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JACOBIAN OF THE GENERALIZED c-HYPERBOLIC COORDINATES

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Abstract: In this paper the transformation by means of the generalized c-hyperbolic coordinates is investigated. We prove of the regularity of this transformation and we use it in search of the solution of the differential system.

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1. Introduction and Preliminaries

In this paper we will investigate of the transformation

$$\Phi_{C_H} : (r, u_1, \dots, u_{n-2}, v) \to (x_1, x_2, \dots, x_n)$$

defined by means of the generalized c-hyperbolic coordinates in the following forms

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$$\begin{aligned} x_1 &= r\cosh u_1; \\ x_k &= r\left(\prod_{i=1}^{k-1}\sinh u_i\right)\cosh u_k,\ \ k=2,3,\dots,n-2; \\ x_{n-1} &= r\left(\prod_{i=1}^{n-2}\sinh u_i\right); \\ x_n &= rv, \end{aligned} \tag{1}$$

where $u_i, r, v \in C^1(\langle t_0,\infty),R)$, $i=1,2,\dots,n-2$, $u_i(t)\neq 0$, $i=1,2,\dots,n-3$ are its angle functions, $r(t)>0$ and $v(t)\in R$. We consider of the functions $\varphi_i : H \to R$, $H\subset R^n$, $i=1,2,\dots,n$, given by

$$\begin{aligned} \varphi_1 &= r\cosh u_1; \\ \varphi_k &= r\left(\prod_{i=1}^{k-1}\sinh u_i\right)\cosh u_k,\ \ k=2,3,\dots,n-2; \\ \varphi_{n-1} &= r\left(\prod_{i=1}^{n-2}\sinh u_i\right); \\ \varphi_n &= rv, \end{aligned} \tag{2}$$

and for each $i=1,2,\dots,n$, $x_i=\varphi_i(r,u_1,\dots,u_{n-2},v)$ holds.

Definition 1. Let $\Phi_{C_H}: R^n\to R^n$ be a transformation given by (1). Jacobian of $x_1,x_2,\dots,x_n$ with respect to $r,u_1,\dots u_{n-2},v$ we denote by $J^n_{\Phi_{C_H}}$ and define by

$$J^n_{\Phi_{C_H}} = \begin{vmatrix} \partial x_1/\partial r & \partial x_2/\partial r & \dots & \partial x_n/\partial r \\ \partial x_1/\partial u_1 & \partial x_2/\partial u_1 & \dots & \partial x_n/\partial u_1 \\ \vdots & \vdots & & \vdots \\ \partial x_1/\partial u_{n-2} & \partial x_2/\partial u_{n-2} & \dots & \partial x_n/\partial u_{n-2} \\ \partial x_1/\partial v & \partial x_2/\partial v & \dots & \partial x_n/\partial v \end{vmatrix}.$$

2. New Results

Theorem 2. *Let Φ_{C_H} be transformation by means of the generalized c-hyperbolic coordinates. Then Jacobian*

$$J^n_{\Phi_{C_H}} = r^{n-1}\prod_{i=1}^{n-3}\sinh^{(n-2-i)}(u_i)\,.$$

Proof. For brevity we will use signification Sh_i, Ch_i, $i = 1, 2, \ldots, n-3$ instead of $\sinh u_i(t)$, $\cosh u_i(t)$. Let $n = 3$ then $J^3_{\Phi_{C_H}} = \begin{vmatrix} Ch_1 & Sh_1 & v \\ rSh_1 & rCh_1 & 0 \\ 0 & 0 & r \end{vmatrix} = r^2$. For every n we can write

$$J^n_{\Phi_{C_H}} = \begin{vmatrix} Ch_1 & S_1Ch_2 & \ldots & \prod_{i=1}^{n-2} Sh_i & v \\ rS_1 & rCh_1Ch_2 & \ldots & rCh_1 \prod_{i=2}^{n-2} Sh_i & 0 \\ 0 & rS_1S_2 & \ldots & rS_1Ch_2 \prod_{i=3}^{n-2} Sh_i & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & rCh_{n-2} \prod_{i=1}^{n-3} Sh_i & 0 \\ 0 & 0 & \ldots & 0 & r \end{vmatrix} = (-1)^{2n} r J^{n-1}_{\Phi_{P_H}}$$

$$= rr^{n-2} \prod_{i=1}^{n-3} (Sh_i)^{(n-2-i)} = r^{n-1} \prod_{i=1}^{n-3} (Sh_i)^{(n-2-i)}.$$

Here $J^{n-1}_{\Phi_{P_H}}$ is Jacobian of the transformation by means of the genegalized p-hyperbolic coordinates [2]. The proof is complete. □

Theorem 3. *Let Φ_{C_H} be a transformation by means of the generalized c-hyperbolic coordinates and Ψ_{C_H} is its inverse transformation. Then Jacobian $J^n_{\Psi_C} \neq 0$.*

Proof. Let Φ_{C_H} be a transformation given by (1) and Ψ_{C_H} is its inverse transformation defined by functions $\psi_i : H' \to R$, $H' \subset R^n$, $i = 1, 2, \ldots, n$, such that for every $(x_1, x_2, \ldots, x_n) \in H'$ is $(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n))$ $= (r, u_1, \ldots u_{n-2}, v) \in H$. It is possible obtain functions ψ_i, $i = 1, 2, \ldots, n$, from the equations (1). We can write

$$(x_1, x_2, \ldots, x_n) = (\varphi_1(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)), \ldots, \varphi_n(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n))), \quad (3)$$

$$(r, u_1, \ldots, u_{n-2}, v) = (\psi_1(\varphi_1(r, u_1, \ldots, u_{n-2}, v), \ldots, \varphi_n(r, u_1, \ldots, u_{n-2}, v)), \ldots, \psi_n(\varphi_1(r, u_1, \ldots, u_{n-2}, v), \ldots, \varphi_n(r, u_1, \ldots, u_{n-2}, v))). \quad (4)$$

Taking the derivative of the equation (3) with respect to $x_1, x_2, \ldots, x_n$, gradually, we get

$$\begin{aligned} (1, 0, \ldots, 0) &= \left(\sum_{i=1}^{n} \frac{\partial \varphi_1}{\partial \psi_i} \frac{\partial \psi_i}{\partial x_1}, \ldots, \sum_{i=1}^{n} \frac{\partial \varphi_n}{\partial \psi_i} \frac{\partial \psi_i}{\partial x_1} \right), \\ (0, 1, \ldots, 0) &= \left(\sum_{i=1}^{n} \frac{\partial \varphi_1}{\partial \psi_i} \frac{\partial \psi_i}{\partial x_2}, \ldots, \sum_{i=1}^{n} \frac{\partial \varphi_n}{\partial \psi_i} \frac{\partial \psi_i}{\partial x_2} \right), \end{aligned} \quad (5)$$

$$\vdots$$

$$(0,0,\ldots,1) \quad = \quad \left(\sum_{i=1}^{n}\frac{\partial\varphi_1}{\partial\psi_i}\frac{\partial\psi_i}{\partial x_n},\ldots,\sum_{i=1}^{n}\frac{\partial\varphi_n}{\partial\psi_i}\frac{\partial\psi_i}{\partial x_n}\right).$$

Analogously by differentiating of the equation (4) with respect to r, u_1, $\ldots u_{n-2}$, v, we obtain

$$\begin{aligned}
(1,0,\ldots,0) &= \left(\sum_{i=1}^{n}\frac{\partial\psi_1}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial r},\ldots,\sum_{i=1}^{n}\frac{\partial\psi_n}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial r}\right),\\
(0,1,\ldots,0) &= \left(\sum_{i=1}^{n}\frac{\partial\psi_1}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial u_1},\ldots,\sum_{i=1}^{n}\frac{\partial\psi_n}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial u_2}\right), \qquad (6)\\
&\vdots\\
(0,\ldots,1,0) &= \left(\sum_{i=1}^{n}\frac{\partial\psi_1}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial u_{n-2}},\ldots,\sum_{i=1}^{n}\frac{\partial\psi_n}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial u_{n-2}}\right),\\
(0,0,\ldots,1) &= \left(\sum_{i=1}^{n}\frac{\partial\psi_1}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial v},\ldots,\sum_{i=1}^{n}\frac{\partial\psi_n}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial v}\right).
\end{aligned}$$

If we denote the matrices

$$M_{\Phi_{C_H}} = \begin{pmatrix} \frac{\partial\varphi_1}{\partial r} & \cdots & \frac{\partial\varphi_n}{\partial r}\\ \vdots & & \vdots\\ \frac{\partial\varphi_1}{\partial u_{n-2}} & \cdots & \frac{\partial\varphi_n}{\partial u_{n-2}}\\ \frac{\partial\varphi_1}{\partial v} & \cdots & \frac{\partial\varphi_n}{\partial v} \end{pmatrix}, M_{\Psi_{C_H}} = \begin{pmatrix} \frac{\partial\psi_1}{\partial x_1} & \cdots & \frac{\partial\psi_n}{\partial x_1}\\ \vdots & & \vdots\\ \frac{\partial\psi_1}{\partial x_n} & \cdots & \frac{\partial\psi_n}{\partial x_n} \end{pmatrix},$$

then multiplying these matrices we get $M_{\Phi_{C_H}}.M_{\Psi_{C_H}} = E$, E is the identity matrix. Moreover, the inverse transformation Ψ_{C_H} is the regular mapping because from relations $J^n_{\Phi_{C_H}}.J^n_{\Psi_{C_H}} = 1$ and $J^n_{\Phi_{C_H}} \neq 0$ follows $J^n_{\Psi_{C_H}} \neq 0$. The theorem is proved. □

Theorem 4. *Let $(v_1(t), v_2(t), \ldots, v_n(t))$, $v_i(t) \neq 0$, $i = 1, 2, \ldots, n$ be a solution of the differential system*

$$x_i^{\cdot} = g_i(x_1, x_2, \ldots x_n). \qquad (7)$$

If we denote the functions $\vartheta_i : J \to R$, $J \subset R$, $i = 1, 2, \ldots, n$

$$\vartheta_i(t) = \psi_i\left(v_1(t), v_2(t), \ldots, v_n(t)\right) \qquad (8)$$

and the functions $h_i : H' \to R$, $H' \subset R^n$, $i = 1, 2, \ldots, n$

$$h_i(r, u_1, \ldots u_{n-2}, v) = \sum_{j=1}^{n}\frac{\partial\psi_i}{\partial x_j}(\varphi_1(r, u_1, \ldots u_{n-2}, v), \ldots, \varphi_n(r, u_1, \ldots u_{n-2}, v))$$

$$\times\, g_j(\varphi_1(r,u_1,\dots u_{n-2},v),\dots,\varphi_n(r,u_1,\dots u_{n-2},v)), \quad (9)$$

then $(\vartheta_1(t),\vartheta_2(t),\dots,\vartheta_n(t))$ *is the solution of the differential system*

$$\begin{array}{rcll} r' &=& h_1(r,u_1,\dots u_{n-2},v), & \\ u_i' &=& h_{i+1}(r,u_1,\dots u_{n-2},v), & i=1,2,\dots,n-2, \\ v' &=& h_n(r,u_1,\dots u_{n-2},v). & \end{array} \quad (10)$$

Proof. Let $(v_1(t),v_2(t),\dots,v_n(t))$ be a solution of the differential system (7) then $dv_i(t)/dt = g_i(v_1(t),v_2(t),\dots,v_n(t))$ holds. The derivative of the functions ϑ_i, $i=1,2,\dots,n$ with respect to t, to get

$$\frac{d\vartheta_i}{dt} = \sum_{j=1}^{n}\frac{\partial\psi_i}{\partial x_j}(v_1,v_2,\dots,v_n)\frac{dv_j}{dt} = \sum_{j=1}^{n}\frac{\partial\psi_i}{\partial x_j}(v_1,v_2,\dots,v_n)g_j(v_1,v_2,\dots,v_n).$$

Substituting $\varphi_i(\vartheta_1,\vartheta_2,\dots,\vartheta_n)$, instead of v_i, $i=1,2,\dots,n$ we obtain

$$\frac{d\vartheta_i}{dt} = \sum_{j=1}^{n}\frac{\partial\psi_i}{\partial x_j}(\varphi_1(\vartheta_1,\vartheta_2,\dots,\vartheta_n),\dots,\varphi_n(\vartheta_1,\vartheta_2,\dots,\vartheta_n))$$
$$\times g_j(\varphi_1(\vartheta_1,\vartheta_2,\dots,\vartheta_n),\dots,\varphi_n(\vartheta_1,\vartheta_2,\dots,\vartheta_n)).$$

Thus $(\vartheta_1(t),\vartheta_2(t),\dots,\vartheta_n(t))$ is the solution of the differential system (10). The theorem is proved. □

Theorem 5. *Let* $(\omega_1(t),\omega_2(t),\dots,\omega_n(t))$ *be a solution of the differential system (10). If we denote the funcions* $w_i : I \to R$, $I \subset R$, $i=1,2,\dots,n$

$$w_i(t) = \varphi_i\left(\omega_1(t),\omega_2(t),\dots,\omega_n(t)\right), \quad (11)$$

where $w_i(t) \neq 0$, *for every* $t \in I$, *then* $(\omega_1(t),\omega_2(t),\dots,\omega_n(t))$ *is the solution of the differential system (7).*

Proof. Let $(\omega_1(t),\omega_2(t),\dots,\omega_n(t))$ be a solution of the differential system (10) then $d\omega_i(t)/dt = h_i(\omega_1(t),\omega_2(t),\dots,\omega_n(t))$ holds. We can write the derivatives of the functions w_i, $i=1,2,\dots,n$ with respect to t in the following forms

$$\frac{dw_1}{dt} = \frac{\partial\varphi_1}{\partial r}(\omega_1,\omega_2,\dots,\omega_n)\frac{d\omega_1}{dt} = \frac{\partial\varphi_1}{\partial r}(\omega_1,\omega_2,\dots,\omega_n)h_1(\omega_1,\omega_2,\dots,\omega_n),$$

$$\frac{dw_i}{dt} = \sum_{j=1}^{n-1}\frac{\partial\varphi_i}{\partial u_j}(\omega_1,\omega_2,\dots,\omega_n)h_{j+1}(\omega_1,\omega_2,\dots,\omega_n), \quad i=2,3,\dots,n-2,$$

$$\frac{dw_n}{dt} = \frac{\partial\varphi_n}{\partial v}(\omega_1,\omega_2,\dots,\omega_n)h_n(\omega_1,\omega_2,\dots,\omega_n).$$

Substituting $\psi_i(w_1,w_2,\dots,w_n)$, instead of ω_i, $i=1,2,\dots,n$ we obtain

$$\frac{dw_1}{dt} = \frac{\partial\varphi_1}{\partial r}(\psi_1(w_1,w_2,\dots,w_n),\dots,\psi_n(w_1,w_2,\dots,w_n))$$

$$\times h_1(\psi_1(w_1, w_2, \ldots, w_n), \ldots, \psi_n(w_1, w_2, \ldots, w_n)),$$

$$\frac{dw_i}{dt} = \sum_{j=1}^{n-1} \frac{\partial \varphi_i}{\partial u_j}(\psi_1(w_1, w_2, \ldots, w_n), \ldots, \psi_n(w_1, w_2, \ldots, w_n)) \times$$

$$\times h_{j+1}(\psi_1(w_1, w_2, \ldots, w_n), \ldots, \psi_n(w_1, w_2, \ldots, w_n)), \quad i = 2, 3, \ldots, n-2,$$

$$\frac{dw_n}{dt} = \frac{\partial \varphi_n}{\partial v}(\psi_1(w_1, w_2, \ldots, w_n), \ldots, \psi_n(w_1, w_2, \ldots, w_n))$$
$$\times h_n(\psi_1(w_1, w_2, \ldots, w_n), \ldots, \psi_n(w_1, w_2, \ldots, w_n)).$$

We denote

$$q_1(x_1, x_2, \ldots, x_n) = \frac{\partial \varphi_1}{\partial r}(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)) \times h_1(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)), \tag{12}$$

$$q_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n-1} \frac{\partial \varphi_i}{\partial u_j}(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)) \times h_{j+1}(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)), \quad i = 2, 3, \ldots, n-2, \tag{13}$$

$$q_n(x_1, x_2, \ldots, x_n) = \frac{\partial \varphi_n}{\partial v}(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)) \times h_n(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)). \tag{14}$$

According to Theorem 4, the functions $w_1, w_2, \ldots, w_n$ are the solutions of the differential system $x_i^{\cdot} = q_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n$. Using the substitution of (9) into (12), (13), (14) to get

$$q_1(x_1, x_2, \ldots, x_n) = \frac{\partial \varphi_1}{\partial r}(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)) \times \sum_{k=1}^{n} \frac{\partial \psi_1}{\partial x_k}(x_1, x_2, \ldots, x_n) g_k(x_1, x_2, \ldots, x_n),$$

$$q_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n-1} \frac{\partial \varphi_i}{\partial u_j}(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)) \times \sum_{k=1}^{n} \frac{\partial \psi_i}{\partial x_k}(x_1, x_2, \ldots, x_n) g_k(x_1, x_2, \ldots, x_n), \quad i = 2, 3, \ldots, n-2,$$

$$q_n(x_1, x_2, \ldots, x_n) = \frac{\partial \varphi_n}{\partial n}(\psi_1(x_1, x_2, \ldots, x_n), \ldots, \psi_n(x_1, x_2, \ldots, x_n)) \times \sum_{k=1}^{n} \frac{\partial \psi_n}{\partial x_k}(x_1, x_2, \ldots, x_n) g_k(x_1, x_2, \ldots, x_n),$$

then from (5) it follows $q_i(x_1, x_2, \ldots, x_n) = g_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$ and the functions w_i, $i = 1, 2, \ldots, n$ are the solutions of the differential system $x_i' = g_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$. The proof of the theorem is completed. □

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