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REDUCTION OF ORDER IN DIFFERENCE EQUATIONS BY SEMICONJUGATE FACTORIZATION

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Abstract: We introduce a method of factoring a difference equation on a group into a triangular system of two difference equations of lower orders. We discuss applications of this method to a variety of nonlinear difference equations of higher orders.

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1. Introduction

In this paper we find sufficient conditions for difference equations of the following type

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_{n-k}) \tag{1}$$

on a group $(G, *)$ to have reductions of order into systems of lower order equations

$$t_{n+1} = g_n(t_n, t_{n-1}, \ldots, t_{n-k+1}), \tag{2}$$

$$x_{n+1} = t_{n+1} * [h(x_n)]^{-1}, \tag{3}$$

where the functions $h : G \to G$ and $g_n : G^k \to G$ are determined by the given functions $f_n : G^{k+1} \to G$ in (1) and the group inversion is denoted by the power -1. Using groups as underlying sets has the theoretical benefit of flexibility in defining order reducing substitutions. The generality gained by going beyond

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real or complex numbers brings potential applicability to discrete models on networks and graphs; see e.g., [6]. Compatible topologies that make the group operation continuous may exist on G and occur naturally in most scientific applications; see, e.g., [3], [5], [11] and the references cited in them.

The system of equations (2) and (3) constitutes a semiconjugate or SC factorization of equation (1). Since the first equation is independent of the second, the above system is *triangular*. This uncoupling feature facilitates the study of the system; in particular, it is possible to determine the structure of periodic solutions of systems that have the triangular property; see [1].

Semiconjugate based reduction in order uses patterns or symmetries inherent in the *form* of the difference equation itself rather than the symmetries in its solutions. Further, groups appear as underlying spaces rather than through group actions. The idea of *form symmetries* has been applied in previous work to obtain reductions of order in many different types of difference equation; see, e.g., [8], [9] and [10]. Some classical methods for reducing the order of a difference equation (e.g., operator factorization of a linear equation, see [4]) can be derived through semiconjugacy; see [9].

2. Factorization Theorem

We begin by defining the semiconjugate relation.

Definition 1. Let S and M be arbitrary nonempty sets and let F, Φ be self maps of S and M, respectively. If there is a surjective (onto) mapping $H : S \to M$ such that $H \circ F = \Phi \circ H$ then we say that the mapping F is *semiconjugate* to Φ. We refer to Φ as a semiconjugate (SC) *factor* of F and call the function H a *link map*.

For a study of semiconjugacy in a topological context see [11]. If H is a bijection (one to one and onto) then F and Φ are *conjugates* and $\Phi = H \circ F \circ H^{-1}$. As we see below, semiconjugacy leads to reduction of order when H is *not* a bijection.

Now, for each f_n in equation (1) let the vector functions $F_n : G^{k+1} \to G^{k+1}$

$$F_n(u_0, \ldots, u_k) = [f_n(u_0, \ldots, u_k), u_0, \ldots, u_{k-1}], \quad u_j \in G \text{ for } j = 0, 1, \ldots, k, \tag{4}$$

denote the *unfoldings* (or associated vector maps) of equation (1). They determine the vector equation

$$(y_{0,n+1}, y_{1,n+1}, \ldots, y_{k,n+1}) = F_n(y_{0,n}, y_{1,n}, \ldots, y_{k,n})$$

in G^{k+1}, i.e., the *state space* of (1), where $(y_{0,n}, y_{1,n}, \ldots, y_{k,n})$ is considered to be the *state of the system* at time n.

Let each F_n in (4) be semiconjugate to a map $\Phi_n : G^k \to G^k$ of the lower dimensional space G^k. Let $H : G^{k+1} \to G^k$ be the link map such that for every n,

$$H \circ F_n = \Phi_n \circ H. \tag{5}$$

We require that factors Φ_n be unfoldings of scalar functions g_n in (2) i.e.,

$$\Phi_n(v_1, \ldots, v_k) = [g_n(v_1, \ldots, v_k), v_1, \ldots, v_{k-1}]. \tag{6}$$

This necessitates some restrictions on H. Since equation (3) can be stated as $t_n = x_n * h(x_{n-1})$ we may substitute this form in (2) to obtain $x_{n+1} * h(x_n) = g_n(x_n * h(x_{n-1}), \ldots, x_{n-k+1} * h(x_{n-k}))$. Thus if H is defined as

$$H(u_0, \ldots, u_k) = [u_0 * h(u_1) \ldots, u_{k-1} * h(u_k)], \tag{7}$$

where $h : G \to G$ then our aim is reached if the functions f_n in (1), g_n in (6) and h satisfy the functional equation

$$f_n(u_0, \ldots, u_k) * h(u_0) = g_n(u_0 * h(u_1) \ldots, u_{k-1} * h(u_k)). \tag{8}$$

For convenience we state the following special case of Theorem 1 in [9] as a lemma.

Lemma 2. *Let $k \geq 1$ and for a sequence of functions f_n in (1) suppose that there are functions $h : G \to G$ and $g_n : G^k \to G$ for $n \geq 1$ that satisfy (8). Then the form symmetry H in (7) is surjective and equation (1) is equivalent to the triangular system of equations (2) and (3); i.e., every solution of the system corresponds uniquely to a solution of (1).*

The next result gives a necessary and sufficient condition for the existence of functions g_n that satisfy (8) provided that the function h is invertible. This result is possibly the most general to imply the reduction of order in the form of equations (2) and (3).

Theorem 3. *Let $h : G \to G$ be a bijection. For $u_0, v_1, \ldots, v_k \in G$ let $\zeta_0 = u_0$ and define*

$$\zeta_j = h^{-1}(\zeta_{j-1}^{-1} * v_j), \quad j = 1, \ldots, k. \tag{9}$$

Then there are functions g_n such that equation (1) is equivalent to the system of equations (2) and (3) if and only if the quantity

$$f_n(u_0, \zeta_1, \ldots, \zeta_k) * h(u_0) \tag{10}$$

is independent of u_0 for every n.

Proof. First assume that the quantity in (10) is independent of u_0 for all $v_1, \ldots, v_k$ so that the functions

$$g_n(v_1, \ldots, v_k) = f_n(u_0, \zeta_1, \ldots, \zeta_k) * h(u_0) \tag{11}$$

are well defined on G^k for each n. Next, for all $u_0, u_1, \ldots, u_k$ define

$$v_j = u_{j-1} * h(u_j), \quad j = 1, \ldots, k. \tag{12}$$

Then by (9)

$$\zeta_1 = h^{-1}(u_0^{-1} * v_1) = h^{-1}(u_0^{-1} * u_0 * h(u_1)) = u_1.$$

In fact $\zeta_j = u_j$ for every j for if by way of induction $\zeta_l = u_l$ for $1 \leq l < j$ then

$$\zeta_j = h^{-1}(\zeta_{j-1}^{-1} * v_j) = h^{-1}(u_{j-1}^{-1} * u_{j-1} * h(u_j)) = u_j.$$

Now by (11)

$$g_n(u_0 * h(u_1), \ldots, u_{k-1} * h(u_k)) = f_n(u_0, \ldots, u_k) * h(u_0).$$

which is (8). Thus by Lemma 2 equation (1) is equivalent to the system of equations (2) and (3).

To prove the converse note that by (8) and the argument leading to it, for every $v_1, \ldots, v_k$ in G, with ζ_j as defined in (9),

$$\begin{aligned} f_n(u_0, \zeta_1, \ldots, \zeta_k) * u_0 &= g_n(u_0 * h(\zeta_1), \zeta_1 * h(\zeta_2), \ldots, \zeta_{k-1} * h(\zeta_k)) \\ &= g_n(v_1, \ldots, v_k) \end{aligned}$$

which is clearly independent of u_0. □

It is worth noting that (9) is a backwards version of the cofactor equation (3) that is obtained by solving it for x_n instead of x_{n+1}. To do this we required h to be invertible. Of course, in (9) it is necessary to iterate only k times.

3. Some Order-Reducing Form Symmetries

Consider two of the simplest possible form symmetries within the context of the previous section that are based on the scalar maps

$$h_1(u) = u \quad \text{and} \quad h_{-1}(u) = u^{-1}.$$

Each of $h_{\pm 1}$ is an invertible self-inverse map.

Definition 4. The form symmetry of type (7) that is generated by h_1 is the *identity form symmetry* and that generated by h_{-1} is the*inversion form symmetry.*

The next result is an immediate consequence of Theorem 3.

Corollary 5. *(a) For every* $u_0, v_1, \dots, v_k \in G$ *let* $\zeta_0 = u_0$ *and define* $\zeta_j = \zeta_{j-1}^{-1} * v_j$ *for* $j = 1, \dots, k$. *Then equation (1) has the identity form symmetry if and only if the quantity*

$$f_n(u_0, \zeta_1, \dots, \zeta_k) * u_0 \tag{13}$$

is independent of u_0 *for every* n.

(b) For every $u_0, v_1, \dots, v_k \in G$ *let* $\zeta_0 = u_0$ *and define* $\zeta_j = v_j^{-1} * \zeta_{j-1}$ *for* $j = 1, \dots, k$. *Then equation (1) has the inversion form symmetry if and only if the quantity*

$$f_n(u_0, \zeta_1, \dots, \zeta_k) * u_0^{-1} \tag{14}$$

is independent of u_0 *for every* n.

Corollary 5 extends previous work on homogeneous equations of degree one in [9] and [10] to larger classes of difference equations.

Example 6. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers neither of which is eventually zero and consider the third order difference equation

$$x_{n+1} = \frac{x_n x_{n-1}}{a_n x_n + b_n x_{n-2}}. \tag{15}$$

Here for all n, $f_n(u_0, u_1, u_2) = u_0 u_1/(a_n u_0 + b_n u_2)$. To check if f_n has the identity form symmetry, we set $\zeta_1 = v_1/u_0$ and $\zeta_2 = v_2 u_0/v_1$ as in Corollary 5(a) and calculate

$$f_n(u_0, \zeta_1, \zeta_2) u_0 = \frac{v_1^2}{a_n v_1 + b_n v_2}$$

which is independent of u_0. Thus equation (15) has the identity form symmetry on the multiplicative group of nonzero real numbers and its SC factorization according to Lemma 2 is

$$t_{n+1} = \frac{t_n^2}{a_n t_n + b_n t_{n-1}}, \quad x_{n+1} = \frac{t_{n+1}}{x_n}. \tag{16}$$

Next, consider the second order factor equation in the system (16) again on the group of nonzero real numbers under multiplication. This equation does not have the identity form symmetry because with $\zeta_1 = v_1/u_0$ and the functions $\phi_n(u_0, u_1) = u_0^2/(a_n u_0 + b_n u_1)$ we obtain

$$\phi_n(u_0, \zeta_1) u_0 = \frac{u_0^2}{a_n u_0 + b_n v_1/u_0} u_0$$

which is not independent of u_0. However, if $\zeta_1 = u_0/v_1$ then

$$\phi_n(u_0, \zeta_1)\frac{1}{u_0} = \frac{u_0^2}{a_n u_0 + b_n u_0/v_1}\frac{1}{u_0} = \frac{v_1}{a_n v_1 + b_n}$$

which is independent of u_0. Thus there is inversion form symmetry by Corollary 5(b) and the SC factorization is

$$\begin{aligned} r_{n+1} &= \frac{r_n}{a_n r_n + b_n}, \\ t_{n+1} &= r_{n+1} t_n . \end{aligned} \tag{17}$$

The substitution $s_n = 1/r_n$ in (17) produces a linear equation $s_{n+1} = a_n + b_n s_n$ and yields a full factorization of equation (15) into the following triangular system of first order equations

$$\begin{aligned} s_{n+1} &= a_n + b_n s_n , \\ t_{n+1} &= \frac{t_n}{s_{n+1}}, \\ x_{n+1} &= \frac{t_{n+1}}{x_n}. \end{aligned}$$

As an alternative approach to factoring equation (15) note that this equation also has the inversion form symmetry. If it were factored that way then its second order factor equation would have the identity form symmetry. Thus again a full factorization into a triangular system of first order equations would be obtained.

In practice, the one-variable function h is often defined using structures that are more complex than a group. In particular, if G is the additive group of a field $\mathcal{F}$ then the function

$$h(u) = -\alpha u,$$

where α is a fixed nonzero element of the field defines a form symmetry

$$H(u_0, u_1, \ldots, u_k) = [u_0 - \alpha u_1, u_1 - \alpha u_2, \ldots, u_{k-1} - \alpha u_k]. \tag{18}$$

For convenience, we represent the field operations here by the ordinary addition and multiplication symbols.

Definition 7. The function H in (18) is the *linear form symmetry.* Relative to the additive group of a field, the linear form symmetry generalizes both the identity form symmetry ($\alpha = -1$) and the inversion form symmetry ($\alpha = 1$).

Of course, the linear form symmetry does not generalize the identity or inversion ones outside the context of fields. Difference equations possessing the linear form symmetry have been studied in [2] and [7], where the SC factorization above has been used to reduce equations of order 2 to pairs of equations of order 1.

The following corollary of Theorem 3 gives a condition for verifying whether equation (1) has the linear form symmetry. For easier reading we denote the multiplicative field inversion by the reciprocal notation.

Corollary 8. *For arbitrary $u_0, v_1, \ldots, v_k$ in a field $\mathcal{F}$ define $\zeta_0 = u_0$ and*

$$\zeta_j = \frac{u_0}{\alpha^j} - \sum_{i=1}^{j} \frac{v_i}{\alpha^{j-i+1}}, \quad j = 1, \ldots, k. \tag{19}$$

Then equation (1) has the linear form symmetry (18) if and only if the quantity

$$f_n(u_0, \zeta_1, \ldots, \zeta_k) - \alpha u_0$$

is independent of u_0.

Note that (19) defines ζ_j in Corollary 8 explicitly rather than recursively. It is obtained from the recursive definition (9) by a simple calculation; since

$$\zeta_j = h^{-1}(\zeta_{j-1}^{-1} * v_j) = -\frac{1}{\alpha}(-\zeta_{j-1} + v_j) = \frac{\zeta_{j-1} - v_j}{\alpha}$$

equality (19) can be established by direct iteration. The next example illustrates Corollary 8.

Example 9. Let $\mathcal{F}$ be a field and $\phi_n : \mathcal{F} \to \mathcal{F}$ be a given sequence of functions. Consider the difference equation

$$x_{n+1} = a x_{n-j} + \phi_n(x_n + c x_{n-k}), \tag{20}$$

where $c \neq 0$, $0 \leq j < k$ and we define $f_n(u_0, u_1, \ldots, u_k) = a u_j + \phi_n(u_0 + c u_k)$. Using (19) we obtain

$$f_n\left(u_0, \frac{u_0 - v_1}{\alpha}, \ldots, \frac{u_0}{\alpha^k} - \sum_{i=1}^{k} \frac{v_i}{\alpha^{k-i+1}}\right) - \alpha u_0$$
$$= \left(-\alpha + \frac{a}{\alpha^j}\right) u_0 - \sum_{i=1}^{j} \frac{a v_i}{\alpha^{j-i+1}} + \phi_n\left(u_0\left[1 + \frac{c}{\alpha^k}\right] - \sum_{i=1}^{k} \frac{c v_i}{\alpha^{k-i+1}}\right).$$

If there is $\alpha \in \mathcal{F}$ such that

$$-\alpha + \frac{a}{\alpha^j} = 0 \text{ or } a = \alpha^{j+1} \quad \text{and} \quad c = -\alpha^k, \tag{21}$$

then by Corollary 8 equation (20) has the linear form symmetry with a SC factorization

$$t_{n+1} = -\sum_{i=1}^{j} \alpha^i t_{n-i+1} + \phi_n\left(\sum_{i=1}^{k} \alpha^{i-1} t_{n-i+1}\right), \tag{22}$$

$$x_{n+1} = t_{n+1} + \alpha x_n = \alpha^n x_0 + \sum_{i=1}^{n} \alpha^{n-i} t_i.$$

If $j = 0$ then the first sum is deleted in (22).

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