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NON-NEGATIVE STEADY STATE SOLUTIONS TO AN ELLIPTIC BIOLOGICAL MODEL

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Abstract: The non-existence and existence of positive solutions for the generalized predator-prey biological model for two species of animals

$$\begin{aligned} &\Delta u + ug(u,v) = 0 \ \text{ in } \ \Omega\,,\\ &\Delta v + vh(u,v) = 0 \ \text{ in } \ \Omega\,,\\ &u = v = 0 \ \text{ on } \ \partial\Omega\,, \end{aligned}$$

is investigated in this paper. The techniques used in this paper are from elliptic theory, the upper-lower solution method, the maximum principles and spectrum estimates. The arguments also rely on detailed properties of solutions to logistic equations.

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1. Introduction

A lot of research has been focused on reaction-diffusion equations modeling the elliptic steady state solutions of predator-prey interacting processes with

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Dirichlet boundary conditions. Our knowledge about the existence of positive solutions is limited to rather specific systems, whose relative growth rates are linear;

$$\begin{aligned}&\Delta u + u(a - bu - cv) = 0 \text{ in } \Omega\,,\\&\Delta v + v(d + eu - fv) = 0 \text{ in } \Omega\,,\\&u = v = 0 \text{ on } \partial\Omega,\end{aligned}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, and where $a, d > 0$ are reproduction rates, $b, f > 0$ are self-limitation rates and $c, e > 0$ are competition rates.

The question in this paper concerns the existence of positive coexistence when all the reproduction, self-limitation and competition rates are nonlinear and combined, more precisely, the existence of the positive steady state of

$$\begin{aligned}&\Delta u + ug(u, v) = 0 \text{ in } \Omega\,,\\&\Delta v + vh(u, v) = 0 \text{ in } \Omega\,,\\&u = v = 0 \text{ on } \partial\Omega,\end{aligned}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, and where $g, h \in C^1$ are such that $g_u < 0, g_v < 0, h_v < 0, h_u > 0$, $g(0, 0) > 0, h(0, 0) > 0$, and there exists $c_0 > 0$ such that $g(u, 0) \le 0$ and $h(0, v) \le 0$ for $u, v \ge c_0$. Also, note that we assume h_u, h_v and g_v are functions that are bounded above.

In Section 3, we provide the coexistence region of the reproduction rates $(g(0, 0), h(0, 0))$ by virtue of Maximum Principles, upper-lower solutions method and the properties of the logistic equation.

2. Preliminaries

In this section, we state some preliminary results which will be useful for our later arguments.

Definition 2.1. (Upper and Lower Solutions)

$$\begin{cases}\Delta u + f(x, u) = 0 \text{ in } \Omega,\\ u|_{\partial\Omega} = 0,\end{cases} \tag{1}$$

where $f \in C^\alpha(\bar\Omega \times R)$ and Ω is a bounded domain in R^n.

(A) A function $\bar u \in C^{2,\alpha}(\bar\Omega)$ satisfying

$$\begin{cases}\Delta \bar u + f(x, \bar u) \le 0 \text{ in } \Omega,\\ \bar u|_{\partial\Omega} \ge 0,\end{cases}$$

is called an upper solution to (3).

(B) A function $\underline{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \underline{u} + f(x, \underline{u}) \geq 0 \text{ in } \Omega, \\ \underline{u}|_{\partial\Omega} \leq 0, \end{cases}$$

is called a lower solution to (3).

Lemma 2.1. *Let $f(x, \xi) \in C^{\alpha}(\bar{\Omega} \times R)$ and let $\bar{u}, \underline{u} \in C^{2,\alpha}(\bar{\Omega})$ be respectively, upper and lower solutions to (3) which satisfy $\underline{u}(x) \leq \bar{u}(x), x \in \bar{\Omega}$. Then (3) has a solution $u \in C^{2,\alpha}(\bar{\Omega})$ with $\underline{u}(x) \leq u(x) \leq \bar{u}(x), x \in \bar{\Omega}$.*

Lemma 2.2. (The First Eigenvalue)

$$\begin{cases} -\Delta u + q(x)u = \lambda u \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain in R^n.

(A) *The first eigenvalue $\lambda_1(q)$, denoted by simply λ_1 when $q \equiv 0$, is simple with a positive eigenfunction ϕ_1.*

(B) *If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.*

Lemma 2.3. (Maximum Principles)

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + \sum_{i=1}^{n} a_i(x) D_i u + a(x) u = f(x) \text{ in } \Omega,$$

where Ω is a bounded domain in R^n.

(M1) $\partial\Omega \in C^{2,\alpha}(0 < \alpha < 1)$.

(M2) $|a_{ij}(x)|_{\alpha}, |a_i(x)|_{\alpha}, |a(x)|_{\alpha} \leq M(i, j = 1, ..., n)$.

(M3) L is uniformly elliptic in $\bar{\Omega}$, with ellipticity constant γ, i.e., for every $x \in \bar{\Omega}$ and every real vector $\xi = (\xi_1, ..., \xi_n)$

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \gamma \sum_{i=1}^{n} |\xi_i|^2.$$

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of $Lu \geq 0 (Lu \leq 0)$ in Ω.

(A) *If $a(x) \equiv 0$, then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u (\min_{\bar{\Omega}} u = \min_{\partial\Omega} u)$.*

(B) *If $a(x) \leq 0$, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+ (\min_{\bar{\Omega}} u \geq -\max_{\partial\Omega} u^-)$, where $u^+ = \max(u, 0), u^- = -\min(u, 0)$.*

(C) *If $a(x) \equiv 0$ and u attains its maximum (minimum) at an interior point of Ω, then u is identically a constant in Ω.*

(D) *If $a(x) \leq 0$ and u attains a nonnegative maximum (nonpositive minimum) at an interior point of Ω, then u is identically a constant in Ω.*

Lemma 2.4. *Let $g_i(u_1, u_2) \in C^1([0,\infty) \times [0,\infty))$ and suppose that there exists a positive constant M such that for every $t \in [0,1]$, if $u = (u_1, u_2)$ is a non-negative solution of the problem*

$$\begin{cases} -\Delta u_1 = tg_1(u_1, u_2) \text{ in } \Omega, \\ -\Delta u_2 = tg_2(u_1, u_2) \text{ in } \Omega, \\ u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

then

$$u_1 \le M, u_2 \le M.$$

Assume that:

(1) Either $g_1(0,0) > \lambda_1, g_2(0,0) \neq \lambda_1$ or $g_1(0,0) \neq \lambda_1, g_2(0,0) > \lambda_1$,

(2)

$$(g_1)_{u_1}(u_1, 0) \le 0 (u_1 \ge 0), (g_1)_{u_1}(u_1, 0) \text{ is not identically zero } (u_1 \in [0, b)),$$
$$(g_2)_{u_2}(0, u_2) \le 0 (u_2 \ge 0), (g_2)_{u_2}(0, u_2) \text{ is not identically zero } (u_2 \in [0, b)),$$

where b is any fixed positive number,

(3) $(u_1^*, 0), (0, u_2^*)$ are any nontrivial non-negative solution with $\lambda_1(-g_2(u_1^*, 0)) < 0$, $\lambda_1(-g_1(0, u_2^*)) < 0$.

Then there is a solution $u_1 > 0, u_2 > 0$ of (3) for $t = 1$.

We also need some information on the solutions of the following logistic equations.

Lemma 2.5.

$$\begin{cases} \Delta u + uf(u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \le 0$ for $u \ge c_0$ and Ω is a bounded domain in R^n.

If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary condition. We denote this unique positive solution as θ_f.

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $a > \lambda_1, b > 0$, we denote $\theta_{\frac{a}{b}}$ as the unique positive solution of

$$\begin{cases} \Delta u + u(a - bu) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0. \end{cases}$$

Hence, $\theta_{\frac{a}{b}}$ is increasing as $a > 0$ is increasing.

3. Existence Region for Steady State

We consider

$$\begin{aligned}&\Delta u+ug(u,v)=0 \text{ in } \Omega\,,\\&\Delta v+vh(u,v)=0 \text{ in } \Omega\,,\\&u=v=0 \text{ on } \partial\Omega,\end{aligned}\tag{4}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $g,h\in C^1$ are such that $g_u<0, g_v<0, h_v<0, h_u>0$, $g(0,0)>0, h(0,0)>0$, and there exists $c_0>0$ such that $g(u,0)\le 0$ and $h(0,v)\le 0$ for $u,v\ge c_0$. Note that we assume h_u, h_v and g_v are functions that are bounded above.

First, we see that the two species can not coexist when the reproduction capacities are not robust enough.

Theorem 3.1. *Suppose* $g(0,0)\le\lambda_1, h(0,0)\le\lambda_1$. *Then* $u=v\equiv 0$ *is the only nonnegative solution to (4).*

Proof. Let (u,v) be a nonnegative solution to (4). By the Mean Value Theorem, there are $\tilde u,\tilde v$ such that

$$g(u,v)-g(u,0)=g_v(u,\tilde v)v\,,\quad h(u,v)-h(0,v)=h_u(\tilde u,v)u.$$

Hence, (4) implies that

$$\begin{aligned}\Delta u+u(g(u,0)+g_v(u,\tilde v)v)&=\Delta u+u(g(u,0)+g(u,v)-g(u,0))\\&=\Delta u+ug(u,v)=0 \text{ in } \Omega,\end{aligned}$$

$$\begin{aligned}&\Delta v+v(h(0,v)+h_u(\tilde u,v)u)\\&\qquad=\Delta v+v(h(0,v)+h(u,v)-h(0,v))=\Delta v+vh(u,v)=0 \text{ in } \Omega.\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta u+u(g(u,0)+\sup(g_v)v)\ge 0 \text{ in } \Omega,\\&\Delta v+v(h(0,v)+\sup(h_u)u)\ge 0 \text{ in } \Omega.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup(h_u)\phi_1\Delta u+\sup(h_u)\phi_1 u(g(u,0)+\sup(g_v)v)\ge 0 \text{ in } \Omega,\\&-\sup(g_v)\phi_1\Delta v-\sup(g_v)\phi_1 v(h(0,v)+\sup(h_u)u)\ge 0 \text{ in } \Omega.\end{aligned}$$

So,

$$\begin{aligned}\int_\Omega -\sup(h_u)\phi_1\Delta u dx&\le\int_\Omega[g(u,0)\sup(h_u)u+\sup(g_v)\sup(h_u)uv]\phi_1 dx,\\ \int_\Omega \sup(g_v)\phi_1\Delta v dx&\le\int_\Omega[-h(0,v)\sup(g_v)v-\sup(g_v)\sup(h_u)uv]\phi_1 dx.\end{aligned}$$

Hence, by Green's Identity, we have

$$\int_\Omega \sup(h_u)\lambda_1\phi_1 u dx \le \int_\Omega [g(u,0)\sup(h_u)u + \sup(g_v)\sup(h_u)uv]\phi_1 dx,$$

$$\int_\Omega -\sup(g_v)\lambda_1\phi_1 v dx \le \int_\Omega [-h(0,v)\sup(g_v)v - \sup(g_v)\sup(h_u)uv]\phi_1 dx.$$

Therefore,

$$\int_\Omega \sup(h_u)(\lambda_1 - g(u,0))u\phi_1 - \sup(g_v)(\lambda_1 - h(0,v))v\phi_1 dx \le 0.$$

Since the left hand side is nonnegative from

$$g(u,0) \le g(0,0) \le \lambda_1\,, \quad h(0,v) \le h(0,0) \le \lambda_1,$$

we conclude that $u = v \equiv 0$. □

Theorem 3.2. *Let $u \ge 0, v \ge 0$ be a solution to (4). If $g(0,0) \le \lambda_1$, then $u \equiv 0$.*

Proof. Proceeding as in the proof of Theorem 3.1, we obtain

$$0 \le \int_\Omega (\lambda_1 - g(u,0))u\phi_1 dx \le \int_\Omega \sup(g_v)vu\phi_1 dx \le 0,$$

and so, $u \equiv 0$.

Theorem 3.2 implies if $g(0,0) \le \lambda_1$ and $h(0,0) > \lambda_1$, then all possible nonnegative solutions to (4) are $(0,0)$ and $(0, \theta_{h(0,\cdot)})$. □

In order to prove further results, we will need the following lemma.

Lemma 3.3. *Let $\tilde{u} \ge 0, \tilde{v} \ge 0$ be a solution of the problem*

$$\begin{cases} -\Delta u = tug(u,v) \ \text{ in } \ \Omega\,, \\ -\Delta v = tvh(u,v) \ \text{ in } \ \Omega\,, \\ u_{\partial\Omega} = v_{\partial\Omega} = 0, \end{cases} \tag{5}$$

where $t \in [0,1]$. Then:

(1) $\tilde{u} \le M_1$, $\tilde{v} \le M_2$, where $M_1 = -\frac{g(0,0)}{\sup(g_u)}$, $M_2 = -\frac{h(M_1,0)+h(0,0)}{\sup(h_v)}$.

(2) For $t = 1$,

$$\tilde{u} \le \theta_{g(\cdot,0)}, \tilde{v} \ge \theta_{h(0,\cdot)}$$

if $\tilde{v} > 0$ in Ω.

Proof. (1) By the Mean Value Theorem and the monotonicity of g, we have

$$g(\tilde{u},\tilde{v}) - g(0,0) \le g(\tilde{u},0) - g(0,0) \le \sup(g_u)\tilde{u},$$

and so,

$$\frac{g(\tilde{u},\tilde{v})-g(0,0)}{\sup(g_u)} \geq \tilde{u}.$$

Hence,

$$\begin{aligned}
&\Delta(-\frac{g(0,0)}{\sup(g_u)}-\tilde{u})+t(-\frac{g(0,0)}{\sup(g_u)}-\tilde{u})(g(\tilde{u},\tilde{v})-g(0,0))\\
&\quad= -\Delta\tilde{u}-t\tilde{u}(g(\tilde{u},\tilde{v})-g(0,0))-t\frac{g(0,0)}{\sup(g_u)}(g(\tilde{u},\tilde{v})-g(0,0))\\
&\quad= t\tilde{u}g(0,0)-t\frac{g(\tilde{u},\tilde{v})-g(0,0)}{\sup(g_u)}g(0,0) \leq tg(0,0)\tilde{u}-tg(0,0)\tilde{u}=0.
\end{aligned}$$

Since $g(\tilde{u},\tilde{v})-g(0,0)\leq 0$, by the Maximum Principles, we conclude

$$\tilde{u} \leq M_1 = -\frac{g(0,0)}{\sup(g_u)}.$$

By the Mean Value Theorem, we have

$$h(\tilde{u},\tilde{v})-h(\tilde{u},0) \leq \sup(h_v)\tilde{v},$$

and so,

$$\frac{h(\tilde{u},\tilde{v})-h(\tilde{u},0)}{\sup(h_v)} \geq \tilde{v}.$$

Hence,

$$\begin{aligned}
&\Delta(-\frac{h(M_1,0)+h(0,0)}{\sup(h_v)}-\tilde{v})+t(-\frac{h(M_1,0)+h(0,0)}{\sup(h_v)}-\tilde{v})(h(\tilde{u},\tilde{v})-h(\tilde{u},0))\\
&\quad= -\Delta\tilde{v}-t\tilde{v}(h(\tilde{u},\tilde{v})-h(\tilde{u},0))\\
&\quad -t\frac{h(M_1,0)(h(\tilde{u},\tilde{v})-h(\tilde{u},0))}{\sup(h_v)}-t\frac{h(0,0)(h(\tilde{u},\tilde{v})-h(\tilde{u},0))}{\sup(h_v)}\\
&= t\tilde{v}h(\tilde{u},0)-t\frac{h(M_1,0)(h(\tilde{u},\tilde{v})-h(\tilde{u},0))}{\sup(h_v)}-t\frac{h(0,0)(h(\tilde{u},\tilde{v})-h(\tilde{u},0))}{\sup(h_v)} \leq 0.
\end{aligned}$$

Since $h(\tilde{u},\tilde{v})-h(\tilde{u},0)\leq 0$, by the Maximum Principles, we conclude

$$\tilde{v} \leq M_2 = -\frac{h(M_1,0)+h(0,0)}{\sup(h_v)}.$$

(2) If $g(0,0)\leq\lambda_1$, then by Theorem 3.2, $\tilde{u}\equiv 0$ and so obviously $\tilde{u}\leq\theta_{g(\cdot,0)}$.

Suppose $g(0,0)>\lambda_1$. Since

$$\begin{cases} \Delta\tilde{u}+\tilde{u}g(\tilde{u},0) = -\tilde{u}(g(\tilde{u},\tilde{v})-g(\tilde{u},0)) \geq 0 \ \text{ in } \ \Omega\,,\\ \tilde{u}|_{\partial\Omega}=0, \end{cases}$$

$\tilde{u}$ is a lower solution to

$$\begin{cases} \Delta u + ug(u,0) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

We can take large enough M such that $M > \tilde{u}$ on $\bar{\Omega}$ and $u = M$ is an upper solution to

$$\begin{cases} \Delta u + ug(u,0) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Since

$$\begin{cases} \Delta u + ug(u,0) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0\,, \end{cases}$$

has a unique positive solution $\theta_{g(\cdot,0)}$, by the upper-lower solution method, we conclude $\tilde{u} \leq \theta_{g(\cdot,0)}$ in Ω.

If $h(0,0) \leq \lambda_1$, then since $\theta_{h(0,\cdot)} = 0$, obviously $\tilde{v} \geq \theta_{h(0,\cdot)}$.

Suppose $h(0,0) > \lambda_1$. Since

$$\begin{cases} \Delta\tilde{v} + \tilde{v}h(0,\tilde{v}) = -\tilde{v}(h(\tilde{u},\tilde{v}) - h(0,\tilde{v})) \leq 0 \text{ in } \Omega, \\ \tilde{v}|_{\partial\Omega} = 0, \end{cases}$$

$\tilde{v} > 0$ is an upper solution to

$$\begin{cases} \Delta v + vh(0,v) = 0 \text{ in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Since $\tilde{v} > 0$, for large enough $n \in N$, $\frac{\theta_{h(0,\cdot)}}{n} < \tilde{v}$ in Ω. Since

$$\begin{aligned} \Delta(\frac{\theta_{h(0,\cdot)}}{n}) + \frac{\theta_{h(0,\cdot)}}{n}h(0,\frac{\theta_{h(0,\cdot)}}{n}) &= \frac{1}{n}[\Delta\theta_{h(0,\cdot)} + \theta_{h(0,\cdot)}h(0,\frac{\theta_{h(0,\cdot)}}{n})] \\ &\geq \frac{1}{n}[\Delta\theta_{h(0,\cdot)} + \theta_{h(0,\cdot)}h(0,\theta_{h(0,\cdot)})] = 0, \end{aligned}$$

$\frac{\theta_{h(0,\cdot)}}{n}$ is a lower solution to

$$\begin{cases} \Delta v + vh(0,v) = 0 \text{ in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Therefore, by the uniqueness of the solution and the upper-lower solution method, we conclude $\theta_{h(0,\cdot)} \leq \tilde{v}$.

Theorem 3.4. *There exist two functions* $M(g), N(h) : [\lambda_1, \infty) \to R$ *such that:*

(A) if $g(0,0) \geq \lambda_1, h(0,0) \leq M(g)$*, then all possible nonnegative solutions to (4) are* $(0,0)$ *and* $(\theta_{g(\cdot,0)}, 0)$*,*

(B) if $\lambda_1 < g(0,0) < N(h), h(0,0) > \lambda_1$*, then all possible nonnegative solutions to (4) are* $(0,0), (\theta_{g(\cdot,0)}, 0)$ *and* $(0, \theta_{h(0,\cdot)})$*,*

(C) if $g(0,0) > \lambda_1, M(g) < h(0,0) < \lambda_1$*, then all possible nonnegative solutions to (4) are* $(0,0), (\theta_{g(\cdot,0)}, 0)$ *and a positive solution* $u^+ > 0, v^+ > 0$*,*

(D) if $h(0,0) > \lambda_1, g(0,0) > N(h)$*, then all possible nonnegative solutions to (4) are* $(0,0), (\theta_{g(\cdot,0)}, 0), (0, \theta_{h(0,\cdot)})$ *and a positive solution* $u^+ > 0, v^+ > 0$*.*

Proof. For $g(0,0) \geq \lambda_1$, let $M(g) = \lambda_1(-h(\theta_{g(\cdot,0)}, 0) + h(0,0))$ and $N(h) = \lambda_1(-g(0, \theta_{h(0,\cdot)}) + g(0,0))$.

(A) Suppose $h(0,0) \leq M(g)$. Let $\tilde{u} \geq 0, \tilde{v} \geq 0$ be a solution to (4). If $\tilde{v} > 0$ in Ω, then $\lambda = h(0,0)$ is the smallest eigenvalue of the problem

$$\begin{cases} -\Delta v + v(-h(\tilde{u}, \tilde{v}) + h(0,0)) = \lambda v \text{ in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

By the monotonicity of h and Lemma 3.3, we have

$$-h(\tilde{u}, \tilde{v}) > -h(\theta_{g(\cdot,0)}, 0),$$

and so

$$h(0,0) = \lambda_1(-h(\tilde{u}, \tilde{v}) + h(0,0)) > \lambda_1(-h(\theta_{g(\cdot,0)}, 0)) + h(0,0) = M(g),$$

which is a contradiction to $h(0,0) \leq M(g)$. Hence, $\tilde{v} \equiv 0$. Therefore, we conclude that if $g(0,0) \geq \lambda_1$ and $h(0,0) \leq M(g)$, then all possible nonnegative solutions to (4) are $(0,0)$ and $(\theta_{g(\cdot,0)}, 0)$.

(B) Suppose $\lambda_1 < g(0,0) \leq N(h)$ and $h(0,0) > \lambda_1$. Let $u \geq 0, v \geq 0$ be a solution to (4) with $v > 0$ in Ω. If $u > 0$ in Ω, then $\lambda = 0$ is the smallest eigenvalue of the problem

$$\begin{cases} -\Delta w - wg(u,v) = \lambda w \text{ in } \Omega, \\ w|_{\partial\Omega} = 0. \end{cases}$$

Since

$$-g(u,v) > -(g(0, \theta_{h(0,\cdot)})$$

from Lemma 3.3 and the monotonicity of g, using Lemma 2.2 we have

$$0 > \lambda_1(-g(0, \theta_{h(0,\cdot)})) = \lambda_1(-g(0, \theta_{h(0,\cdot)}) + g(0,0)) - g(0,0) = N(h) - g(0,0).$$

This contradicts $g(0,0) \leq N(h)$. Hence $u = 0$, so all possible nonnegative solutions to (4) are $(0,0), (0, \theta_{h(0,\cdot)})$ and $(\theta_{g(\cdot,0)}, 0)$.

(C) Suppose $g(0,0) > \lambda_1$ and $M(g) < h(0,0) < \lambda_1$. Let $u \geq 0, v \geq 0$ be a solution to (4) in which one component is zero. Then $u = 0, v = 0$ or $u = \theta_{g(\cdot,0)}, v = 0$. Since $\lambda_1(-h(\theta_{g(\cdot,0)}, 0)) = \lambda_1(-h(\theta_{g(\cdot,0)}, 0) + h(0,0)) - h(0,0) = M(g) - h(0,0) < 0$, by the combination of Lemmas 2.4 and 3.3, there is a positive solution to (4) $u^+ > 0, v^+ > 0$.

(D) Suppose $g(0,0) > N(h)$ and $h(0,0) > \lambda_1$. Let $u \geq 0, v \geq 0$ be a solution to (4) in which one component is zero. Then since

$$\begin{aligned} g(0,0) &> N(h) = \lambda_1(-(g(0,\theta_{h(0,\cdot)}) - g(0,0))) \\ &> \lambda_1(-(g(0,0) - g(0,0))) = \lambda_1(0) = \lambda_1, \end{aligned}$$

from Lemma 2.2 and the monotonicity of g, we have $u = 0, v = 0$ or $u = 0, v = \theta_{h(0,\cdot)}$ or $u = \theta_{g(\cdot,0)}, v = 0$. Since $\lambda_1(-g(0,\theta_{h(0,\cdot)})) = \lambda_1(-g(0,\theta_{h(0,\cdot)}) + g(0,0)) - g(0,0) = N(h) - g(0,0) < 0$, by the combination of Lemmas 2.4 and 3.3, there is a positive solution to (4) $u^+ > 0, v^+ > 0$. □

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