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ON THE MAXIMAL RANK CONJECTURE IN $\mathbb{P}^4$

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Abstract: Here (following old joint work with Ph. Ellia and using an inductive method due to A. Hirschowitz) we prove that a general embedding in $\mathbb{P}^4$ of a curve with general moduli has maximal rank, i.e. it has good postulation.

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1. Introduction

Several years ago we wrote jointly with Ph. Ellia a series of papers on the postulation of curves in projective spaces (see [3], [4], [5], [6], [7], [8]), developed under the guidance of A. Hirschowitz and using a key method that he introduced (see [12], [11]). Here we improve one of our old results and prove the following result.

Theorem 1. *Fix integers d, g such that $g \geq 0$ and either $d \geq g+4$ or $d < g+4$ and $5d \geq 4g+4$. Let $C \subset \mathbb{P}^4$ be a general degree d non-degenerate embedding of a general smooth curve of genus g. Then C has maximal rank, i.e. for every integer t either $h^1(\mathbb{P}^4, \mathcal{I}_C(t)) = 0$ or $h^0(\mathbb{P}^4, \mathcal{I}_C(t)) = 0$.*

In the statement of Theorem 1 the case $d \geq g+4$ (the non-special embeddings) was proved in [4] and [7] and hence we do not consider it here. In the last part of the statement of Theorem 1 we only need to consider the integers $t \geq 2$. Let $C \subset \mathbb{P}^4$ be any embedding of a curve with general moduli. A consequence of Gieseker-Petri Theorem gives $h^1(C, \mathcal{O}_C(2)) = 0$ (see [1], Corollary 5.7). Hence Riemann-Roch shows that we need to prove $h^1(\mathbb{P}^4, \mathcal{I}_C(t)) = 0$ if

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$t \geq 2$ and $td+1-g \leq \binom{t+4}{4}$ and $h^0(\mathbb{P}^4, \mathcal{I}_C(t)) = 0$ if $t \geq 2$ and $td+1-g \geq \binom{t+4}{4}$.

Key lemmas are [8], Lemma 5.2, and [4], Lemma 1.

An essential tool is the following component of the Hilbert scheme of a projective space. Fix integers r, d, g such that $r \geq 3$, $g \geq 0$ and either $d \geq g+r$ or $d-r < g \leq d-r+\lfloor (d-r-2)/(r-2) \rfloor$. There is an irreducible component $W(d,g;r)$ of the Hilbert scheme of $\mathbb{P}^r$ which is generically smooth and of dimension $(r+1)d-(r-3)(g-1)$ such that a general $C \in W(d,g;r)$ has the following properties (see [5] for the case $r=3$, [8] for the case $r \geq 4$):

(a) C is a smooth and connected non-degenerate curve with degree d, genus g and $h^1(C, N_C) = 0$, where N_C denote the normal bundle of C in $\mathbb{P}^r$;

(b) if $d \geq g+r$, then $h^1(C, \mathcal{O}_C(1)) = 0$;

(c) if $d < g+r$, then C is linearly normal and $h^1(C, \mathcal{O}_C(2)) = 0$;

(d) if $\rho(g,r,d) \geq 0$, then C has general moduli;

(e) if $\rho(g,r,d) < 0$, then the general fiber of the natural rational map $\gamma_{d,g,r} : W(d,g;r) \dashrightarrow \mathcal{M}_g$ has dimension $\dim(\mathrm{Aut}(\mathbb{P}^r)) = r^2+2r$, i.e. $W(d,g;r)$ has the right number of moduli in the sense of [16].

If $U = \mathbb{P}^x$, $x \geq 3$, and U is a linear subspace of another projective, then we often write $W(d,g;U)$ instead of $W(d,g;x)$.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. Preliminaries

Fix integers r, d, g such that $r \geq 3$, $d \geq r$ and $0 \leq g \leq d-r+\lfloor (d-r-2)/(r-2) \rfloor$. If $(d,g,,r) \neq (r,0,r)$, then we say that the triple (d,g,r) has as *critical value* the first integer $k \geq 2$ such that $kd+1-g \leq \binom{r+k}{r}$. We say that the triple $(r,0,r)$ has critical value 1. Let k be the critical value of (d,g,r). It is easy to check that $td+1-g < \binom{r+t}{r}$ for every integer $t \geq k+1$. Hence $C \in W(d,g;r)$ has maximal rank if and only if $h^1(\mathbb{P}^r, \mathcal{I}_C(k)) = 0$ (i.e. the restriction map $\rho_{C,k,r} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(C, \mathcal{O}_C(k))$ is surjective) and $h^0(\mathbb{P}^r, \mathcal{I}_C(k-1)) = 0$ (i.e. the restriction map $\rho_{C,k-1,r} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k-1)) \to H^0(C, \mathcal{O}_C(k-1))$ is injective). If $kd+1-g = \binom{r+k}{r}$, then it is sufficient to check that $h^i(\mathbb{P}^r, \mathcal{I}_C(k)) = 0$ for one of the integers $i \in \{0,1\}$.

Let $H \subset \mathbb{P}^r$, $r \geq 4$, be a hyperplane. In Section 4 we collect the numerical lemmas which we will use.

We need the following well-known lemma (the so-called Horace Lemma) (see [12]).

Lemma 1. *Let $H \subset \mathbb{P}^r$ be a hyperplane. Fix any projective scheme $T \subset \mathbb{P}^r$. Let $\mathrm{Res}_H(T)$ be the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_T : \mathcal{I}_H$ as its ideal sheaf. Then*

$$h^i(\mathbb{P}^r, \mathcal{I}_T(t)) \leq h^i(\mathbb{P}^r, \mathcal{I}_{\mathrm{Res}_H(T)}(t-1)) + h^i(H, \mathcal{I}_{T\cap H,H}(t))$$

for all integers $i \geq 0$ and $t \geq 0$.

Proof. The definition of the residual scheme $\mathrm{Res}_H(T)$ gives the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(T)}(t-1) \to \mathcal{I}_T(t) \to \mathcal{I}_{T\cap H,H}(t) \to 0,$$

whose long cohomological exact sequence gives the lemma. □

Lemma 2. *Fix integers r, g such that $g-1 \geq r \geq 3$. Let $U_{r,g}$ be the set of all smooth, connected and non-degenerate curves $C \subset \mathbb{P}^r$ such that $p_a(C) = g$, C is linearly normal and $h^1(C, \mathcal{O}_C(1)) = 1$. Then $U_{r,g}$ is irreducible and non-empty.*

Proof. Obviously $U_{g-1,g}$ is the set of all canonical embeddings of all smooth non-hyperelliptic curves. Thus the irreducibility of $U_{g-1,g}$ follows from the irreducibility of $\mathcal{M}_g$ and the irreduciblity of the projective linear group $\mathrm{Aut}(\mathbb{P}^{g-1})$. Now assume $r < g-1$. Any element of $U_{r,g}$ may be obtained taking a linear projection of any $X \in U_{g-1,g}$ from $g-r-1$ sufficiently general points of X. Since $U_{g-1,g}$ is irreducible and the symmetric product of $g-r-1$ copies of any irreducible curve is non-empty, irreducible and $(g-r-1)$-dimensional, $U_{r,g}$ is irreducible. □

Remark 1. Let $D \subset \mathbb{P}^r$, $r \geq 2$, be a rational normal curve. Then N_D is a direct sum of $r-1$ line bundles of degree $r+2$ (see e.g. [15] or [14]).

Lemma 3. *Let $D \subset \mathbb{P}^r$ be a linearly normal elliptic curve.*

(a) N_D is semistable.

(b) For any $A \subset D$ such that $\sharp(A) \leq 2$ we have $h^1(D, N_D(-1)(-A)) = 0$.

(c) Assume $r \leq 4$. Then $h^1(D, N_D(-1)(-A)) = 0$ for every $A \subset D$ such that $\sharp(A) = 3$.

Proof. Part (a) is proved in [9]. Fix $A \subset D$ such that $\sharp(A) \leq 2$ and assume $h^1(D, N_D(-1)(-A)) > 0$. Since $\omega_D \cong \mathcal{O}_D$, Serre duality gives the existence of a non-zero morphism $\beta : N_D(-1)(-A) \to \mathcal{O}_D$. Since N_D is semistable, $N_D(-1)(-A)$ is a rank $r-1$ semistable vector bundle with degree $2(r+1) - (r-1)\cdot\sharp(A)$. Since $\deg(N_D(-1)(-A)) > 0$ and $\beta \neq 0$, the vector bundle

$N_D(-1)(-A)$ is not semistable, contradiction. If $r \le 4$, then the obtain a similar contradiction even if $\sharp(A) = 3$. $\square$

Remark 2. Fix integers $d \ge g+3 \ge 3$ and a general $S \subset \mathbb{P}^3$ such that $\sharp(S) \le 2d$ (we only need a weaker case, say $\sharp(S) \le d+3$). There is a smooth and connected curve $C \subset \mathbb{P}^3$ such that $\deg(C) = d$, $p_a(C) = g$, $h^1(C, \mathcal{O}_C(1)) = 0$, $S \subset C$ and $h^1(C, N_C(-S)) = 0$ (use [13], Theorem 1.5).

Lemma 4. *Let H be a hyperplane of $\mathbb{P}^4$. Fix integers g, d such that $d \ge g+4 \ge 4$. Write $g = 4m+e$ with m a non-negative integer and $0 \le e \le 3$. If $d \le 4m+7$, set $f := d-4m-4$. If $d \ge 4m+8$ set $f := d-4m-8$. Fix an integer s such that $0 \le s \le 4+3m+f$. Let $S \subset H$ be a general subset such that $\sharp(S) = s$. Then there exists a smooth $C \in W(d,g;4)$ such that $S \subset C \cap H$, C intersects transversally H, $h^1(C, \mathcal{O}_C(1)) = 0$ and $h^1(C, N_C(-S)) = 0$.*

Proof. It is sufficient to do the case $s = 4+3m+f$. It is sufficient to find a nodal and connected curve $X \subset \mathbb{P}^4$ such that X intersects transversally H, $S \subset X_{reg}$, $h^1(X, O_X(1)) = 0$ and $h^1(X, N_X(-S)) = 0$. First assume $d = 4$. Hence $g = 0$ and $s = c$. Take as C a general rational normal curve of H and use that any two serts of 4 points of H spanning H are projectively equivalent. Remark 1 shows that N_C is a direct sum of 3 line bundles of degree 6. Hence $h^1(C, N_C(-E)) = 0$ for any $E \subset C$ such that $\sharp(E) \le 7$. Apply [13], Theorem 1.5, to $E := C \cap H$. Now assume $e = 0$ and $d = g+4 > 0$, i.e. $d = 4m+4$, $g = 4m$ and $s = 4+3m$ for some integer $m > 0$. We use induction on m, the case $m = 0$ being just checked. Fix a general $S' \subset H$ such that $\sharp(S') = 3m$. Let A be a general element of $W(4m, 4m-4; 4)$ such that $S' \subset A \cap H$. By the inductive assumption there is such a curve A and it is smooth, connected, $h^1(A, \mathcal{O}_A(1)) = 0$ and $h^1(A, N_A(-S')) = 0$. Fix a general $E \subset A$ such that $\sharp(E) = 5$. Let B a general rational normal curve of H such that $E \subset B$. Set $X := A \cup B$. X is nodal and $X \in W(d,g;4)$ (apply several times [8], Lemma 2.2; for a far stronger statement, see [8], Lemma 2.3). To prove the result using X it is sufficient to prove $h^1(X, N_X(-S'-S'')) = 0$, where S'' is a general union of 3 points of B. The Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_{A\cup B}(t) \to \mathcal{O}_A(t) \oplus \mathcal{O}_B(t) \to \mathcal{O}_{A\cap B}(t) \to 0\,, \tag{1}$$

gives $h^1(A \cup B, \mathcal{O}_{A\cup B}(1)) = 0$, because $h^1(A, \mathcal{O}_A(1)) = h^1(B, \mathcal{O}_B(1)) = 0$ and the vanishing of $h^1(B, \mathcal{O}_B(1)(-(A\cap B)))$ gives the surjectivity of the restriction map $H^0(B, \mathcal{O}_B(1)) \to H^0(A \cap B, \mathcal{O}_{A\cap B}(1))$ (indeed, $\sharp(A \cap B) = 5$ and that B is a rational normal curve of $H \cong \mathbb{P}^4$). Consider the Mayer-Vietoris exact sequence

$$0 \to N_{A\cup B}(-S'-S'') \to N_{A\cup B}(-S'-S'')|A \oplus N_{A\cup B}(-S'-S'')|B \\ \to N_{A\cup B}(-S'-S'')|A\cap B \to 0\,. \qquad (2)$$

Since $N_{A\cup B}(-S'-S'')|A \cong N_{A\cup B}(-S')|A$ is obtained from $N_A(-S')$ making 5 positive elementary transformations and $h^1(A, N_A(-S')) = 0$, we have $h^1(A, N_{A\cup B}(-S'-S'')|A)$. Since $S'\cap B = \emptyset$, $N_X(-S'-S'')|B \cong N_X(-S'')|B$. The vector bundle $N_X(-S'')|B$ is obtained from the vector bundle $N_B(-S'')$ making 5 positive elementary transformations and $N_B(-S'')$ is isomorphic to the direct sum of 3 line bundles of degree 2, $h^1(B, N_{A\cup B}(-S'-S'')|B) = 0$. Hence (1) shows that to prove $h^1(X, N_X(-S'-S'')) = 0$ it is sufficient to prove the surjectivity of the restriction map $H^0(B, N_X(-S'-S'')|B) \to H^0(E, N_X(-S'-S'')|E)$. Hence it is sufficient to prove $h^1(B, N_X(-S'-S'')|B(-E)) = 0$. Hence it is sufficient to prove that every rank 1 direct summand of $N_X|B$ has degree at least 7. This is true, because we may do sufficiently general 3 of the 5 positive elementary transformations needed to obtain $N_X|B$ for B. Now assume that either $g/4 \notin \mathbb{Z}$ or $d > g+4$. Write $g = 4m+e$ and $d = (4m+4)e+a$. Fix a general $S' \subset H$ such that $\sharp(S') = 4+3m$. By the case $(d,g,s) = (4m{+}4, 4m, 4{+}3m)$ just done there is a smooth $A \in W(4m{+}4, 4m; 4)$ such that $h^1(A, \mathcal{O}_A(1)) = 0$ and $h^1(A, N_A(-S')) = 0$. Fix a general $F \subset A$ such that $\sharp(F) = e+1$. Let $B \subset \mathbb{P}^4$ be a general smooth rational curve of degree $d-4m-4$. Take a general $S'' \subset B$ such that $\sharp(S'') = s-3m-4$. We saw that it is sufficient to prove $h^1(B, N_{A\cup B}(-F-S'')) = 0$ and that this vanishing is true if $h^1(B, N_B(-F-S'')) = 0$. First assume $d-4m-4 \geq 4$. Since $\sharp(F) \leq 4$, B may be considered as a general degree $d-4m-4$ rational curve of $\mathbb{P}^4$. Let $a_1 \geq a_2 \geq a_3$ be the splitting type of N_B. We have $a_3 = \lfloor \deg(N_B)/3 \rfloor$ (see [15] or [14]). Since $\deg(N_B) = 5\cdot\deg(B) - 2 = 5d-20m-22$, and $s-(4+3m) = \lfloor (5d-20m-22)/3 \rfloor$, we are done. □

We lift the following joint lemma with C. Fontanari from a joint paper in preparation.

Lemma 5. *Fix integers r, m, e such that $r \geq 4$, $r \geq m \geq 2$, and $e \in \{0,1\}$. Let $H \subset \mathbb{P}^r$ be a hyperplane and $V \subseteq \mathbb{P}^r$ an m-dimensional linear subspace such that $V\cap H \neq V$. Let $Y \subset H$ be a nodal and connected curve such that $h^1(Y, N_{Y,H}) = 0$. Set $c := h^1(Y, \mathcal{O}_Y(1))$. Assume $m+e \geq c$. If $e = 1$ assume $m = r$. Fix $S \subset Y_{reg}$ such that $\sharp(S) = m+e+1$ and $h^1(Y, \mathcal{O}_Y(1)(S)) = 0$. Let $D \subset V$ be a smooth curve of genus e and degree $m+e$ spanning V such that D intersects transversally $V\cap H$ and $S = Y\cap D$. Then $h^1(Y\cup D, N_{Y\cup D}) = 0$. If $m+e \geq c+1$ and $h^1(Y, \mathcal{O}_Y(1)(S')) = 0$ for all $S' \subset S$ such that $\sharp(S') = m+e-1$, then $Y\cup D$ is smoothable.*

Lemma 6. *Fin integers t, a, b, c such that $t \geq 3$, $0 \leq a \leq t-1$, $b \geq 2t$, and $tc + 1 + a(t+1) + b \leq \binom{t+3}{3}$. Let $A \subset H$ be a general union of a degree c smooth rational curve and a disjoint lines. Then $h^1(H, \mathcal{I}_A(t)) = 0$.*

Proof. The quickest way is to follow the proof in [6] with (at some point) taking only surjectivity of a certain restriction map instead of bijectivity and inserting a of the lines without intersecting the rational curve. □

3. Proof of Theorem 1

For all integers $k \geq 2$ let g_k be the maximal integer such that $k(\lceil 4g_k/5 \rceil + 4) + 1 - g_k \leq \binom{k+4}{4}$. Set $d_k := \lceil 4g_k/5 \rceil + 4$ and $a_k := \binom{k+4}{k} - k \cdot d_k - 1 + g_k$. We have

$$k \cdot d_k + 1 - g_k + a_k = \binom{k+4}{4}, \ 0 \leq a_k \leq k-2\,. \tag{3}$$

Remark 3. We have $g_k \geq k-2$ for every integer $k \geq 2$ (Remark 5). Hence $g_k \geq a_k$ for every integer $k \geq 2$.

Remark 3 justifies the introduction of the following assertion $A(k)$, $k \geq 2$:

$A(k)$, $k \geq 2$. There is $X \in W(d_k, g_k - a_k; 4)$ such that $h^i(\mathbb{P}^4, \mathcal{I}_X(k)) = 0$, $i = 0, 1$.

As $A(1)$ we take the assertion that a rational normal curve of $\mathbb{P}^4$ is linearly normal. Hence $A(1)$ is true.

Lemma 7. *Fix an integer $k \geq 2$. Assume the existence of $X \in W(d_k, g_k; 4)$ such that $h^1(\mathbb{P}^4, \mathcal{I}_X(k)) = 0$ (or, equivalently, $h^0(\mathbb{P}^4, \mathcal{I}_X(k)) = a_k$) and $h^0(\mathbb{P}^4, \mathcal{I}_X(k-1)) = 0$ and that no irreducible component of X is contained in H. Fix a general $W \subset H$ such that $\sharp(W) = a_k$. Then $h^0(\mathbb{P}^4, \mathcal{I}_{X \cup W}(k)) = 0$.*

Proof. Since $h^0(\mathbb{P}^4, \mathcal{I}_X(k-1)) = 0$ and $\mathrm{Res}_H(X) = X$, we have $h^0(\mathbb{P}^4, \mathcal{I}_{X \cup H}(k)) = 0$ (Lemma 1). Since $h^0(\mathbb{P}^4, \mathcal{I}_X(k)) = a_k$, we may take a_k general points of H instead of H in the previous relation. □

Lemma 8. *Fix integers d, g such that $g \geq 0$, $5d \geq 4g + 20$ and (d, g) has critical value 2. Then there exists a smooth and non-degenerate curve $C \in \mathbb{P}^4$ with degree d, genus g and maximal rank.*

Proof. Since the case $d \geq g + 4$ is true (see [4]), we may assume $d < g + 4$. Since $h^1(C, \mathcal{O}_C(1)) > 0$ for any a smooth $C \in W(d, g; 4)$, we have $g \geq 5$, with equality if and only if $d = 10$ and C is a canonically embedded genus 5 curve. Since a canonically embedded smooth curve is projectively normal and

$2d+1-g \le \binom{6}{2} = 15$, it is sufficient to prove the lemma for the following pairs of integers (d,g): $(9,6)$, $(10,7)$, $(11,8)$, $(12,10)$. Since our curves are non-degenerate, it is sufficient to check that $h^1(\mathbb{P}^4, \mathcal{I}_C(2)) = 0$ for a general $C \in W(d,g;4)$. Fix a hyperplane $H \subset \mathbb{P}^4$. Let $Y \subset H$ be a general smooth curve of genus 5 and degree 7. We have $h^i(H, \mathcal{I}_{Y,H}(2)) = 0$, $i = 0,1$ (see [5]). Fix an integer s such that $2 \le s \le 5$ and a general $S \subset Y$ such that $\sharp(S) = s$. If $s \ge 4$, then set $V := \mathbb{P}^4$. If $s \le 3$ take as V an s-dimensional linear subspace of $\mathbb{P}^4$ such that $V \cap H = \langle S \rangle$. First assume $2 \le s \le 4$. Let D be a rational normal curve of V such that $S \subset V$. Thus $\deg(D) = s$, D intersects transversally H and $h^1(\mathbb{P}^4, \mathcal{I}_D(1)) = 0$. Since $\mathrm{Res}_H(Y \cup D) = D$ and $h^1(H, \mathcal{I}_{Y,H}(2)) = 0$, Lemma 1 gives $h^1(\mathbb{P}^4, \mathcal{I}_{Y\cup D}(2)) = 0$. We have $\deg(Y \cup D) = 7+s$ and $p_a(Y) = 5+s$. To conclude for the pair $(d,g) = (7+s, 5+s)$ it is sufficient to prove $Y \cup D \in W(6+s, 3+s; 4)$. Lemma 5 gives $h^1(Y \cup D, N_{Y\cup D}) = 0$ and that $Y \cup D$ is smoothable. Since $h^1(Y \cup D, \mathcal{O}_{Y\cup D}(1)) = 1$ (use a Mayer-Vietoris exact sequence) we have $h^1(C, \mathcal{O}_C(1)) \le 1$ for a general smoothing C of $Y \cup D$. Apply Lemma 2. Now we consider the case $(d,g) = (12,10)$. We take $s = 5$ and as curve D a linearly normal elliptic curve of $\mathbb{P}^4$ containing S. Hence D intersects transversally H and $S = D \cap Y$. Another joint lemma with C. Fontanari (omitted here) gives $Y \cup D \in W(12,10;4)$. Now we consider the case $(d,g) = (9,6)$. Here we make the previous construction with $s = 3$, except that here Y is a canonically embedded curve $Y \subset \mathbb{P}^3$ with degree 6 and genus 4. □

Lemma 9. *Fix integers d, g such that $g \ge 0$, $5d \ge 4g + 20$ and (d,g) has critical value 3. Then there exists a smooth and non-degenerate curve $C \in \mathbb{P}^4$ with degree d, genus g and maximal rank.*

Proof. Since the case $d \ge g+4$ is true (see [4]), we may assume $d < g+4$. Let C be a general element of $W(d,g;4)$. To prove that C has maximal rank it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_C(3)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_C(2)) = 0$. Since $W(d,g;4)$ is irreducible, the semicontinuity theorem for cohomology shows that it is sufficient to find $C, C' \in W(d,g;4)$ such that $h^1(\mathbb{P}^4, \mathcal{I}_C(3)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{C'}(2)) = 0$. Since the existence of C' is easy (take the union of some curve C'' given by Lemma 8 and another curve), we just consider the condition $h^1(\mathbb{P}^4, \mathcal{I}_C(3)) = 0$. Write $d = g + 4 - c$ with $c > 0$. The inequality $\rho(g,4,d) \ge 0$ is equivalent to the inequality $5c \le g$. Since $3d + 1 - g \le \binom{7}{4} = 35$ and $2d + 1 - g > \binom{6}{2} = 15$, the pair (d,g) is one of the following pairs: $(13,10)$, $(13,11)$, $(14,11)$, $(14,12)$, $(15,12)$, $(15,13)$, $(15,14)$, $(16,14)$, $(16,15)$.

Let $D(d',g') \subset \mathbb{P}^4$ be a general element of $W(d',g';4)$, where (d',g') is one of the pairs of integers listed in the proof of Lemma 8. Hence D intersects transversally H. Fix an integer a such that $0 \le a \le 4$. Let $Y_a \subset H$ be a general curve of degree $4+a$ and genus $1+a$. We have $h^1(H,\mathcal{I}_Y(3))=0$ (see [6]). We may take Y_a passing through $2\cdot\deg(Y_a)$ points of H. Fix an integer s such that $1\le s\le \min\{d',8+2a\}$ and a general $S\subset H$ such that $\sharp(S)=s$. We may take $D(d',g')$ and Y_a with the additional condition $S=D(d',g')\cap Y_a$ (Lemma 4 and Remark 10). We have $\deg(D(d',g')\cup Y_a)=d'+3+a$ and $p_a(D(d',g')\cup Y_a)=g'+a+s-1$. If $(d,g)=(13,10)$, then we take $(d',g')=(9,6)$, $a=1$ and $s=1$. If $(d,g)=(13,11)$, then we take $(d',g')=(9,6)$, $a=1$ and $s=2$. If $(d,g)=(14,11)$, then we take $(d',g')=(11,8)$, $a=0$ and $s=4$. If $(d,g)=(14,12)$, then we take $(d',g')=(11,8)$, $a=0$ and $s=5$. If $(d,g)=(15,12)$, then we take $(d',g')=(11,8)$, $a=1$ and $s=4$. If $(d,g)=(15,13)$, then we take $(d',g')=(11,8)$, $a=1$ and $s=5$. If $(d,g)=(15,14)$, then we take $(d',g')=(11,8)$, $a=1$ and $s=6$. If $(d,g)=(16,14)$, then we take $(d',g')=(12,10)$, $a=1$ and $s=4$. If $(d,g)=(16,15)$, then we take $(d',g')=(12,10)$, $a=1$ and $s=5$. To obtain $h^1(Y\cup D,\mathcal{I}_{Y\cup D}(3))=0$, it is sufficient to prove $h^1(H,\mathcal{I}_{Y_a\cup(D\cap H)}(3))=0$. We have $\sharp((Y\cap H)\backslash S)=d'-s$. Hence we certain need $d'-s\le h^0(H,\mathcal{I}_{Y_a}(3))$. This inequality is also sufficient (see [4], Lemma 1, and [8], Lemma 5.2). We have $h^0(H,\mathcal{I}_{Y_a}(3))=20-9-3a-1+a=10-2a$. Hence our construction works in all cases. □

Lemma 10. *For every integer $k\ge 1$ the assertion $A(k)$ is true .*

Proof. By Lemmas 2 and 1) we may assume $k\ge 4$ and that the lemma is true for the integer $k':=k-1$. Notice that $g_k-g_{k-1}\ge d_k-d_{k-1}-3$. Fix a general $S\subset H$ such that $\sharp(S)=g_k-g_{k-1}-d_k+d_{k-1}+4+a_{k-1}-a_k$. Lemma 4 gives the existence of a smooth $U\in W(d_{k-1},g_{k-1}-a_{k-1};4)$ intersecting transversally H and containing S. Since S is general, we may also assume that U is general in $W(d_{k-1},g_{k-1};4)$. By the inductive assumption we may assume $h^i(\mathbb{P}^4,\mathcal{I}_U(k-1))=0$, $i=0,1$. Since $k\ge 4$, we have $2(d_k-d_{k-1})\ge g_k-g_{k-1}+k$ (Lemma 13). There is a smooth $Y\in W(d_k-d_{k-1},d_k-d_{k-1}-3;H)$ such that $Y\cap(A\cap H)=S$. By [8], Lemma 2.3, $U\cup Y\in W(d_k,g_k-a_k,g;4)_{reg}$. Apply Lemma 1. □

To make easier steps (e) and (f) of the proof of Theorem 10 we give the following variation of the statement of Lemma 10.

Lemma 11. *For every integer $k\ge 2$ a general $X\in W(d_k,g_k;4)$ has maximal rank, i.e. $h^1(\mathbb{P}^4,\mathbb{P}^4,\mathcal{I}_X(k))=a_k$ and $h^0(\mathbb{P}^4,\mathbb{P}^4,\mathcal{I}_X(k-1))=0$.*

Proof. Since the pair (d_k,g_k) has critical value k, the two formulations of the lemma are equivalent. Since $W(d_k,g_k;4)$ is irreducible, it is sufficient to prove the existence of curves $X,X'\in W(d_k,g_k;4)$ (even diffrent) such that $h^1(\mathbb{P}^4,\mathbb{P}^4,\mathcal{I}_X(k))=a_k$ and $h^0(\mathbb{P}^4,\mathbb{P}^4,\mathcal{I}_{X'}(k-1))=0$. We first prove the h^1-

part of the lemma. Since the cases $k = 2, 3$ are true by Lemmas 2 and 1, we may assume $k \geq 4$. Fix a general $A \in W(d_{k-1}, g_{k-1} - a_{k-1}; 4)$. Since $A(k-1)$ is true (Lemma 10), $h^i(\mathbb{P}^4, \mathcal{I}_A(k-1)) = 0$, $i = 0, 1$. Since A is general, it intersects transversally H. Fix $S \subset A \cap H$ such that $\sharp(A) = 3 + a_{k-1}$. Take a general $Y \in W(d_k - d_{k-1}, d_k - d_{k-1} - 3; H)$ such that $S = Y \cap (A \cap H)$. As in the previous lemma use $A \cup Y$. It is obvious that the same construction works even for the h^0-part, because $h^0(\mathbb{P}^4, \mathbb{P}^4, \mathcal{I}_{A\cup Y}(k-1)) \leq h^0(\mathbb{P}^4, \mathbb{P}^4, \mathcal{I}_A(k-1)) = 0$. □

Proof of Theorem 1. Fix integers d, g such that $W(d, g; 4)$ is defined and $\rho(g, 4, d) \geq 0$, i.e. such that $W(d, g; 4)$ is the component of $\mathrm{Hilb}(\mathbb{P}^4)$ containing non-degenerate curves of degree d and genus g with general moduli. By the maximal rank conjecture for non-special curves (see [7]) we may assume $d < g + 4$. Let k be the critical value of the pair (d, g). By Lemmas 8 and 9 we may assume $k \geq 4$. Let m be the maximal non-negative integer such that $g_m \leq g$. Hence $g < g_{m+1}$. Since $\rho(d, 4, g) \geq 0$, we have $d_m \leq d$.

(a) Here and in steps (b), (c), and (d) we assume $m \leq k - 2$. For every integer t such that $m + 1 \leq t \leq k - 1$ we define the integers u_t and v_t by the relations

$$tu_t + 1 - g + v_t = \binom{t+4}{4}, \ 0 \leq v_t \leq t - 1\,. \tag{4}$$

For every integer t such that $m+1 \leq t \leq k-1$ we define the following assertion $B(t)$:

Assertion $B(t)$: Let $U \cup T \subset \mathbb{P}^4$ be a general union of a general $U \in W(u_t - v_t, g; 4)$ and a disjoint union T of v_t lines. Then $h^1(\mathbb{P}^4, \mathcal{I}_{U\cup T}(t)) = 0$, $i = 1, 2$.

For every integer t such that $k+1 \leq t \leq m-1$ we have $g \leq u_t = v_t + \lfloor (u_t - v_t - 6)/2 \rfloor$ (Lemma 15). Hence the component $W(u_t - v_t, g; 4)$ is defined. Since $\chi(\mathcal{O}_{U\cap T}(t)) = \binom{t+4}{4}$, $B(t)$ is well-defined.

(b) Here we prove $B(m+1)$. Take a general $A \in W(d_m, g_m - a_m; 4)$. Hence A intersects transversally H. Since $A(m)$ is true, $h^i(\mathbb{P}^4, \mathcal{I}_B(m)) = 0$, $i = 0, 1$. Lemma +15 gives $u_{m+1} - v_{m+1} - d_m \geq m$. Fix a general $S \subset H$ such that $\sharp(S) = v_m + 1 + g - g_m$, general $B \in W(u_{m+1} - v_{m+1} - d_m, 0; H)$ such that $S \subset B$ and a general disjoint union $T \subset H$ of v_{m+1} lines. Lemma 6 gives $h^1(H, \mathcal{I}_{B\cup T}(m+1)) = 0$. First applying several times [8], Lemma 5.2, and then applying [4], Lemma 1, we get $h^i(H, \mathcal{I}_{B\cup T\cup(A\cap H)}(m+1)) = 0$, $i = 0, 1$. Hence a smoothing of $A \cup B \cup T$ proves $B(m+1)$.

(c) Here we prove that $B(t)$ is true for every integer t such that $m+1 \leq t \leq k-1$. Since the case $t = m+1$ was proved in step (b), we may assume $k \geq m+3$ and that $B(t-1)$ is true. Fix $U \cup T$ satisfying $B(t-1)$. By the generality of

$U \cup T$ we may assume that it intersects transversally H. First assume $v_t \ge v_{t-1}$. Let U' be a general element of $W(u_t - u_{t-1} - v_t, 0; H)$ intersecting $U \cap H$ at one point and $T' \subset H$ a general udisjoint union of $v_t - v_{t-1}$ lines. Since $v_t - v_{t-1} \le v_t \le t-1$, and $u_{t-1} \ge 2t$ (Lemmas 17 and 18), we may apply Lemma 6 and get $h^1(H, \mathcal{I}_{U' \cup T'}(t)) = 0$. By construction $\sharp((U \cap T)|(H \backslash U' \cup T')) = u_{t-1} - 1$. Hence $\sharp((U \cap T)|(H \backslash U' \cup T')) + v_t - v_{t-1} + tv_t + 1 = \binom{t+3}{3}$. Using first [8], Lemma 5.2, and then [4], Lemma 1, we get $h^i(H, \mathcal{I}_{(U \cup T) \cup H \cup U' \cup T'}(t)) = 0$, $i = 0, 1$. Lemma 1 gives $B(t)$. Now assume $v_{t-1} > v_t$. We take a general $U' \in W(u_t - u_{t-1}, 0; H)$ intersecting U at one point and intersecting exactly $v_{t-1} - v_t$ lines of T'. These are not constrains to the generality of U, because $T \cap H$ may be a general set of v_{t-1} points of H. Hence [12] gives $h^i(H, \mathcal{I}_{(U \cup T) \cup H \cup U'}(t)) = 0$, $i = 0, 1$. Lemma 1 gives $B(t)$.

(d) Now we check the $h^1 = 0$ part of Theorem 1 for the pair (d, g) under the assumption $m \le k-2$. First assume $d \ge u_{k-1} + v_{k-1}$. Take a general $U \cup T$ satisfying $B(k-1)$. Take a general $U' \in W(d - u_{k-1}, 0; H)$ such that $\sharp(A \cap U) = 1$ and $T \cap H \subset U'$. As in the second part of (c) we see that $U \cup T \cup T' \in W(d, g; 4)$ and $h^1(\mathbb{P}^4, \mathcal{I}_{U \cup U' \cup T}(k)) = 0$. Now assume $d < u_{k-1} + v_{k-1}$. The pair (d, g) has critical value k and u_k is the maximal integer such that the pair (u_k, g) has critical value k, $u_k \ge d$. Since $u_k - u_{k-1} - v_k \ge k$ and $d < u_{k-1} + v_{k-1}$ Lemma 17 gives $kd + 1 - g \le \binom{k+4}{k} - 2k$. Hence in this case we take $U_1 \in W(u_t, g; 4)$ with $h^1(\mathcal{I}_{U_1}(k-1)) = 0$ (and hence $h^0(\mathbb{P}^4, \mathcal{I}_{U_1}(k-1)) = v_{k-1}$. Since (d, g) has critical value k, we have $d > u_{k-1}$. A solution is given by $U_1 \cup U'$ with U' general in H and intersecting U_1 at eactly one point.

(e) Here we assume $m = k-1$. First assume $d - d_{k-1} \ge g - (g_t - a_t)$. Take a general $A \in W(d_{k-1}, g_{k-1} - a_{k-1}; 4)$. Hence $h^i(\mathbb{P}^4, \mathcal{I}_A(k-1)) = 0$, $i = 0, 1$, and A intersects transversally H. Fix a general $B \in W(d - d_{k-1}, 0; H)$ intersecting A at $g - (g_t - a_t) + 1$ points and apply [4], Lemma 1, and [8], Lemma 5.2, we get $h^1(\mathbb{P}^4, \mathcal{I}_{A \cup B}(k) = 0$. Now assume $d - d_{k-1} < g - (g_t - a_t)$, but $2(d - d_{k-1}) + 1 \ge g - (g_t - a_t)$. In this case we make the same construction taking $B \in W(d - d_{k-1}, 0; H)$ with $q := g - (g_t - a_t) - (d - d_{k-1})$. If $2(d - d_{k-1}) + 1 < g - (g_t - a_t)$, then $\binom{k+4}{4} - kd - 1 + g$ is very large; we only need that it is at least a_{k-1}. Instead of using $A(k-1)$ we take $A \in W(d_{k-1}, g_{k-1}; 4)$ such that $h^1(\mathbb{P}^4, \mathcal{I}_A(k-1)) = 0$, i.e. $h^0(\mathbb{P}^4, \mathcal{I}_A(k-1)) = a_{k-1}$ (Lemma 11 for the integer $k-1$) and then add a suitable curve in H.

(f) Here we assume $m \ge k$. Since $g_m \le g < g_{m+1}$, $d_m = \lceil 4g_m/5 \rceil + 4$, $md_m + 1 - g_m \ge \binom{m+4}{m} - m + 1$, (d, g) has critical value k and $\lceil 4g/5 \rceil + 4 \ge g$, Lemma 12 gives $m = k$, $d = d_m$ and $g_m = g$. Apply Lemma 11.

(g) Here we show how to modify the construction to cover the h^0-part of

Theorem 1. First assume $k \geq m+3$. We start with Y satisfying $B(k-2)$ and then instead of adding something to get a reducible curve W with $h^i(\mathbb{P}^4, \mathcal{I}_W(k-1)) = 0$, $i = 0, 1$, whose general smoothing satisfies $B(k-1)$ we add something more in H to get a nodal element of $W(d, g; 4)$ not contained in any degree k hypersurface. If $k = m+2$ or $k = m+1$, then we start with a curve Y satisfying $B(k-2)$. □

4. Numerical Lemmas

In this section we give the numerical results which we used in this paper.

Remark 4. We have $(d_2, g_2, a_2) = (12, 10, 0)$, $(d_3, g_3, a_3) = (16, 15, 1)$, $(d_4, g_4, a_4) = (23, 24, 1)$, $(d_5, g_5, a_5) = (34, 37, 1)$, $(d_6, g_6, a_6) = (42, 47, 4)$, $(d_7, g_7, a_7) = (56, 65, 2)$, $(d_8, g_8, a_8) = (71, 83, 5)$,

Remark 5. Fix an integer $x \geq 2$. Since $d_x = \lceil 4g_x/5 \rceil + 4 \leq g_x + 4$ and $a_x \leq x - 2$, (3) gives

$$g_x \geq (\binom{x+4}{4} - 5x + 1)/(x-1)\,. \tag{5}$$

From (5) we get $g_x \geq 4$ for all $x \geq 4$ and $d_x \geq 4$ for all $x \geq 3$.

Remark 6. Since $g_x \leq 2d_x - 1$, (3) gives

$$d_x \leq \binom{x+4}{4}/(x-2) \tag{6}$$

for all integers $x \geq 3$.

Lemma 12. *We have $g_x - g_{x-1} \geq x$ and $d_x - d_{x-1} \geq x$ for all $x \geq 3$.*

Proof. Subtracting (3) for the integer $k := x - 1$ from (3) for the integer $k := x$ we get

$$(x-1)(d_x - d_{x-1}) - (g_x - g_{x-1}) + d_x = \binom{x+3}{3}\,. \tag{7}$$

Since $d_x = \lceil 4g_x/5 \rceil + 4$ and $d_{x-1} = \lceil 4g_{x-1}/5 \rceil + 4$, we have

$$4(g_x - g_{x-1})/5 - 2 \leq d_x - d_{x-1} \leq 4(g_x - g_{x-1})/5 + 2\,. \tag{8}$$

We get from (7) and (8) we get

$$(x - 1 - 4/5)(d_x - d_{x-1}) \geq \binom{x+3}{3} - \binom{x+4}{4}/(x-2) - 2\,. \tag{9}$$

Hence $d_x - d_{x-1} \geq x$ for all $x \geq 9$. Similarly (or by (7) we get $g_x \geq g_{x-1} \geq x$ for all $x \geq 9$. For low x use the explicit values of the integers g_x, g_{x-1}, d_x and

d_{x-1} given in Remark 4. □

Lemma 13. *We have $2(d_x - d_{x-1}) \geq g_x - g_{x-1} + x$ for all $x \geq 3$.*

Proof. If $x \geq 9$, then use (8). For $3 \leq x \leq 8$ ise the explict values of the integers g_x, g_{x-1}, d_x and d_{x-1} given in Remark 4. □

Lemma 14. *We have $d_{x-1} \geq 2(g_x - g_{x-1} + a_{x-1} - a_x + 1)$ for all $x \geq 3$.*

Proof. For $3 \leq x \leq 8$ use the explict values of the integers g_x, g_{x-1}, d_x and d_{x-1} given in Remark 4. □

From now on we take the set-up of step (a) of the proof of Theorem 1. We first assume $k \geq m+2$. By assumption the pair (d, g) has critical value $k \geq m+2$ and $d < g+4$. Taking the difference of equation (4) for the integer $t' := t$ with the same equation for the integer $t' := t-1$ we get

$$(t-1)(u_t - u_{t-1}) + u_t + v_t - v_{t-1} = \binom{t+3}{3}. \tag{10}$$

The pair (d_t, g_t) has critical value t and

$$\binom{t+4}{4} - (t-2) \leq td_t + 1 - g_t \leq \binom{t+4}{4}. \tag{11}$$

The pair (u_t, g) has critical value t. Since $0 \leq v_t \leq t-1$, (4) gives

$$\binom{t+4}{4} - (t-1) \leq tu_t + 1 - g \leq \binom{t+4}{4}. \tag{12}$$

Moreover, $g_t > g$, $d_t = \lceil 4g_t/5 \rceil + 4$ and $u_t \geq \lceil 4g/5 \rceil + 4$.

Lemma 15. *Assume $k \geq m+2$. Then $u_{m+1} - v_{m+1} \geq d_m + m$.*

Proof. Subtracting (3) for the integeb $k := m$ from (4) for the integer $t := m+1$ we get

$$m(u_{m+1} - d_m) + u_{m+1} + v_{m+1} - a_m + g_m - g = \binom{m+4}{3}. \tag{13}$$

Subtracting (3) for the integeb $k := m+1$ from (4) for the integer $t := m+1$ we get

$$(m+1)(d_{m+1} - u_{m+1}) = g_{m+1} + a_{m+1} - g - v_{m+1}\,. \tag{14}$$

Recall that $g_m \leq g < g_{m+1}$, $0 \leq a_{m+1} \leq m-1$ and $0 \leq v_{m+1} \leq m$. □

Lemma 16. *For every integer t such that $k+1 \leq t \leq m-1$ we have $d_t \geq u_t$ and $\lceil 4u_t/5 \rceil + 4 \geq g$.*

Proof. The proof is omitted. □

Lemma 17. *Assume $k \geq m+3$ and fix an integer t such that $k+2 \leq t \leq$*

$m-1$. *Then* $u_t - u_{t-1} \geq t+1$.

Proof. Subtracting (4) for the integer $t' := t-1$ from the same equation for the integer t we get

$$(t-1)(u_t - u_{t-1}) + u_t + v_t - v_{t-1} = \binom{t+3}{3}. \tag{15}$$

From (4) we also get $u_t \leq \binom{t+3}{3}/(t-2)$. Hence (15) gives the lemma. □

From Lemmas 15 and 17 we get by induction on t the following lemma.

Lemma 18. *For every integer t such that $k+1 \leq t \leq m-1$ we have $u_t \geq 2t$ and $u_t - v_t \geq 4g/5+4$. Hence the component $W(u_t - v_t, g; 4)$ is defined and contains curves with general moduli.*

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References

[1] E. Arbarello, M. Cornalba, On a conjecture of Petri, *Comment. Math. Helv.*, **56**, No. 1 (1981), 1-38.

[2] E. Ballico, Ph. Ellia, Generic curves of small genus in $\mathbb{P}^3$ are of maximal rank, *Math. Ann.*, **264**, No. 2 (1983), 211-225.

[3] E. Ballico, Ph. Ellia, On postulation of curves: Embeddings by complete linear series, *Arch. Math.*, Basel, **43** (1984), 244-249.

[4] E. Ballico, Ph. Ellia, On the postulation of curves in $\mathbb{P}^4$, *Math. Zeit.*, **188**, No. 2 (1985), 215-223.

[5] E. Ballico, Ph. Ellia, Beyond the maximal rank conjecture for curves in $\mathbb{P}^3$, In: *Space Curves, Proceedings Rocca di Papa*, Lecture Notes in Math., **1266**, Springer, Berlin (1985), 1-23.

[6] E. Ballico, Ph. Ellia, The maximal rank conjecture for non-special curves in $\mathbb{P}^3$, *Invent. Math.*, **79**, No. 3 (1985), 541-555.

[7] E. Ballico, Ph. Ellia, The maximal rank conjecture for non-special curves in $\mathbb{P}^n$, *Math. Zeit.*, **196**, No. 3 (1987), 585-599.

[8] E. Ballico, Ph. Ellia, On the existence of curves with maximal rank in $\mathbb{P}^n$, *J. Reine Angew. Math.*, **397** (1989), 1-22.

[9] L. Ein, R. Lazarsfeld, Stability and restrictions of Picard bundles, with an application to the normal bundles of elliptic curves, In: *Complex Projective Geometry*, Trieste-Bergen (1989), 149-156; *London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge, **179** (1992).

[10] R. Hartshorne, A. Hirschowitz, Droites en position générale dans l'espace projectif, In: *Algebraic Geometry. Proceedings*, La Rabida (1981), 169-188; *Lect. Notes in Math.*, **961**, Springer, Berlin (1982).

[11] R. Hartshorne, A. Hirschowitz, Smoothing algebraic space curves, *Algebraic Geometry*, Sitges (1983), 98-131; Lecture Notes in Math., **1124**, Springer, Berlin (1985).

[12] A. Hirschowitz, Sur la postulation générique des courbes rationnelles, *Acta Math.*, **146** (1981), 209-230.

[13] D. Perrin, Courbes passant par m points généraux de $\mathbb{P}^3$, *Bull. Soc. Math. France*, Mémoire 28/29 (1987).

[14] Z. Ran, Normal bundles of rational curves in projective spaces, *Asian J. Math.*, **11**, No. 4 (2007), 567-608.

[15] G. Sacchiero, Normal bundles of rational curves in projective space, *Ann. Univ. Ferrara Sez.*, **VII**, No. 26 (1980), 33-40.

[16] E. Sernesi, On the existence of certain families of curves, *Invent. Math.*, **75**, No. 1 (1984), 25-57.