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COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES RELATED TO SĂLĂGEAN OPERATOR AND APPLICATIONS

Toshio Hayami[1], Shigeyoshi Owa[2 §]
[1,2]Department of Mathematics
Kinki University
Higashi-Osaka, Osaka, 577-8502, JAPAN
[1]e-mail: ha_ya_to112@hotmail.com
[2]e-mail: owa@math.kindai.ac.jp

Abstract: For functions $f(z)$ which are analytic in the open unit disk $\mathbb{U}$, generalization operator $D^j f(z)$ of Sălăgean operator is introduced. With the operator $D^j f(z)$, the subclass $\mathcal{S}_j^m(\alpha)$ is defined. Some interesting sufficient conditions involving coefficient inequalities for $f(z)$ to be in the class $\mathcal{S}_j^m(\alpha)$ are discussed. Also, consequences of coefficient inequalities for differential subordination are considered.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (a_0 = 0,\ a_1 = 1) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.2}$$

which are analytic in $\mathbb{U}$. If $p(z) \in \mathcal{P}$ satisfies Re $p(z) > 0$ $(z \in \mathbb{U})$, then we say

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§Correspondence author

that $p(z)$ is the Carathéodory function (cf. [1]).

If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some α $(0 \leqq \alpha < 1)$, then $f(z)$ is said to be starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are starlike of order α in $\mathbb{U}$. Similary, we say that $f(z)$ is a member of the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.4}$$

for some α $(0 \leqq \alpha < 1)$.

As usual, in the present investigation, we write

$$\mathcal{S}^* \equiv \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} \equiv \mathcal{K}(0).$$

Classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [4].

We define the following differential operater due to Sălăgean [5].

For a function $f(z) \in \mathcal{A}$,

$$D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n ,$$

$$D^j f(z) = D\big(D^{j-1} f(z)\big) = z + \sum_{n=2}^{\infty} n^j a_n z^n .$$

Also, we meditate the following integral operator

$$D^{-1} f(z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta = z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n ,$$

$$D^{-j} f(z) = D^{-1}\big(D^{-(j-1)} f(z)\big) = z + \sum_{n=2}^{\infty} \frac{1}{n^j} a_n z^n$$

for any negative integers.

Then, for $f(z) \in \mathcal{A}$ given by (1.1), we know that

$$D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 0,\ \pm 1,\ \pm 2,\ \cdots).$$

Using the above operator $D^j f(z)$, we consider the subclass $\mathcal{S}_j^m(\alpha)$ of $\mathcal{A}$ as follows:

$$\mathcal{S}_j^m(\alpha) = \left\{ f(z) \in \mathcal{A} \;:\; \operatorname{Re}\left(\frac{D^m f(z)}{D^j f(z)}\right) > \alpha \quad (z \in \mathbb{U};\; m \neq j,\; 0 \leqq \alpha < 1) \right\}.$$

Remark 1.1.

$$\mathcal{S}_0^1(\alpha) \equiv \mathcal{S}^*(\alpha), \quad \mathcal{S}_1^2(\alpha) \equiv \mathcal{K}(\alpha).$$

Furthermore, we consider the subclass $\mathcal{STS}(\mu)$ and $\mathcal{STC}(\mu)$ of $\mathcal{A}$ as follows:

$$\mathcal{STS}(\mu) = \left\{ f(z) \in \mathcal{A} \;:\; \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\mu}{2} \quad (0 < \mu \leqq 1) \right\}$$

and

$$\mathcal{STC}(\mu) = \left\{ f(z) \in \mathcal{A} \;:\; \left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\mu}{2} \quad (0 < \mu \leqq 1) \right\}.$$

If a function $f(z)$ is a member of the class $\mathcal{STS}(\mu)$ or $\mathcal{STC}(\mu)$, we say that $f(z)$ is strongly starlike or strongly convex of order μ, respectively.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$, such that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ and $f(z) = g(w(z))$.

If $g(z)$ is univalent in $\mathbb{U}$, then it is known that

$$f(z) \prec g(z) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We apply the following lemma to obtain our results.

Lemma 1.2. *A function* $p(z) \in \mathcal{P}$ *satisfies* $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$ *if and only if*

$$p(z) \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U}) \tag{1.5}$$

for all $|x| = 1$.

Then, by using Lemma 1.2, various conditions for starlike functions are studied. The following results are enumerated as the some example.

Lemma 1.3. *A function* $f(z) \in \mathcal{A}$ *is in* $\mathcal{S}^*(\alpha)$ *if and only if*

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \qquad (z \in \mathbb{U};\; |x| = 1), \tag{1.6}$$

where

$$A_n = \frac{n+1-2\alpha+(n-1)x}{2-2\alpha} a_n.$$

Silverman, Silvia, and Telage [7] have given

Remark 1.4. The relation (1.6) of Lemma 1.3 is equivalent to

$$\frac{1}{z}\left(f(z)*\frac{z+\dfrac{x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}\right)\neq 0 \qquad (z\in\mathbb{U},\ |x|=1).$$

Furthermore, letting $\alpha=0$ in Lemma 1.8, Nezhmetdinov and Ponnusamy [3] have given the sufficient conditions for coefficients of $f(z)$ to be in the class $\mathcal{S}^*$.

Hayami, Owa and Sirivastava [2] have shown the following results.

Theorem 1.5. *If $f(z)\in\mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j+1-2\alpha)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right]\leqq 2(1-\alpha)$$

for some α $(0\leqq\alpha<1)$, $\beta\in\mathbb{R}$, and $\gamma\in\mathbb{R}$, then $f(z)\in S^*(\alpha)$.

Theorem 1.6. *If $f(z)\in\mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}j(j+1-2\alpha)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}j(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right]\leqq 2(1-\alpha)$$

for some α $(0\leqq\alpha<1)$, $\beta\in\mathbb{R}$, and $\gamma\in\mathbb{R}$, then $f(z)\in\mathcal{K}(\alpha)$.

The next theorem due to Silverman [6] is a well-known result.

Theorem 1.7. *A function $f(z)\in\mathcal{A}$ satisfies the following conditions*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leqq 1-\alpha \quad \text{or} \quad \sum_{n=2}^{\infty}n(n-\alpha)|a_n|\leqq 1-\alpha,$$

then $f(z)\in\mathcal{S}^*(\alpha)$ or $f(z)\in\mathcal{K}(\alpha)$, respectively.

The object of the present paper is to give some generalizations for the results in [2] and [3]. We also briefly discuss the improvement of Lemma 1.2, and several (new or known) interesting corollaries.

2. Coefficient Inequalities for $\mathcal{S}_j^m(\alpha)$

In this section, we discuss the improvement of Lemma 1.2.

Lemma 2.1. *A function* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ *satisfies* $\operatorname{Re} p(z) > \alpha$ $(z \in \mathbb{U})$ *if and only if*

$$\frac{p(z)-\alpha}{\beta} \neq \frac{x-1}{x+1} \qquad (0 \leqq \alpha < 1,\ \beta > 0;\ z \in \mathbb{U})$$

for any $x \in \mathbb{C}$ *satisfying* $|x| = 1$.

Proof. First, a simple computation gives us

$$\operatorname{Re}\left(\frac{x-1}{x+1}\right) = 0$$

for all $|x| = 1$. On the other hand, if $\operatorname{Re} p(z) > \alpha$ $(z \in \mathbb{U})$, then it follows that

$$\operatorname{Re}\left(\frac{p(z)-\alpha}{\beta}\right) > 0.$$

Thus, we know that

$$\frac{p(z)-\alpha}{\beta} \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U})$$

for all $|x| = 1$. Further, noting that $\frac{p(0)-\alpha}{\beta} = \frac{1-\alpha}{\beta} > 0$ for $p(z) \in \mathcal{P}$, we see that $\operatorname{Re} p(z) > \alpha$ $(z \in \mathbb{U})$ if it follows that

$$\frac{p(z)-\alpha}{\beta} \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U})$$

for all $|x| = 1$. □

With the help of Lemma 2.1, we have

Theorem 2.2. *If* $f(z) \in \mathcal{A}$ *satisfies any one of the following inequalities*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k^{m-j}-\alpha+\beta)k^{j}(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$
$$\left.+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k^{m-j}-\alpha-\beta)k^{j}(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right] \leqq 2\beta$$

$$(0 \leqq \alpha < 1,\ 0 < \beta \leqq 1-\alpha), \tag{2.1}$$

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k^{m-j}-\alpha+\beta)k^{j}(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$

$$+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k^{m-j}-\alpha-\beta)k^{j}(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\Bigg] \leqq 2(1-\alpha)$$

$$(0 \leqq \alpha < 1,\ \beta > 1-\alpha) \tag{2.2}$$

for some $\gamma \in \mathbb{R}$ *and* $\delta \in \mathbb{R}$, *then* $f(z) \in \mathcal{S}_j^m(\alpha)$.

Proof. Let us define the function $p(z)$ by $p(z) = \frac{D^m f(z)}{D^j f(z)}$ for $f(z) \in \mathcal{A}$.

Applying Lemma 2.1, $f(z) \in \mathcal{S}_j^m(\alpha)$ if and only if

$$\frac{p(z)-\alpha}{\beta} = \frac{\dfrac{D^m f(z)}{D^j f(z)} - \alpha}{\beta} \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U}) \tag{2.3}$$

for all $|x| = 1$.

Then, we need not to consider Lemma 2.1 for $z = 0$, because it follows that

$$\frac{p(0)-\alpha}{\beta} = \frac{1-\alpha}{\beta} \neq \frac{x-1}{x+1} \qquad (|x| = 1).$$

The relation (2.3) is equivalent to

$$(x+1)\Big(D^m f(z) - \alpha D^j f(z)\Big) \neq (x-1)\beta D^j f(z),$$

that is, that

$$\Big\{(1-\alpha+\beta)+x(1-\alpha-\beta)\Big\}z+\sum_{n=2}^{\infty}\Big\{(n^{m-j}-\alpha+\beta)+x(n^{m-j}-\alpha-\beta)\Big\}n^j a_n z^n \neq 0. \tag{2.4}$$

Now, setting $\varphi(x) = (1-\alpha+\beta)+x(1-\alpha-\beta)$, we know

$$\begin{cases} \text{(i) } |\varphi(x)| \geqq 2\beta \quad \text{for } 0 < \beta \leqq 1-\alpha\,, \\ \text{(ii) } |\varphi(x)| \geqq 2(1-\alpha) \quad \text{for } \beta > 1-\alpha\,. \end{cases}$$

Dividing the both sides of (2.4) by $\varphi(x)z$ $(z \neq 0)$, we obtain that

$$1 + \sum_{n=2}^{\infty} B_n z^{n-1} \neq 0\,,$$

where

$$B_n = \frac{(n^{m-j}-\alpha+\beta)+x(n^{m-j}-\alpha-\beta)}{\varphi(x)} n^j a_n \quad (n \geqq 2).$$

Therefore, it is sufficient that

$$1 + \sum_{n=2}^{\infty}\left[\sum_{l=1}^{n}\left\{\sum_{k=1}^{l} B_k(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right] z^{n-1} \neq 0,$$

where $B_0 = 0$ and $B_1 = 1$. Thus, if $f(z)$ satisfies

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k^{m-j}-\alpha+\beta)k^{j}(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$

$$\left.+|x|\cdot\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k^{m-j}-\alpha-\beta)k^{j}(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right] \leqq |\varphi(x)|$$

then $f(z) \in \mathcal{S}_j^m(\alpha)$. This completes the proof of Theorem 2.2. □

Putting $m = 1$ and $j = 0$ in Theorem 2.2, we obtain

Corollary 2.3. *If $f(z) \in \mathcal{A}$ satisfies any one of the following inequalities*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k-\alpha+\beta)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$

$$\left.+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k-\alpha-\beta)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right] \leqq 2\beta$$

$$(0 \leqq \alpha < 1,\ 0 < \beta \leqq 1-\alpha),$$

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k-\alpha+\beta)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$

$$\left.+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k-\alpha-\beta)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right] \leqq 2(1-\alpha)$$

$$(0 \leqq \alpha < 1,\ \beta > 1-\alpha)$$

for some $\gamma \in \mathbb{R}$ and $\delta \in \mathbb{R}$, then $f(z) \in \mathcal{S}^*(\alpha)$.

Also, considering $m = 2$ and $j = 1$ in Theorem 2.2, we have

Corollary 2.4. *If $f(z) \in \mathcal{A}$ satisfies any one of the following inequalities*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(k-\alpha+\beta)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$

$$+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(k-\alpha-\beta)(-1)^{l-k}\left(\begin{array}{c}\gamma\\ l-k\end{array}\right)a_k\right\}\left(\begin{array}{c}\delta\\ n-l\end{array}\right)\right|\Bigg]\leqq 2\beta$$

$$(0\leqq\alpha<1,\ 0<\beta\leqq 1-\alpha),$$

$$\sum_{n=2}^{\infty}\Bigg[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(k-\alpha+\beta)(-1)^{l-k}\left(\begin{array}{c}\gamma\\ l-k\end{array}\right)a_k\right\}\left(\begin{array}{c}\delta\\ n-l\end{array}\right)\right|$$

$$+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(k-\alpha-\beta)(-1)^{l-k}\left(\begin{array}{c}\gamma\\ l-k\end{array}\right)a_k\right\}\left(\begin{array}{c}\delta\\ n-l\end{array}\right)\right|\Bigg]\leqq 2(1-\alpha)$$

$$(0\leqq\alpha<1,\ \beta>1-\alpha)$$

for some $\gamma\in\mathbb{R}$ *and* $\delta\in\mathbb{R}$, *then* $f(z)\in\mathcal{K}(\alpha)$.

Taking $\gamma=\delta=0$ in Theorem 2.2, we derive

Corollary 2.5. *If* $f(z)\in\mathcal{A}$ *satisfies the either of condition*

$$\sum_{n=2}^{\infty}(n^m-\alpha n^j)|a_n|\leqq 1-\alpha \qquad (m>j) \tag{2.5}$$

or

$$\sum_{n=2}^{\infty}\Big\{|(n^{m-j}-\alpha+\beta)n^j a_n|+|(n^{m-j}-\alpha-\beta)n^j a_n|\Big\}\leqq 2(1-\alpha)$$

$$(\beta>1;\ m>j) \tag{2.6}$$

for some α $(0\leqq\alpha<1)$, *then* $f(z)\in\mathcal{S}_j^m(\alpha)$.

Proof. When $\gamma=\delta=0$, the relation (2.1) or (2.2) is equivalent to

$$\sum_{n=2}^{\infty}\Big\{|(n^{m-j}-\alpha+\beta)n^j a_n|+|(n^{m-j}-\alpha-\beta)n^j a_n|\Big\}\leqq 2\beta$$

$$(0\leqq\alpha<1,\ 0<\beta\leqq 1-\alpha),$$

or

$$\sum_{n=2}^{\infty}\Big\{|(n^{m-j}-\alpha+\beta)n^j a_n|+|(n^{m-j}-\alpha-\beta)n^j a_n|\Big\}\leqq 2(1-\alpha)$$

$$(0\leqq\alpha<1,\ \beta>1-\alpha)$$

respectively. It is easy to obtain that $(n^{m-j}-\alpha+\beta)$ is positive for any n, α and β.

(i) For the case $0 < \beta \leqq 1 - \alpha$, we see that $(n^{m-j} - \alpha - \beta) > 0$ for $n \geqq 2$ and

$$\sum_{n=2}^{\infty} \Big\{ |(n^{m-j} - \alpha + \beta)n^j a_n| + |(n^{m-j} - \alpha - \beta)n^j a_n| \Big\}$$
$$= \sum_{n=2}^{\infty} 2(n^m - \alpha n^j)|a_n| \leqq 2\beta \leqq 2(1 - \alpha).$$

(ii) For the case $1 - \alpha < \beta \leqq 1$, we know that $(n^{m-j} - \alpha - \beta) > 0$ for $n \geqq 2$ and

$$\sum_{n=2}^{\infty} \Big\{ |(n^{m-j} - \alpha + \beta)n^j a_n| + |(n^{m-j} - \alpha - \beta)n^j a_n| \Big\}$$
$$= \sum_{n=2}^{\infty} 2(n^m - \alpha n^j)|a_n| \leqq 2(1 - \alpha).$$

Hence, for the case $0 < \beta \leqq 1$, we have the inequality (2.5). Further, for the case $\beta > 1$, we have the condition (2.6). □

In particular, setting $\gamma = \delta = 0$ in Corollary 2.3 or Corollary 2.4, we deduce the following corollaries including the well-known result due to Silverman [6].

Corollary 2.6. *If $f(z) \in \mathcal{A}$ satisfies the either of condition*

$$\sum_{n=2}^{\infty} (n - \alpha)|a_n| \leqq 1 - \alpha$$

or

$$\sum_{n=2}^{\infty} \Big\{ |(n - \alpha + \beta)a_n| + |(n - \alpha - \beta)a_n| \Big\} \leqq 2(1 - \alpha) \qquad (\beta > 1)$$

for some α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{S}^*(\alpha)$.

Corollary 2.7. *If $f(z) \in \mathcal{A}$ satisfies the either of condition*

$$\sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leqq 1 - \alpha$$

or

$$\sum_{n=2}^{\infty} \Big\{ |n(n - \alpha + \beta)a_n| + |n(n - \alpha - \beta)a_n| \Big\} \leqq 2(1 - \alpha) \qquad (\beta > 1)$$

for some α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathcal{K}(\alpha)$.

3. Applications to Differential Subordinations

Next, we apply Lemma 1.2 to differential subordinations.

Remark 3.1. Setting $x = -\frac{1}{\zeta}$ for $\zeta \in \mathbb{C}$ satisfying $|\zeta| = 1$, the inequality (1.5) is equivalent to

$$p(z) \neq \frac{-\frac{1}{\zeta} - 1}{-\frac{1}{\zeta} + 1} \qquad (z \in \mathbb{U})$$

or

$$p(z) \neq \frac{1+\zeta}{1-\zeta} \qquad (z \in \mathbb{U}).$$

Now, in consideration of this fact, we think about Lemma 1.2 again.

Lemma 3.2. *A function $p(z) \in \mathcal{P}$ satisfies $p(\mathbb{U}) \subset \psi(\mathbb{U})$ if and only if*

$$p(z) \neq \psi(\zeta) \qquad (z \in \mathbb{U}),$$

where

$$\psi(\zeta) = \frac{1+\zeta}{1-\zeta}$$

for all $|\zeta| = 1$.

Proof. For $|\xi| < 1$, we set $\psi(\xi) = \frac{1+\xi}{1-\xi}$. Then,

$$|\xi| = \left| \frac{\psi(\xi) - 1}{\psi(\xi) + 1} \right| < 1.$$

This means that

$$\text{Re } \psi(\xi) > 0$$

for any $|\xi| < 1$. Therefore, we see that $p(\mathbb{U}) \subset \psi(\mathbb{U})$. The proof of Lemma 3.2 is completed. □

In view of Lemma 3.2, we derive

Lemma 3.3. *For a function $p(z) \in \mathcal{P}$, the subordination $p(z) \prec \frac{1+Az}{1+Bz}$ holds true if and only if*

$$p(z) \neq \frac{1+Ax}{1+Bx} \qquad (z \in \mathbb{U})$$

for any $|x| = 1$, where $A, B \in \mathbb{C}, A \neq B$, and $|B| \leqq 1$.

Proof. Let us define $q(z)$ by

$$q(z) = \frac{1+Az}{1+Bz} \qquad (|B| \leqq 1,\ A \neq B;\ z \in \mathbb{U}).$$

Since $q(z)$ is univalent in $\mathbb{U}$ and $p(0) = q(0) = 1$,

$$p(z) \prec q(z) \quad \Longleftrightarrow \quad p(\mathbb{U}) \subset q(\mathbb{U}).$$

Thus, it is easily observed from Lemma 3.2 that the assertion of Lemma 3.3 is correct. □

We next derive the coefficient condition for functions $f(z)$ to be subordinate to certain univalent functions.

Theorem 3.4. *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k-1)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right.$$
$$\left.+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(A-kB)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right] \leqq |A-B| \qquad (3.1)$$

for some $A, B \in \mathbb{C}$ ($|B| \leqq 1,\ A \neq B$), $\gamma \in \mathbb{R}$ and $\delta \in \mathbb{R}$, then $\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$.

Proof. We should prove that

$$\frac{zf'(z)}{f(z)} \neq \frac{1+Ax}{1+Bx} \qquad (z \in \mathbb{U}) \qquad (3.2)$$

for all $|x| = 1$.

The relation (3.2) implies that

$$(1+Ax)\left(z+\sum_{n=2}^{\infty}a_n z^n\right)-(1+Bx)\left(z+\sum_{n=2}^{\infty}na_n z^n\right) \neq 0,$$

that is, that

$$x(A-B)z\left(1+\sum_{n=2}^{\infty}\frac{-(n-1)+x(A-nB)}{x(A-B)}a_n z^{n-1}\right) \neq 0. \qquad (3.3)$$

Dividing the both sides of (3.3) by $x(A-B)z$ ($z \neq 0$), we obtain

$$1+\sum_{n=2}^{\infty}C_n z^{n-1} \neq 0\,, \qquad (3.4)$$

where

$$C_n = \frac{-(n-1)+x(A-nB)}{x(A-B)}a_n \quad (n \geqq 2).$$

Furthermore, letting $C_0 = 0$ and $C_1 = 1$, and multiplying the both sides of (3.4) by $(1-z)^{\gamma}(1+z)^{\delta} \neq 0$ in $\mathbb{U}$ for $\gamma,\ \delta \in \mathbb{R}$, we know that

$$1+\sum_{n=2}^{\infty}\left[\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}C_k(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right]z^{n-1} \neq 0.$$

Therefore, if the following condition

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k-1)(-1)^{l-k}\left(\begin{array}{c}\gamma\\ l-k\end{array}\right)a_k\right\}\left(\begin{array}{c}\delta\\ n-l\end{array}\right)\right|\right.$$

$$\left.+|x|\cdot\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(A-kB)(-1)^{l-k}\left(\begin{array}{c}\gamma\\ l-k\end{array}\right)a_k\right\}\left(\begin{array}{c}\delta\\ n-l\end{array}\right)\right|\right]\leqq|A-B|$$

is satisfied, then it follows that

$$1+\sum_{n=2}^{\infty}C_n z^{n-1}\neq 0.$$

This completes the proof of Theorem 3.4. □

By Theorem 3.4, we know that

Corollary 3.5. *If $f(z)\in\mathcal{A}$ satisfies the following condition* (3.1) *for some real parameters A,B ($|A|\leqq 1,\ |B|\leqq 1,\ A\neq B$), then*

$$f(z)\in\mathcal{STS}\left(\frac{2}{\pi}\sin^{-1}\frac{|A-B|}{1-AB}\right).$$

Proof. Considering $q(z)=\frac{1+Az}{1+Bz}$, we have that

$$|z|=\left|\frac{1-q(z)}{Bq(z)-A}\right|<1,$$

which implies that

$$\left|q(z)-\frac{1-AB}{1-B^2}\right|<\frac{|A-B|}{1-B^2}.$$

This shows that $|\arg q(\mathbb{U})|<\sin^{-1}\frac{|A-B|}{1-AB}$. Therefore, the assertion of Corollary 3.5 holds true. □

Replacing $f(z)$ by $zf'(z)$ in Theorem 3.4, we also have

Theorem 3.6. *If $f(z)\in\mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(k-1)(-1)^{l-k}\left(\begin{array}{c}\gamma\\ l-k\end{array}\right)a_k\right\}\left(\begin{array}{c}\delta\\ n-l\end{array}\right)\right|\right.$$

$$\left.+\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(A-kB)(-1)^{l-k}\left(\begin{array}{c}\gamma\\ l-k\end{array}\right)a_k\right\}\left(\begin{array}{c}\delta\\ n-l\end{array}\right)\right|\right]\leqq|A-B| \qquad (3.5)$$

for some $A,B\in\mathbb{C}$ ($|B|\leqq 1,\ A\neq B$), $\gamma\in\mathbb{R}$ and $\delta\in\mathbb{R}$, then $1+\frac{zf''(z)}{f'(z)}\prec\frac{1+Az}{1+Bz}$.

Similarly, by Theorem 3.6, we deduce that

Corollary 3.7. *If* $f(z) \in \mathcal{A}$ *satisfies the condition* (3.5) *for some real parameters* A, B ($|A| \leqq 1,\ |B| \leqq 1, A \neq B$), *then* $f(z) \in \mathcal{STC}\left(\frac{2}{\pi}\sin^{-1}\frac{|A-B|}{1-AB}\right)$.

When $A = -B = r$ for some real number r $(0 < r \leqq 1)$ in Theorem 3.4 and Theorem 3.6, we obtain the following two corollaries.

Corollary 3.8. *If* $f(z) \in \mathcal{A}$ *satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k-1)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right| + r\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(k+1)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right] \leqq 2r$$

for some r $(0 < r \leqq 1)$, $\gamma \in \mathbb{R}$ *and* $\delta \in \mathbb{R}$, *then* $f(z) \in \mathcal{S}^*\left(\frac{1-r}{1+r}\right)$ *and* $f(z) \in \mathcal{STS}\left(\frac{2}{\pi}\sin^{-1}\frac{2r}{1+r^2}\right)$.

Corollary 3.9. *If* $f(z) \in \mathcal{A}$ *satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(k-1)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right| + r\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}k(k+1)(-1)^{l-k}\binom{\gamma}{l-k}a_k\right\}\binom{\delta}{n-l}\right|\right] \leqq 2r$$

for some r $(0 < r \leqq 1)$, $\gamma \in \mathbb{R}$ *and* $\delta \in \mathbb{R}$, *then* $f(z) \in \mathcal{K}\left(\frac{1-r}{1+r}\right)$ *and* $f(z) \in \mathcal{STC}\left(\frac{2}{\pi}\sin^{-1}\frac{2r}{1+r^2}\right)$.

Remark 3.10. If, in the hypothesis (3.1) or (3.5), we set

$$A = 1 - 2\alpha \quad \text{and} \quad B = -1 \qquad (0 \leqq \alpha < 1),$$

we arrive Theorem 1.5 or Theorem 1.6 (for detail, [2]).

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