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ON (k,l)-KERNELS IN THE CORONA OF DIGRAPHS

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Abstract: In this paper we give the necessary and sufficient conditions for the existence of (k,l)-kernels in the corona of digraphs. Next we prove results for the corona to be (k,l)-kernel perfect. Moreover, we determine the total number of (k,l)-kernels, k-independent sets and l-dominating sets in this product.

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1. Introduction

For concepts not defined here see [1]. Let D be a finite, directed graph (for short: a digraph) without loops and multiple arcs, where $V(D)$ is the set of vertices and $A(D)$ is the set of arcs of D. By the empty digraph we mean a digraph with $V(D)=\emptyset$. A path from a vertex x_1 to a vertex x_n, $n \geq 2$, is a sequence of vertices $x_1, ..., x_n$ and arcs $(x_i, x_{i+1}) \in A(D)$, for $i = 1, 2, ..., n-1$. By $d_D(x_i, x_j)$ we denote the length of the shortest path from x_i to x_j in D. If there is no a path from x_i to x_j in D, then we put $d_D(x_i, x_j) = \infty$. For any $X \subseteq V(D)$ and $x \in V(D)$ we put $d_D(x, X) = \min_{y \in X} d_D(x, y)$.

Let k, l be integers $k \geq 2$ and $l \geq 1$. We say that a subset $J \subset V(D)$ is a (k,l)-kernel of D if:

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(1) for each $x_i, x_j \in J$ and $i \neq j$, $d_D(x_i, x_j) \geq k$ and

(2) for each $x_i \notin J$ there exists $x_j \in J$ such that $d_D(x_i, x_j) \leq l$.

From the definition of the (k,l)-kernel immediately follows that if J is a (k,l)-kernel of D, then J is also a (k_0, l_0)-kernel of D, where $k_0 \leq k$ and $l_0 \geq l$. If the set J satisfies the condition in (1) or in (2), then we shall call it a k-independent set of D (also named as a k-stable set of D) or an l-dominating set of D, respectively. We notice that 2-independent set is an independent set and 1-dominating set is a dominating set of D. In addition we assume that a subset containing only one vertex and the empty set also are meant as k-independent sets. The set $V(D)$ is an l-dominating set of D. We assume that for the empty digraph D the empty set is a unique k-independent set of D, also the empty set is a unique l-dominating set of D. Consequently we assume that the empty set is a unique (k,l)-kernel of the empty digraph, for every k, l. A digraph D whose every induced subdigraph has a (k,l)-kernel is a (k,l)-kernel perfect digraph. The concept of (k,l)-kernels was introduced by Kwaśnik in [7]. A $(2,1)$-kernel is a kernel in Berge's sense. Moreover $(k, k-1)$-kernels (also named as k-kernels), $k \geq 2$ were considered for instance in [2], [7], [10]. Sufficient conditions for the existence of (k,l)-kernels in digraphs have been investigated for instance in [4], [5], [6], [8]. By $NkI(D)$, $NlD(D)$ and $NklK(D)$ we mean the number of all k-independent sets, l-dominating sets and (k,l)-kernels in the digraph D, respectively. The total number of k-independent sets and (k,l)-kernels in graphs and in some their products were studied for instance in [8] and [10].

Let D be a digraph with $V(D) = \{x_1, ..., x_n\}$, $n \geq 1$ and $\mathcal{H} = (H_i)_{i \in \mathcal{I} = \{1,...,n\}}$ be a sequence of vertex disjoint digraphs. For a nonempty digraph H_i, $i \in \mathcal{I}$ from the sequence $\mathcal{H}$ we put $V(H_i) = \{y_1^i, ..., y_{p_i}^i\}$, $p_i \geq 1$. The corona of the digraph D and the sequence $\mathcal{H}$ is a digraph $D \circ \mathcal{H}$ such that $V(D \circ \mathcal{H}) = V(D) \cup \bigcup_{i \in \mathcal{I}} V(H_i)$ and $A(D \circ \mathcal{H}) = A(D) \cup \bigcup_{i \in \mathcal{I}} A(H_i)) \cup \bigcup_{i \in \mathcal{I}} \{(y_t^i, x_i); t = 1, ..., p_i\}$.

The corona $G \circ \mathcal{H}$ of a graph G and sequence $\mathcal{H}$ of vertex disjoint graphs is known in literature, see for instance [9]. If all graphs in the sequence $\mathcal{H}$ are isomorphic to the same graph H, then we obtain the corona of two graphs, see [3]. We have applied this definition for digraphs.

In this paper first we study the existence of (k,l)-kernels in $D \circ \mathcal{H}$. Next we prove the necessary and sufficient condition for (k,l)-kernel perfectness of corona. In the last section we calculate the total number of k-independent sets, l-dominating sets and (k,l)-kernels in corona of digraphs.

2. The Existence of (k,l)-Kernels in Corona of Digraphs

In this section we establish necessary and sufficient conditions for the existence of (k,l)-kernels in $D \circ \mathcal{H}$ based upon factors. Our first theorem describes k-independent sets in the corona. Next we characterize l-dominating sets in $D\circ\mathcal{H}$.

Theorem 1. *Let $k \geq 2$ be integer. A subset $S^* \subset V(D \circ \mathcal{H})$ which meets $V(D)$ is a k-independent set of $D \circ \mathcal{H}$ if and only if $S \subset V(D)$ is a nonempty k-independent set of D such that $S^* = S \cup \bigcup_{i\in\mathcal{I}_1} S_i$, where $\mathcal{I}_1 = \{j; d_D(x_j, S) \geq k-1\}$ and S_i is an arbitrary k-independent set of H_i, for every $i \in \mathcal{I}_1$.*

Proof. 1. Let $S \subseteq V(D)$ be a k-independent set of a digraph D, $\mathcal{I}_1 = \{j; d_D(x_j, S) \geq k-1\}$ and let S_i be an arbitrary k-independent set of H_i, for every $i \in \mathcal{I}_1$. We shall prove that the set $S^* = S \cup \bigcup_{i\in\mathcal{I}_1} S_i$ is a k-independent set of $D \circ \mathcal{H}$. Evidently by assumptions of S and S_i with $i \in \mathcal{I}_1$, and by the definition of $D \circ \mathcal{H}$ immediately follows that S and S_i are k-independent sets of $D \circ \mathcal{H}$. Hence to prove that S^* is a k-independent set of $D \circ \mathcal{H}$ the following cases should be considered.

$$x_p \in S \text{ and } y_t^i \in S_i \text{ with } i \in \mathcal{I}_1. \tag{1.1}$$

First let observe that if $y_t^i \in S_i$ and $i \in \mathcal{I}_1$, then $d_D(x_i, S) \geq k-1$. Consequently from the definition of the corona we receive that $(y_t^i, x_i) \in A(D)$, so $d_{D\circ\mathcal{H}}(y_t^i, S) \geq 1 + k - 1 = k$ and this implies that $d_{D\circ\mathcal{H}}(y_t^i, x_p) \geq k$. Moreover in view of the definition of the corona $D \circ \mathcal{H}$ there is no a path from x_p to any vertex of $V(H_p)$, whence $d_{D\circ\mathcal{H}}(x_p, y_t^i) > k$.

$$y_t^i \in S_i,\ y_m^j \in S_j, \text{ for } i, j \in \mathcal{I}_1 \text{ with } i \neq j. \tag{1.2}$$

The definition of the corona gives that for every $i, j \in \mathcal{I}_1$ with $i \neq j$ there is no path between vertices $y_t^i \in V(H_i)$, $y_m^j \in V(H_j)$, so $d_{D\circ\mathcal{H}}(y_t^i, y_m^j) = \infty > k$.

Taking the above possibilities into considerations we obtain that S^* is a k-independent set of $D \circ \mathcal{H}$.

2. Let $S^* \subset V(D \circ \mathcal{H})$ be a k-independent set of $D \circ \mathcal{H}$ which meets $V(D)$. Denote $S = S^* \cap V(D)$. Evidently the set S is a nonempty k-independent set of D. Moreover the definition of the set S and the definition of $D \circ \mathcal{H}$ imply that we can depict the set S^* as follows $S^* = S \cup \bigcup_{i\in\mathcal{I}_1} S_i$, where $\mathcal{I}_1 = \{i; d_D(x_i, S) \geq k-1\}$ and $S_i \subseteq V(H_i)$. Because $S_i \subset S^*$ so S_i is k-independent. Consequently S_i is an arbitrary k-independent set of H_i, for every $i \in \mathcal{I}_2$.

Thus the theorem is proved. □

The next theorem directly follows from the definition of corona.

Theorem 2. *Let $k \geq 2$ be integer and $\mathcal{I}$ be a set of indexes of vertices belonging to D. A subset $S^* \subset V(D \circ \mathcal{H})$ which does not meet $V(D)$ is a k-independent set of $D \circ \mathcal{H}$ if and only if $S^* = \bigcup_{i \in \mathcal{I}_1} S_i$, where $\mathcal{I}_1$ is an arbitrary subset of $\mathcal{I}$ and S_i is an arbitrary k-independent set of H_i, for every $i \in \mathcal{I}_1$.*

Theorem 3. *Let $l \geq 1$ be integer and $\mathcal{I}$ be a set of indexes of vertices belonging to D. A subset $Q^* \subseteq V(D \circ \mathcal{H})$ is an l-dominating set of $D \circ \mathcal{H}$ if and only if $Q \subseteq V(D)$ is an l-dominating set of D such that $Q^* = Q \cup \bigcup_{i \in \mathcal{I}_1} Q_i \cup \bigcup_{j \in (\mathcal{I} \setminus \mathcal{I}_1)} Q_j$, where $\mathcal{I}_1 = \{i; d_D(x_i, Q) = l\}$ and Q_i is an arbitrary l-dominating set of H_i, for every $i \in \mathcal{I}_1$, and Q_j is an arbitrary subset of $V(H_j)$, for every $j \in (\mathcal{I} \setminus \mathcal{I}_1)$.*

Proof. 1. Assume that $Q \subseteq V(D)$ is an l-dominating set of D. Denote $\mathcal{I}_1 = \{i; d_D(x_i, Q) = l\}$ and let Q_i be an arbitrary l-dominating set of H_i, for every $i \in \mathcal{I}_1$. Moreover let Q_j be an arbitrary subset of H_j, for every $j \in (\mathcal{I} \backslash \mathcal{I}_1)$. We shall prove that the set $Q^* = Q \cup \bigcup_{i \in \mathcal{I}_1} Q_i \cup \bigcup_{j \in (\mathcal{I} \backslash \mathcal{I}_1)} Q_j$ is an l-dominating set of $D \circ \mathcal{H}$. To prove it we distinguish possible cases:

(1.1) $x_p \notin Q^*$. Then from the definition of the set Q^* we have that $x_p \notin Q$. Because Q is l-dominating of D, so $d_D(x_p, Q) \leq l$. Moreover by the definition of the corona and by the set Q^* we have that $d_{D \circ \mathcal{H}}(x_p, Q^*) = d_D(x_p, Q) \leq l$.

(1.2) $y_s^i \notin Q^*$, where $i \in \mathcal{I}_1$ and $1 \leq s \leq p_i$. Because $i \in \mathcal{I}_1$, so in view of the set Q^* we have that there exists an l-dominating set Q_i of H_i such that $Q_i \subseteq Q^*$. Hence $d_{D \circ \mathcal{H}}(y_s^i, Q^*) = d_{H_i}(y_s^i, Q_i) \leq l$.

(1.3) $y_m^j \notin Q^*$, where $j \in (\mathcal{I} \setminus \mathcal{I}_1)$ and $1 \leq m \leq p_j$. Then from the fact that $j \notin \mathcal{I}_1$ it follows that for the vertex $x_j \in V(D)$ holds $d_D(x_j, Q) < l$. Of course $Q \subseteq Q^*$. Moreover for every $m = 1, ..., p_j$ there exists an arc (y_m^j, x_j) in $D \circ \mathcal{H}$, whence $d_{D \circ \mathcal{H}}(y_m^j, Q^*) = 1 + d_D(x_j, Q) \leq l$.

All this together gives that Q^* is an l-dominating set of $D \circ \mathcal{H}$.

2. Let Q^* be an l-dominating set of $D \circ \mathcal{H}$. Denote $Q = Q^* \cap V(D)$. First we shall prove that Q is an l-dominating set of D. Assume on the contrary that Q is not l-dominating. Then there exists a vertex $x_p \in V(D)$ such that $d_D(x_p, Q) > l$. Moreover by the definition of the corona there is no a path from x_p to $(Q^* \setminus Q)$, a contradiction with the assumption of Q^*. Consequently $Q^* \subseteq Q \cup \bigcup_{i \in \mathcal{I}} V(H_i)$, where Q is an l-dominating set of D. Let $\mathcal{I}_1 \subset \mathcal{I}$ and

$\mathcal{I}_1 = \{i; d_D(x_i, Q) = l\}$. If $i \in \mathcal{I}_1$, then there exists a vertex $x_i \in V(D)$ such that $d_D(x_i, Q) = l$. Hence for every vertex $y^i_s \in V(H_i)$, $s = 1, ..., p_i$ we have that $d_{D\circ\mathcal{H}}(y^i_t, Q) = l+1$. Because Q^* is an l-dominating set of $D\circ\mathcal{H}$, so a subset $Q^* \cap V(H_i) = Q_i$ is an l-dominating set of H_i, for every $i \in \mathcal{I}_1$. If $j \in (\mathcal{I} \setminus \mathcal{I}_1)$ then for every vertex $x_j \in V(D)$ we have that $d_D(x_j, Q) < l$. Therefore from the definition of corona for every vertex $y^j_m \in V(H_j)$, $m = 1, ..., p_j$ we calculate that $d_{D\circ\mathcal{H}}(y^j_m, Q) = 1 + d_D(x_j, Q) < l + 1 \leq l$. This implies that a subset $Q_j = Q^* \cap V(H_j)$ can be an arbitrary subset of $V(H_j)$. Finally the set Q^* can be depicted in the following way $Q^* = Q \cup \bigcup_{i\in\mathcal{I}_1} Q_i \cup \bigcup_{j\in(\mathcal{I}\setminus\mathcal{I}_1)} Q_j$, where Q, Q_i and Q_j are as in the statement of the theorem.

Thus the theorem is proved. □

Corollary 1. *Let $k \geq 2$, $l \geq 1$ be integers. If $D \circ \mathcal{H}$ has a (k,l)-kernel J then J meets $V(D)$.*

Lemma 1. *Let $k \geq 3$, $l < k - 1$ be integers. If $D \circ \mathcal{H}$ has a (k,l)-kernel J, then $J \subseteq V(D)$.*

Proof. If $|J| = 1$, then the lemma immediately follows by Corollary 1. Let $|J| \geq 2$. We proceed by contradiction. Assume that $D \circ \mathcal{H}$ has a (k,l)-kernel J, for $l < k - 1$ and there exists $i \in \mathcal{I}$ such that $y^i_p \in J$, $1 \leq t \leq p_i$. From Corollary 1 the set J meets $V(D)$. Moreover the definition of corona implies that $J' = J \cap V(D)$ is a (k,l)-kernel of D. This means that for every vertex $x_i \in V(D)$ holds $d_D(x_i, J') = d_{D\circ\mathcal{H}}(x_i, J) \leq l < k - 1$. Hence for every $i \in \mathcal{I}$ and y^i_t with $t = 1, ..., p_i$, $d_{D\circ\mathcal{H}}(y^i_t, J') \leq l + 1 < k$, a contradiction with the assumption about the set J, which completes the proof. □

Theorem 4. *Let $k \geq 3$, $l < k - 1$ be integers. Let $\mathcal{H} = (H_i)_{i\in\mathcal{I}}$ be a sequence of arbitrary nonempty vertex disjoint digraphs H_i. A digraph $D \circ \mathcal{H}$ has a (k,l)-kernel if and only if a digraph D has a $(k, l-1)$-kernel.*

Proof. Let k, l be as in the statement of the theorem. Assume that the digraph D has a $(k, l-1)$-kernel J. It is obvious that J is a (k,l)-kernel of digraph $D \circ \mathcal{H}$. Conversely let J be a (k,l)-kernel of $D \circ \mathcal{H}$ with $l < k - 1$. From Lemma 1 we have that $J \subseteq V(D)$. Of course J is a k-independent set of D. It suffices to prove that J is an $(l-1)$-dominating set of D. By assumption of J, for every vertex $y^i_t \in V(H_i)$, $i \in \mathcal{I}$ holds $d_{D\circ\mathcal{H}}(y^i_t, J) \leq l$. Consequently $d_{D\circ\mathcal{H}}(x_i, J) = d_D(x_i, J) \leq l - 1$, which completes the proof. □

Theorem 5. *Let $k \geq 2$ be integer. A subset J^* is a $(k, k-1)$-kernel of $D\circ\mathcal{H}$ if and only if there exists a $(k, k-1)$-kernel J of D such that $J^* = J \cup \bigcup_{i\in\mathcal{I}_1} J_i$,* *where* $\mathcal{I}_1 = \{i; d_D(x_i, J) = k-1\}$ *and* J_i *is a* $(k, k-1)$*-kernel of* H_i*, for every* $i \in \mathcal{I}_1$.

Proof. 1. Let $J \subseteq V(D)$ be a $(k, k-1)$-kernel of D. Let $\mathcal{I}_1 = \{i; d_D(x_i, J) = k-1\}$ and J_i be a $(k, k-1)$-kernel of H_i, for every $i \in \mathcal{I}_1$. We shall prove that $J^* = J \cup \bigcup_{i \in \mathcal{I}_1} J_i$ is a $(k, k-1)$-kernel of $D \circ \mathcal{H}$. By Theorem 1 the set J^* is k-independent of $D \circ \mathcal{H}$. Proving analogously as in Theorem 3 we obtain that J^* is a $(k-1)$-dominating set of $D \circ \mathcal{H}$. Consequently J^* is a $(k, k-1)$-kernel of $D \circ \mathcal{H}$.

2. Let $J^* \subseteq V(D \circ \mathcal{H})$ be a $(k, k-1)$-kernel of $D \circ \mathcal{H}$. By Corollary 1 the set J^* meets $V(D)$ whence denote $J = J^* \cap V(D)$. It is clear that the set J is a nonempty k-independent set of the digraph D. Moreover proving analogously as in Theorem 3 we have that J is a $(k-1)$-dominating set of the digraph D. Evidently by Theorem 1 we deduce that $(J^* \setminus J) \subseteq \bigcup_{i \in \mathcal{I}_1} V(H_i)$, where $\mathcal{I}_1 = \{i; d_D(x_i, J) = k-1\}$. If $i \in \mathcal{I}_1$, then for the vertex x_i holds $d_D(x_i, J) = d_{D \circ \mathcal{H}}(x_i, J^*) = k-1$. Hence for every vertex $y_t^i \in V(H_i)$, $t = 1, ..., p_i$ we have that $d_{D \circ \mathcal{H}}(y_t^i, J) = k$. Because J^* is a $(k, k-1)$-kernel of $D \circ \mathcal{H}$ so immediately follows that $J_i = J^* \cap V(H_i)$ is a $(k, k-1)$-kernel of H_i. Finally $J^* = J \cup \bigcup_{i \in \mathcal{I}_1} J_i$, where $\mathcal{I}_1 = \{i; d_D(x_i, J) = k-1\}$ and J_i is a $(k, k-1)$-kernel of H_i, for every $i \in \mathcal{I}_1$.

All this together completes the proof. □

Theorem 6. *Let* $k \geq 2$, $l \geq k$ *be integers. A subset* J^* *is a* (k, l)*-kernel of* $D \circ \mathcal{H}$ *if and only if there exists a* (k, l)*-kernel* J *of* D *such that* $J^* = J \cup \bigcup_{i \in \mathcal{I}_1} J_i \cup \bigcup_{j \in \mathcal{I}_2} J_j$*, where* $\mathcal{I}_1 = \{i; d_D(x_i, J) = l\}$ *and* J_i *is a* (k, l)*-kernel of* H_i *for every* $i \in \mathcal{I}_1$*, and* $\mathcal{I}_2 = \{j; k-1 \leq d_D(x_j, J) < l\}$*, where* J_j *is an arbitrary* k*-independent subset of* H_j*, for every* $j \in \mathcal{I}_2$.

Proof. 1. Let $J \subseteq V(D)$ be a (k, l)-kernel of D, $\mathcal{I}_1 = \{i; d_D(x_i, J) = l\}$ and let J_i be a (k, l)-kernel of H_i, for every $i \in \mathcal{I}_1$. Moreover let $\mathcal{I}_2 = \{j; k-1 \leq d_D(x_j, J) < l\}$ and let J_j be an arbitrary k-independent set of H_j, for every $j \in \mathcal{I}_2$. We now show that $J^* = J \cup \bigcup_{i \in \mathcal{I}_1} J_i \cup \bigcup_{j \in \mathcal{I}_2} J_j$ is a (k, l)-kernel of $D \circ \mathcal{H}$. Clearly by Theorem 1 the set J^* is k-independent of $D \circ \mathcal{H}$. Proving analogously as in Theorem 3 we obtain that J^* is an l-dominating set of $D \circ \mathcal{H}$. Consequently J^* is a (k, l)-kernel of $D \circ \mathcal{H}$.

2. Let $J^* \subseteq V(D \circ \mathcal{H})$ be a (k, l)-kernel of $D \circ \mathcal{H}$, where $l \geq k$. By Corollary 1 the set J^* meets $V(D)$ so let $J = J^* \cap V(D)$. Similarly as in Theorem 5

the set J is a (k,l)-kernel of D. Evidently $(J^* \setminus J) \subseteq \bigcup_{i\in(\mathcal{I}_1\cup\mathcal{I}_2)} V(H_i)$, where $\mathcal{I}_1 = \{i; d_D(x_i, J) = l\}$ and $\mathcal{I}_2 = \{j; k-1 \le d_D(x_j, J) < l\}$. To describe the set J^* we consider the following cases:

(2.1) $i \in \mathcal{I}_1$. This means that for the vertex x_i, $i \in \mathcal{I}_1$, holds $d_D(x_i, J) = d_{D\circ\mathcal{H}}(x_i, J^*) = l$. So for every $y_t^i \in V(H_i)$, $t = 1, ..., p_i$ we have that $d_{D\circ\mathcal{H}}(y_t^i, J) = l + 1$. From the fact that J^* is a (k,l)-kernel kernel of $D \circ \mathcal{H}$ immediately follows that $J^* \cap V(H_i) = J_i$ is a (k,l)-kernel of H_i.

(2.2) $j \in \mathcal{I}_2$. It is clear that for every vertex $x_j \in V(D)$, $j \in \mathcal{I}_2$ we have that $k-1 \le d_{D\circ\mathcal{H}}(x_j, J) < l$. Moreover for every $y_m^j \in V(H_j)$, $m = 1, ..., p_j$ immediately follows that $k < d_{D\circ\mathcal{H}}(y_m^j, J^*) = d_{D\circ\mathcal{H}}(y_m^j, J) < l+1$. This implies that a subset $J_j = J^* \cap V(H_j)$ can be an arbitrary k-independent set of $V(H_j)$, for every $j \in \mathcal{I}_2$. All this together gives that $J^* = J \cup \bigcup_{i\in\mathcal{I}_1} J_i \cup \bigcup_{j\in\mathcal{I}_2} J_j$, where J, J_i, J_j are as in the statements of theorem.

Thus the theorem is proved. ☐

Corollary 2. *Let $k \ge 2$, $l \ge k-1$ be integers. A digraph $D \circ \mathcal{H}$ has a (k,l)-kernel, $l \ge k-1$ if and only if there exists a (k,l)-kernel J of a digraph D such that for every $i \in \mathcal{I}_1$, where $\mathcal{I}_1 = \{i; d_D(x_i, J) = l\}$ a digraph H_i has a (k,l)-kernel.*

3. (k,l)-Kernel Perfect Corona of Digraph

From the definition of $D \circ \mathcal{H}$ the following proposition is obvious.

Proposition 1. *Let $\mathcal{I}$ be a set of indexes of vertices belonging to $V(D)$. Every induced subdigraph of $D \circ \mathcal{H}$ is:*

(a) an induced subdigraph of H_i for some $i \in \mathcal{I}$;

(b) an induced subdigraph of D;

(c) a digraph of the form $\tilde{D}\circ\tilde{\mathcal{H}}$, where $\tilde{D}$ is an induced subdigraph of D with $V(\tilde{D}) = \{x_t; t \in \tilde{\mathcal{I}}\}$, where $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ and $\tilde{\mathcal{H}} = (\tilde{H}_t)_{i\in\tilde{\mathcal{I}}}$, where $\tilde{H}_t$ is an induced subdigraph of H_t;

(d) the union of the digraphs from (a), (b), (c).

Theorem 7. *Let $k \ge 2$, $l \ge k-1$ be integers and let $\mathcal{I}$ be a set of indexes of vertices belonging to $V(D)$. A digraph $D \circ \mathcal{H}$ is (k,l)-kernel perfect if and only if D and H_i are (k,l)-kernel perfect, for every $i \in \mathcal{I}$.*

Proof. If a digraph $D \circ \mathcal{H}$ is (k,l)-kernel perfect, then D and H_i, for every $i \in \mathcal{I}$, are (k,l)-kernel perfect as induced subdigraphs of $D \circ \mathcal{H}$. Conversely assume that D and H_i, $i \in \mathcal{I}$ are (k,l)-kernel perfect. By Proposition 1 we need only to prove that $D \circ \mathcal{H}$ has a (k,l)-kernel. Let J be a (k,l)-kernel of the digraph D and let $\mathcal{I}_1 = \{i; d_D(x_i, J) = l\}$. Because every H_i, $i \in \mathcal{I}$ has a (k,l)-kernel, so H_i, $i \in \mathcal{I}_1 \subset \mathcal{I}$ has a (k,l)-kernel and by Corollary 2 the theorem follows. □

4. The Total Number of (k,l)-kernels in the Corona

In this section we calculate the total number of k-independent sets, l-dominating sets and (k,l)-kernels of corona of digraphs.

Theorem 8. *Let $k \geq 2$ be integer. Let D be a digraph on n vertices, $n \geq 1$ and $\mathcal{H} = (H_i)_{i \in \mathcal{I}}$ be a sequence of vertex disjoint digraphs H_i, where $\mathcal{I}$ is a set of indexes of vertices belonging to $V(D)$. Let $\mathcal{S} = \{S_1, ..., S_t\}$, $t \geq 1$ be a family of all nonempty k-independent sets of the digraph D. Let $S_r \in \mathcal{S}$ and $\mathcal{I}_r = \{j; d_D(x_j, S_r) \geq k-1\}$. Then*

$$NkI(D \circ \mathcal{H}) = \sum_{r=1}^{t} f_1(\mathcal{I}_r) + \prod_{i \in \mathcal{I}} NkI(H_i)\,,$$

where

$$f_1(\mathcal{I}_r) = \begin{cases} \prod\limits_{j \in \mathcal{I}_r} NkI(H_j)\,, & \text{if } \mathcal{I}_r \neq \emptyset\,, \\ 1\,, & \text{otherwise}\,. \end{cases}$$

Proof. Let D be a given digraph on n vertices, $n \geq 1$. Assume that $\mathcal{S}_1^*$ is a family of all k-independent sets of $D \circ \mathcal{H}$ which meets $V(D)$ and let $\mathcal{S}_2^*$ be a family of all k-independent sets of $D \circ \mathcal{H}$ which does not meet $V(D)$. Evidently $NkI(D \circ \mathcal{H}) = |\mathcal{S}_1^*| + |\mathcal{S}_2^*|$. Theorem 1 implies that to obtain a k-independent set of $D \circ \mathcal{H}$ from the family $|\mathcal{S}_1^*|$ first we have to obtain a k-independent set of D. Let $\mathcal{S} = \{S_1, ..., S_t\}$, $t \geq 1$ be a family of all nonempty k-independent sets of the digraph D. Assume that $S_r \in \mathcal{S}$ and $\mathcal{I}_r = \{j; d_D(x_j, S_r) \geq k-1\}$. If $\mathcal{I}_r \neq \emptyset$, then in each H_j, $j \in \mathcal{I}_r$ we have to choose an arbitrary k-independent set. Evidently we can do it on $NkI(H_j)$ ways. If $\mathcal{I}_r = \emptyset$, then S_r is a k-independent set from the family $\mathcal{S}_1^*$. Hence from the fundamental combinatorial statements we have $\sum_{r=1}^{t} f_1(\mathcal{I}_r)$ sets belonging to the family $\mathcal{S}_1^*$, where

$$f_1(\mathcal{I}_r) = \begin{cases} \prod_{j\in\mathcal{I}_r} NkI(H_j) & \text{if } \mathcal{I}_r \neq \emptyset\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Moreover Theorem 2 immediately gives that $|\mathcal{S}_2^*| = \prod_{i\in\mathcal{I}} NkI(H_i)$. All this together gives that $NkI(D\circ\mathcal{H}) = \sum_{r=1}^{t} f_1(\mathcal{I}_r) + \prod_{i\in\mathcal{I}} NkI(H_i)$. Thus the theorem is proved. □

Theorem 9. *Let $l \geq 1$ be integer. Let D be a digraph on n vertices and $\mathcal{H} = (H_i)_{i\in\mathcal{I}}$ be a sequence of vertex disjoint digraphs H_i, where $\mathcal{I}$ is the set of indexes of the vertices belonging to $V(D)$ and $|V(H_i)| = p_i$, $p_i \geq 0$. Let $\mathcal{Q} = \{Q_1, ..., Q_t\}$, $t \geq 1$ be a family of all l-dominating sets of the digraph D. Let $Q_r \in \mathcal{Q}$ and $\mathcal{I}_r = \{j; d_D(x_j, S_r) = l\}$. Then*

$$NlD(D\circ\mathcal{H}) = \sum_{r=1}^{t}\left(f_2(\mathcal{I}_r) \prod_{s\in(\mathcal{I}\setminus\mathcal{I}_r)} 2^{p_s} \right),$$

where

$$f_2(\mathcal{I}_r) = \begin{cases} \prod_{j\in\mathcal{I}_r} NlD(H_j) & \textit{if } \mathcal{I}_r \neq \emptyset\,, \\ 1 & \textit{otherwise}\,. \end{cases}$$

Proof. Let D be a given digraph on n vertices and $\mathcal{H} = (H_i)_{i\in\mathcal{I}}$ be a sequence of vertex disjoint digraphs H_i, $i \in \mathcal{I}$ on p_i vertices, $p_i \geq 0$. Let Q^* be a family of all l-dominating sets of $D\circ\mathcal{H}$. Of course $|Q^*| = NlD(D\circ\mathcal{H})$. Theorem 3 implies that to obtain an l-dominating set of $D\circ\mathcal{H}$ first we have to choose an l-dominating set of D. Let $\mathcal{Q} = \{Q_1, ..., Q_t\}$, $t \geq 1$ be a family of all l-dominating sets of the digraph D. Let $Q_r \in \mathcal{Q}$ and $\mathcal{I}_r = \{j; d_D(x_j, Q_r) = l\}$. If $\mathcal{I}_r \neq \emptyset$, then by Theorem 3 in each H_j, $j \in \mathcal{I}_r$ we choose an arbitrary l-dominating set. Of course we can do it on $NlD(H_j)$ ways. If $\mathcal{I}_r = \emptyset$, then Q_r is an l-dominating set from the family Q^*. Moreover Theorem 3 implies that in each H_s, $s \in (\mathcal{I}\setminus\mathcal{I}_r)$ we can choose an arbitrary subset of $V(H_s)$. Evidently we have 2^{p_s} such subsets, so from the fundamental combinatorial statements we have that $NlD(D\circ\mathcal{H}) = \sum_{r=1}^{t}\left(f_2(\mathcal{I}_r) \prod_{s\in(\mathcal{I}\setminus\mathcal{I}_r)} 2^{p_s} \right)$, where

$$f_2(\mathcal{I}_r) = \begin{cases} \prod_{j\in\mathcal{I}_r} NlD(H_j) & \text{if } \mathcal{I}_r \neq \emptyset\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Thus the theorem is proved. □

Using the same method as in Theorem 8 and Theorem 9 we can prove.

Theorem 10. *Let $k \geq 2$, $l \geq k-1$ be integers. Let D be a digraph on n vertices and $\mathcal{H} = (H_i)_{i \in \mathcal{I}}$ be a sequence of vertex disjoint digraphs H_i, where $\mathcal{I}$ be a set of indexes of vertices from $V(D)$. Let $\mathcal{J} = \{J_1, ..., J_t\}$, $t \geq 1$ be a family of all (k,l)-kernels of the digraph D. Let $J_r \in \mathcal{J}$ and $\mathcal{I}_r = \{j; d_D(x_j, J_r) = l\}$ and $\mathcal{I}_r^* = \{s; k-1 \leq d_D(x_s, J_r) < l\}$. Then*

$$NklK(D \circ \mathcal{H}) = \begin{cases} \sum_{r=1}^{t} f_3(\mathcal{I}_r) & \text{if } l = k-1\,, \\ \sum_{r=1}^{t} f_3(\mathcal{I}_r) f_4(\mathcal{I}_r) & \text{if } l \geq k\,, \end{cases}$$

where

$$f_3(\mathcal{I}_r) = \begin{cases} \prod_{j \in \mathcal{I}_r} NlD(H_j) & \text{if } \mathcal{I}_r \neq \emptyset\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

and

$$f_4(\mathcal{I}_r) = \begin{cases} \prod_{s \in \mathcal{I}_r^*} NkI(H_s) & \text{if } \mathcal{I}_r^* \neq \emptyset\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

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