \quad (\varepsilon \to 0). \quad (3.9)$$

Notice that the sequence $\{m^{-1-\varepsilon}\}$ is monotonously decreasing, it follows from (3.8) and (3.9) that

$$S(\tilde{a}, \tilde{b}) = \sum_{m=1}^{\infty} \{\zeta_p - (\varphi(m) - \tilde{\circ}(1))\} m^{-1-\varepsilon}$$
$$= \zeta_p \sum_{m=1}^{\infty} m^{-1-\varepsilon} - \sum_{m=1}^{\infty} (\varphi(m) - \tilde{\circ}(1)) m^{-1-\varepsilon}$$
$$> \zeta_p \int_1^{\infty} x^{-1-\varepsilon} dx - \sum_{m=1}^{\infty} (\varphi(m) - \tilde{\circ}(1)) m^{-1-\varepsilon}$$
$$= \frac{\zeta_p}{\varepsilon} - \sum_{m=1}^{\infty} (\varphi(m) - \tilde{\circ}(1)) m^{-1-\varepsilon} = \frac{\zeta_p}{\varepsilon} - O(1) \quad (\varepsilon \to 0). \quad (3.10)$$

Below we will prove that the series $\sum\limits_{m=1}^{\infty} (\varphi(m) - \tilde{\circ}(1)) m^{-1-\varepsilon}$ in (3.10) is bounded. In fact, it is known from Lemma 2.1 that $\varphi(m) \downarrow 0 (m \to \infty)$. In other words, $\forall 0 < \varepsilon < 1$, there exists m_0, when $m > m_0$, $|\varphi(m) - \tilde{\text{o}}(1)| < \varepsilon$. Hence we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} (\varphi(m) - \tilde{\text{o}}(1))\, m^{-1-\varepsilon} \\
&\quad = \sum_{m=1}^{m_0} (\varphi(m) - \tilde{\text{o}}(1))\, m^{-1-\varepsilon} + \sum_{m=m_0+1}^{\infty} (\varphi(m) - \tilde{\text{o}}(1))\, m^{-1-\varepsilon} \\
&\quad < \sum_{m=1}^{m_0} (\varphi(m) - \tilde{\text{o}}(1))\, m^{-1-\varepsilon} + \sum_{m=m_0+1}^{\infty} \varepsilon m^{-1-\varepsilon} \\
&\quad < \sum_{m=1}^{m_0} (\varphi(m) - \tilde{\text{o}}(1))\, m^{-1-\varepsilon} + \varepsilon \int_{m_0}^{\infty} x^{-1-\varepsilon} dx \\
&\quad = \sum_{m=1}^{m_0} (\varphi(m) - \tilde{\text{o}}(1))\, m^{-1-\varepsilon} + \frac{1}{m_0^{\varepsilon}} \quad (\varepsilon \to 0)\,. \qquad (3.11)
\end{aligned}$$

It shows that the series $\sum_{m=1}^{\infty} (\varphi(m) - \tilde{\text{o}}(1))\, m^{-1-\varepsilon}$ is bounded. Thereby when m is large enough and ε is small enough, it follows from (3.10) that

$$S(\tilde{a}, \tilde{b}) > \frac{\zeta_p}{\varepsilon} (1 + \circ(1)) \quad (m \to \infty \text{ and } \varepsilon \to 0)\,. \qquad (3.12)$$

It is obvious that the inequality (3.12) is in contradiction with the inequality (3.7). This shows that the constant factor ζ_p in (3.1) is the best possible. In particular, it is known from (1.5) that when $s > 0$, ζ_p can be expressed by the Riemann zeta function. The proof of theorem is completed. □

When $p = 2$, we can obtain the following important result.

Theorem 3.2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers, and s be a nonnegative integer If $\sum_{n=1}^{\infty} a_n^2 < +\infty$ and $\sum_{n=1}^{\infty} b_n^2 < +\infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^{2s} a_m b_n}{m+n} \le \left(\pi^{2s+1} E_s\right) \left\{\sum_{n=1}^{\infty} a_n^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\infty} b_n^2\right\}^{\frac{1}{2}}, \qquad (3.13)$$

where the coefficient $\pi^{2s+1}E_s$ is the best possible, and that $E_0 = 1$ and E_s is the Euler number, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$,, etc.

Proof. We need only to show that $\zeta_2 = \pi^{2s+1}E_S$. Based on (1.3) and (1.4), we have

$$\zeta_2 = (2s)!2^{2r+2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2s+1}} \quad (s \in N_0)\,. \qquad (3.14)$$

It is known from the paper [7] that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2s+1}} = \frac{\pi^{2s+1}}{2^{2s+2}(2s)!} E_s\,, \tag{3.15}$$

where E_s is the Euler number, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521, \cdots$.

Notice that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\cdots = \frac{\pi}{4}$. Hence we can define $E_0 = 1$, It follows from (3.14) and (3.15) that $\zeta_2 = \pi^{2s+1} E_s$, where $s \in N_0$. $\square$

In particular, when $s = 1$, we have the following result.

Theorem 3.3. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $\sum\limits_{n=1}^{\infty} a_n^p < +\infty$ and $\sum\limits_{n=1}^{\infty} b_n^q < +\infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^2 a_m b_n}{m+n} \le \tilde{\zeta}_p \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{3.16}$$

where $\tilde{\zeta}_p = \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^3 (2 - \sin^2 \frac{\pi}{p})$, and the coefficient $\tilde{\zeta}_p$ is the best possible.

Proof. For case $s = 1$, we need only compute the integral in (3.4)

$$\tilde{\zeta}_p = \tilde{\zeta}_q = \int_0^{\infty} \frac{\left(\ln \frac{1}{u}\right)^2}{(1+u)} \left(\frac{1}{u}\right)^{\frac{1}{q}} du.$$

By substitution $u = e^{-t}$, and then by using the result of the paper [10] (pp. 229, formula (1110)) we have

$$\tilde{\zeta}_p = \int_{-\infty}^{+\infty} \frac{t^2 e^{-\frac{1}{p}t}}{1+e^{-t}} dt = \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^3 (2 - \sin^2 \frac{\pi}{p}). \qquad \square \tag{3.17}$$

Specially, when $p = 2$, $\tilde{\zeta}_p$ in (3.17) is reduced to π^3. Hence we have the following result.

Corollary 3.4. *With the assumptions as in Theorem 3.3, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^2 a_m b_n}{m+n} \le \pi^3 \left\{\sum_{n=1}^{\infty} a_n^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\infty} b_n^2\right\}^{\frac{1}{2}}, \tag{3.18}$$

where the coefficient π^3 is the best possible.

4. Applications

As applications, we will build some equivalent inequalities each other.

Theorem 4.1. *Let s be a nonnegative integer. $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. If $\sum\limits_{m=1}^{\infty} a_m^p < +\infty$, then*

$$\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m}{m+n}\right\}^p \le \zeta_p^p \sum_{m=1}^{\infty} a_m^p, \tag{4.1}$$

where ζ_p is defined by (1.4) and the constant factor ζ_p^p is the best possible. And the inequality (4.1) is equivalent to (3.1).

Proof. Assume that the inequality (3.1) is valid. Setting a sequence as

$$b_n = \left(\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m}{m+n}\right)^{p-1} \qquad (n=1,2,\cdots).$$

By using (3.1), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m}{m+n}\right\}^p &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m}{m+n}\left(\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2r}a_m}{m+n}\right)^{p-1}\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m b_n}{m+n} \le \zeta_p\left\{\sum_{m=1}^{\infty}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}\\ &= \zeta_p\left\{\sum_{m=1}^{\infty}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\left(\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m}{m+n}\right)\right)^p\right\}^{\frac{1}{q}}. \end{aligned} \tag{4.2}$$

It follows from (4.2) that the inequality (4.1) keeps valid after some simplifications.

On the other hand, assume that the inequality (4.1) keeps valid. By applying in turn Hölder's inequality and (4.1), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m b_n}{m+n} &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m}{m+n}\right)b_n\\ &\le \left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{\left(\ln\frac{n}{m}\right)^{2s}a_m}{m+n}\right)^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}\end{aligned}$$

$$\leq \left\{ \zeta_p^p \sum_{m=1}^{\infty} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}} = \zeta_p \left\{ \sum_{m=1}^{\infty} a^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}} . \quad (4.3)$$

Hence the inequality (3.1) is valid. So the inequality (4.1) is equivalent to (3.1).

If the constant factor ζ_p^p in (4.1) is not the best possible, then it is known from (4.3) that the constant factor ζ_p in (3.1) is also not the best possible. This is a contradiction. Theorem 4.1 is proved. □

Theorem 4.2. *Let* $\{a_n\}$ *be a sequence of real numbers, and* s *be a nonnegative integer. If* $\sum_{n=1}^{\infty} a_n^2 < +\infty$, *then*

$$\sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^{2s} a_m}{m+n} \right\}^2 \leq \left(\pi^{2s+1} E_s\right)^2 \sum_{m=1}^{\infty} a_m^2, \quad (4.4)$$

where the coefficient $(\pi^{2s+1}E_s)^2$ *is the best possible, and that* $E_0 = 1$ *and* E_s *is the Euler number, viz.* $E_1 = 1, E_2 = 5,\ E_3 = 61, E_4 = 1385$, *etc. And the inequality (4.4) is equivalent to (3.13).*

Theorem 4.3. *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers,* $\frac{1}{p}+\frac{1}{q} = 1$ *and* $p > 1$. *If* $\sum_{n=1}^{\infty} a_n^p < +\infty$, *then*

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^2 a_m}{m+n} \right)^p \leq \tilde{\zeta}_p^p \sum_{m=1}^{\infty} a_m^p, \quad (4.5)$$

where $\tilde{\zeta}_p = \left(\frac{\pi}{\sin \frac{\pi}{p}}\right)^3 (2-\sin^2 \frac{\pi}{p})$, *and the coefficient* $\tilde{\zeta}_p^p$ *is the best possible. And the inequality (4.5) is equivalent to (3.16).*

Corollary 4.4. *Let* $\{a_n\}$ *be a sequence of real numbers. If* $\sum_{m=1}^{\infty} a_m^2 < +\infty$, *then*

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{\left(\ln \frac{n}{m}\right)^2 a_m}{m+n} \right)^2 \leq \pi^6 \sum_{m=1}^{\infty} a_m^2, \quad (4.6)$$

where the coefficient π^6 *is the best possible. And the inequality (4.6) is equivalent to (3.18).*

The proofs of Theorems 4.2-4.3 and Corollary 4.4 are similar to the one of Theorem 4.1, they are omitted here.

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