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A VALUATION MODEL FOR PERPETUAL CONVERTIBLE BONDS WITH MARKOV REGIME-SWITCHING MODELS

Na Song[1], Yue Jiao[2], Wai-Ki Ching[3 §], Tak-Kuen Siu[4], Zhen-Yu Wu[5]

[1,2,3]Advanced Modeling and Applied Computing Laboratory
Department of Mathematics
The University of Hong Kong
Pokfulam Road, HONG KONG, P.R. CHINA
[1]e-mail: smynmath@163.com
[2]e-mail: jiaoyue@hkusua.hku.hk
[3]e-mails: wching@hkusua.hku.hk, wkc@maths.hku.hk

[4]Department of Mathematics and Statistics
Curtin University of Technology
Perth, W.A. 6845, AUSTRALIA
e-mail: ktksiu2005@gmail.com

[5]Department of Finance and Management Science
N. Murray Edwards School of Business
University of Saskatchewan
25 Campus Drive, Saskatoon, SK S7N 5A7, CANADA
e-mail: wu@edwards.usask.ca

Abstract: This paper develops a valuation model for a perpetual convertible bond when the price dynamics of the underlying share are governed by continuous-time Markovian regime-switching models. We suppose that the appreciation rate and the volatility of the underlying share are modulated by a continuous-time, finite-state, observable Markov chain. The states of this chain are interpreted as the states of an economy. Here the valuation problem of the perpetual convertible bond can be viewed as that of valuing a perpetual stock loan, or a perpetual American option with time-dependent strike price. With the presence of the regime-switching effect, the market in the model is, in general, incomplete. To provide a convenient method to determine a price kernel for valuation, we employ the regime-switching Esscher transform introduced in Elliott, Chan and Siu (2005) [4]. We then adopt the differential equation

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§Correspondence author

approach in Guo and Zhang (2004) [7] to solve the optimal stopping problem associated with the valuation of the perpetual convertible bond. Numerical examples are presented to illustrate the practical implementation of the proposed model.

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1. Introduction

A convertible bond is a kind of bond, which gives its holder an option to convert the bond in shares of the stock issued by the same firm at a certain pre-specified ratio. It is also called a convertible debenture and is a hybrid financial instrument with both equity and bond features. The contractual structure of a convertible bond may be more complex than a vanilla warrant in the sense that the convertible bond pays a periodic coupon and involves a dual option at the same time. On one hand, the bondholder has the option to convert the bond in shares of common stock at his discretion. On the other hand, the issuing firm has the right to call back the bond for redemption and the bondholder retains the right to convert the bond or to redeem it (see, for example, Brennan and Schwartz (1977) [1]). A common restriction on the call option is usually that the bond may not be called for five years. Since the bondholders' conversion strategy depends on the firm's call strategy, and vice versa, these two optimal strategies must be solved simultaneously. Brennan and Schwartz (1977) [1] showed that the optimal call strategy of an issuer of a convertible is to call the bond when the value of the bond if it is called is identical to the value of the bond if it is not called. So, the optimal call strategy and the optimal conversion strategy can be analyzed from a perspective of a two-person zero-sum game. The value of the bond may be related to an equilibrium of the game. A more simple approach to deal with the pricing problem of a convertible bond is through the relationship between the convertible bond and an American option. Indeed, under certain assumptions about the contractual structure of a convertible bond, the pricing problem of the convertible bond can be transformed into the pricing of an American option with time-dependent strike price, which has its similarity with the fair valuation of a stock loan, or a security load, considered in the paper by Xia and Zhou (2007) [12]. However, as pointed out in Xia and Zhou (2007) [12], the valuation of an American option with time-dependent strike price and a finite maturity is rather complicated and intractable. To simplify the problem and make the problem more mathematically tractable, they assume the maturity of the stock loan to be infinite and consider the valuation problem of a perpetual American option with time-dependent strike price. By establishing the relationship between a perpetual convertible bond and a perpetual American option with time-dependent strike price, the valuation of a convertible bond can be further simplified to that of valuing a perpetual American option with time-dependent strike price or a perpetual stock loan. Due to the relatively long maturity of a convertible bond, the perpetual convertible bond cannot only provides a mathematical idealization of the convertible bond, but also provides a reasonably good proxy for a convertible bond.

In this paper, we develop a valuation model for a perpetual convertible bond when the price dynamics of the underlying share are governed by Markovian regime-switching models. We suppose that the appreciation rate and the volatility of the underlying share are modulated by a continuous-time, finite-state, observable Markov chain. We interpret the states of the chain as the states of an economy. The valuation of the perpetual convertible bond can be viewed as the valuation of a perpetual stock loan, or a perpetual American option with time-dependent strike price. With the regime-switching effect incorporated, the market, in general, is incomplete. Hence, there is more than one price kernel or stochastic discount factor. To provide a convenient way to determine a price kernel, we therefore employ the regime-switching Esscher transform introduced in Elliott, Chan and Siu (2005) [4]. Indeed, the use of the Esscher transform to specify a price kernel can be justified by the Minimal Entropy Martingale Measure (MEMM) in the context of the regime-switching model (see Elliott, Chan and Siu (2005)[4]). Instead of adopting the probabilistic approach to solve the valuation problem, which has been fully explored in Xia and Zhou (2007) [12], we employ the differential equation approach in Guo and Zhang (2004) [7] to solve the optimal stopping problem associated with the valuation problem. We illustrate the practical implementation of the proposed model via some numerical examples.

The next section establishes the relationship between a perpetual convertible bond and a perpetual American option with time-dependent strike price. Section 3 presents the asset price dynamics governed by the Markovian regime-switching model. In Section 4, we illustrate how to use the regime-switching Esscher transform to determine a price kernel in the incomplete market. Section 5 provides a solution to the valuation problem using the differential equation approach in Guo and Zhang (2004) [7]. Numerical examples are then presented in Section 6. The final section summarizes the paper.

2. Perpetual American Option with Time-Dependent Strike Price

In this section, we establish the link between a perpetual convertible bond and a perpetual American option with time-dependent strike price. Firstly, we establish the link between a convertible bond and an American option with time-dependent strike price with finite maturity. Then the perpetual case can be obtained by letting the maturity tend to infinite.

Assume that a bond has a maturity T, with an annual interest rate c payable at the end of each year. Thus the interest rate c is referred to as the coupon rate in finance, and the market discount rate is r determined by the market. The face value is \$1000, and it will be paid back to bondholders at the maturity T if they do not want to convert the bond in a share of the issuing firm. If there is no possibility to be converted to a share, but remains a regular bond, the current selling price of the bond can be easily calculated using *MS Excel* or a financial calculator. When it is a convertible bond, however, the bondholders can maximize their benefits by choosing to become shareholders.

In a special case that the coupon rate $c = 0$, it is clear that the current selling price of zero-coupon bond should be $B_0 = Fe^{-rT}$, where F is the face value of the bond and is a constant.

As a convertible bond, however, the current selling price should be $CB_0 = Fe^{-rT} + C$, where $C > 0$ is the price of the American call option embedded in the convertible bond. This option gives the bondholders the right, but not the obligation, to convert bonds into shares at any time point t before the maturity T of the convertible bond. In other words, when the bondholders exercise this option, they are converted to shareholders of the issuing firms. Therefore, the strike price of this American call option, the number of shares converted to, is time-dependent.

At a time point t, $t \in (0, T]$, the payoff of this American call option with time-dependent strike price B_0e^{rt} is $Y(t) = (S(t) - B_0e^{rt})_+$, assuming that one bond can be converted to one share. Here $(x)_+ = \max\{x, 0\}$. $Y(t)$ is called the intrinsic value of the American call option at time t. By letting $T \to \infty$, the relationship between the perpetual convertible bond and the perpetual American option becomes clear.

3. The Model

We consider a continuous-time economy with two primitive securities, namely, a risk-free money market account B and a risky share S. These securities can be traded continuously over time. We also assume a perfect market where there is no transaction cost, income tax and capital gain tax; there is no restriction on short-selling and assets are divisible. We fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ represents a real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, \infty)$ of the economy. We assume that the states of the economy are modeled by a continuous-time, finite-state, observable Markov chain $X := \{X_t\}_{t\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{X} := (x_1, x_2 \ldots, x_N)$. Following Elliott, Aggoun and Moore (1994) [6], we can write X as its canonical representation with loss of generality; that is, the state space of X is a finite set of unit vectors $\{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathcal{R}^{\mathcal{N}}$. Let $Q(t)$ for the Q-matrix or the generator $[q_{ij}(t)]_{i,j=1,2,\ldots,N}$ of the Markov chain X. Then, under the canonical representation for X, Elliott (1993) [5] and Elliott, Aggoun and Moore (1994) [6] derived the following semi-martingale representation theorem for X:

$$X_t = X_0 + \int_0^t QX_s ds + M_t, \tag{3.1}$$

where $\{M_t\}_{t\in\mathcal{T}}$ is an $\mathcal{R}^N$-valued martingale with respect to the right-continuous, $\mathcal{P}$-completed, filtration generated by X.

Let r be the constant instantaneous market interest rate of the money market account, where $r > 0$. Then, the evolution of the bond price is governed by: $B_t = e^{rt}$ and $t \in \mathcal{T}$.

We suppose that the appreciation rate $\{\mu_t\}_{t\in\mathcal{T}}$ and the volatility rate $\{\sigma_t\}_{t\in\mathcal{T}}$ of S also depend on X and are given by: $\mu_t := \mu(t, X_t) = \langle \mu, X_t \rangle$, $\sigma_t := \sigma(t, X_t) = \langle \sigma, X_t \rangle$ where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)$ and $\sigma := (\sigma_1, \sigma_2 \ldots, \sigma_N)$ with $\sigma_i > 0$ for each $i = 1, 2, \ldots, N$.

Let $W := \{W_t\}_{t\in\mathcal{T}}$ denote a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. We suppose that X and W are stochastically independent. The share price process S is then governed by the following Markov-modulated geometric Brownian motion (GBM):

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \ , \quad S_0 = s \ . \tag{3.2}$$

4. The Valuation Method

In this section, we determine a price kernel by the regime-switching Esscher transform in Elliott, Chan and Siu (2005) [4].

First, let Z_t denote the logarithmic return $\ln(S_t/S_0)$ from S over the interval $[0, t]$. Write $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t\in\mathcal{T}}$ and $\mathcal{F}^Z := \{\mathcal{F}_t^Z\}_{t\in\mathcal{T}}$ for the right-continuous, $\mathcal{P}$-completed, filtration generated by X and Z, respectively. For each $t \in \mathcal{T}$, $\mathcal{G}_t$ is defined as the σ-algebra $\mathcal{F}_t^X \vee \mathcal{F}_t^Z$. Let $\theta_t := \theta(t, X_t)$ denote the regime switching Esscher parameter, which can be written as follows:

$$\theta(t, X_t) = \langle\theta, X_t\rangle = \sum_{i=1}^{N} \theta_i \langle X_t, e_i\rangle \ , \tag{4.1}$$

where $\theta := (\theta_1, \theta_2, \ldots, \theta_N) \in \mathcal{R}^N$.

Then, as in Elliott, Chan and Siu (2005) [4], the regime switching Esscher transform $\mathcal{Q}_\theta \sim \mathcal{P}$ on $\mathcal{G}_t$ with respect to $\{\theta_s\}_{s\in[0,t]}$ is defined as:

$$\left.\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\right|_{\mathcal{G}_t} := \frac{\exp\left(\int_0^t \theta_s dZ_s\right)}{E_{\mathcal{P}}\left[\exp\left(\int_0^t \theta_s dZ_s\right)\middle|\mathcal{F}_t^X\right]}$$
$$= \exp\left(\int_0^t \theta_s\sigma_s dW_s - \frac{1}{2}\int_0^t \theta_s^2\sigma_s^2 ds\right), \tag{4.2}$$

where $t \in \mathcal{T}$. Harrison and Kreps (1979) [8], Harrison and Pliska (1981, 1983) [9, 10] established the relationship between the absence of arbitrage and the existence of equivalent martingale measures. This is called the fundamental theorem of asset pricing and extended by some authors in more general settings. Delbaen and Schachermayer (1994) [3] showed that the absence of arbitrage opportunities is "essentially" equivalent to the existence of an equivalent martingale measure under which the discounted stock price process is a martingale. We call the second statement a martingale condition.

For each $t, s \in \mathcal{T}$ with $s \leq t$, let $\tilde{\mathcal{G}}_{t,s}$ denote a double-index σ-field defined by $\tilde{\mathcal{G}}_{t,s} := \mathcal{F}_t^X \vee \mathcal{F}_s^Z$. Write $\tilde{\mathcal{G}}$ for the filtration $\{\tilde{\mathcal{G}}_{t,s}\}_{t\in\mathcal{T},s\in[0,t]}$. Then the martingale condition in our setting is that the discounted stock price process is a martingale with respect to $\tilde{\mathcal{G}}$ under a risk-neutral Esscher transform, denoted as $\mathcal{Q}_{\tilde{\theta}}$. In particular,

$$S_0 = E_{\mathcal{Q}_{\tilde{\theta}}}[e^{-rt}S_t|\mathcal{F}_t^X] \ , \quad \text{for any } t \in \mathcal{T} \ , \tag{4.3}$$

where $\{\tilde{\theta}_t\}_{t\in\mathcal{T}}$ denote the family of risk-neutral regime-switching Esscher parameters.

Then, for each $t \in \mathcal{T}$, Elliott, Chan and Siu (2005) [4] determined the risk-neutral regime-switching Esscher parameter $\tilde{\theta}_t$ uniquely from the martingale condition as follows:

$$\tilde{\theta}_t = \left\langle \tilde{\theta}, X_t \right\rangle = \frac{r - \mu_t}{\sigma_t^2} = \sum_{i=1}^{N} \left(\frac{r - \mu_i}{\sigma_i^2} \right) \langle X_t, e_i \rangle \ . \tag{4.4}$$

Hence, the Radon-Nikodym derivative of $\mathcal{Q}_{\tilde{\theta}}$ is given by:

$$\left. \frac{d\mathcal{Q}_{\tilde{\theta}}}{d\mathcal{P}} \right|_{\mathcal{G}_t} = \exp \left[\int_0^t \left(\frac{r - \mu_s}{\sigma_s} \right) dW_s - \frac{1}{2} \int_0^t \left(\frac{r - \mu_s}{\sigma_s} \right)^2 ds \right] . \tag{4.5}$$

By Girsanov's Theorem, $\tilde{W}_t = W_t + \int_0^t \left(\frac{r-\mu_s}{\sigma_s} \right) ds$ is a standard Brownian motion with respect to $\tilde{\mathcal{G}}$ under $\mathcal{Q}_{\tilde{\theta}}$. Hence, the stock price dynamic under $\mathcal{Q}_{\tilde{\theta}}$ is governed by:

$$dS_t = rS_t dt + \sigma_t S_t d\tilde{W}_t \ . \tag{4.6}$$

Write τ for a stopping time on $(\Omega, \mathcal{F}, \mathcal{P})$. So given $\tilde{\mathcal{G}}_{\infty,t}$, a conditional price of a perpetual convertable bond at time t is:

$$V(t) := \operatorname*{ess\,sup}_{\tau \geq t} E_{\mathcal{Q}_{\tilde{\theta}}}[e^{-r(\tau - t)}(S(\tau) - B(\tau))^+ | \tilde{\mathcal{G}}_{\infty,t}] \ ,$$

where $B(\tau) := B_0 e^{r\tau}$, and B_0 is the initial bond price.

Let $V(S, X)$ denote the price of a perpetual convertible bond given $S_t = S$ and $X_t = X$. Then, $V(S, X)$ is given by:

$$V^p(S, X) = \operatorname*{ess\,sup}_{\tau \geq t} E_{\mathcal{Q}_{\tilde{\theta}}}[e^{-r(\tau - t)}(S(\tau) - B(\tau))^+ | (S_t, X_t) = (S, X)] \,. \tag{4.7}$$

Let $f(t, S, X) := (S(t) - B(t))^+$. Then we have

$$V^p(S, X) = \operatorname*{ess\,sup}_{\tau \geq t} E_{\mathcal{Q}_{\tilde{\theta}}}[e^{-r(\tau - t)} f(\tau, S, X) | (S_t, X_t) = (S, X)]. \tag{4.8}$$

5. Solution to the Pricing Problem

We adopt the differential equation approach in Guo and Zhang (2004) [7] to solve the optimal stopping problem associated with the valuation of perpetual convertible bonds presented in the last section. Note that in [7] a put option was considered while here the convertible bond pricing resembles a call option.

Let $\mathcal{C}^p$ and $\mathcal{S}^p$ denote the continuation region and the stopping region, respectively, which are defined as follows:

$$\mathcal{C}^p := \{(S,X)|V^p(S,X) > f(S,X)\}$$

and

$$\mathcal{S}^p := \{(S,X)|V^p(S,X) = f(S,X)\}.$$

We let $\tau^p := \inf\{u \geq t|(S_u, X_u) \in \overline{\mathcal{C}^p}\}$, where $\overline{\mathcal{C}^p}$ is the complement of the set $\mathcal{C}^p$. Since (S_u, X_u) is a two-dimensional Markov process with respect to the enlarged filtration $\mathcal{G}$, the optimal stopping rule is τ^p; i.e.,

$$V^p(S,X) = E_{\mathcal{Q}_{\tilde{\theta}}}[e^{-r(\tau^p - t)} f(\tau^p, S, X)|(S_t, X_t) = (S,X)]. \tag{4.9}$$

We note that $V^p(S,X)$ and $f(t,S,X)$ are increasing function of S, for each fixed X and t. As in Guo and Zhang (2004) [7], there exists two threshold levels $S_1^p, S_2^p \leq F$, where F is the face value of the convertible, such that the continuation region $\mathcal{C}^p$ can be represented by:

$$\mathcal{C}^p = \{(S,e_1)|S \in (0,S_1^p)\} \cup \{(S,e_2)|S \in (0,S_2^p)\}\ . \tag{4.10}$$

There are three possible cases, namely, (I) $S_1^p < S_2^p$, (II) $S_1^p > S_2^p$ and (III) $S_1^p = S_2^p$.

We follow the mathematical treatment in Guo and Zhang [7] and consider the following three cases:

Case I. $S_1^p < S_2^p$. When $S \in [S_1^p, S_2^p]$: Let $V_i^p := V^p(S, e_i)$, for $i = 1,2$; $\mathbf{V}^p := (V_1^p, V_2^p)$. Then, by Itô's differentiation rule, $\mathbf{V}^p$ satisfies:

$$\left\{ \begin{array}{rcl} V_1^p &=& S - B(t)\,, \\ q_{22}V_2^p &=& rS\frac{\partial V_2^p}{\partial S} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 V_2^p}{\partial S^2} + q_{22}(S - B(t)). \end{array} \right. \tag{4.11}$$

When $S \in [0, S_1^p]$:

$$\left\{ \begin{array}{rcl} q_{11}V_1^p &=& rS\frac{\partial V_1^p}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1^p}{\partial S^2} + q_{11}V_2^p\,, \\ q_{22}V_2^p &=& rS\frac{\partial V_2^p}{\partial S} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 V_2^p}{\partial S^2} + q_{22}V_1^p\,. \end{array} \right. \tag{4.12}$$

When $S \in [S_2^p, \infty]$:

$$V_1^p = V_2^p = S - B(t). \tag{4.13}$$

Now, we consider equation (4.11). The equation is an inhomogeneous equation whose solution can be written as

$$V_2(S) = C_1 S^{\gamma_1} + C_2 S^{\gamma_2} + \phi(S)\,, \tag{4.14}$$

where $\phi(S)$ is a special solution and γ_1, γ_2 are the two real roots of

$$r\gamma + \frac{1}{2}\sigma_2^2\gamma(\gamma - 1) = q_{22}. \tag{4.15}$$

In particular, when $q_{22}-r \neq 0$, one can choose

$$\phi(S) = \frac{q_{22}S}{q_{22}-r} - B(t). \tag{4.16}$$

Next, we consider equation (4.12). It has an associated characteristic function

$$g_1(\beta)g_2(\beta) = q_{11}q_{22}\,, \tag{4.17}$$

where

$$\begin{cases} g_1(\beta) &= q_{11} - (r - \frac{1}{2}\sigma_1^2)\beta - \frac{1}{2}\sigma_1^2\beta^2\,, \\ g_2(\beta) &= q_{22} - (r - \frac{1}{2}\sigma_2^2)\beta - \frac{1}{2}\sigma_2^2\beta^2\,. \end{cases}$$

Moreover, this characteristic function has four distinct roots, the general form of the solution is given by

$$\begin{cases} V_1(S) &= \sum_{i=1}^4 A_i S^{\beta_i}\,, \\ V_2(S) &= \sum_{i=1}^4 B_i S^{\beta_i}\,, \end{cases} \tag{4.18}$$

with

$$B_i = l_i A_i \quad \text{and} \quad l_i = l(\beta_i) = \frac{g_1(\beta_i)}{q_{11}} = \frac{q_{22}}{g_2(\beta_i)}. \tag{4.19}$$

We note that when $S \to 0$, we have $v_1^p \to 0$ and $v_2^p \to 0$. While when $S \to \infty$, we have $v_1^p \to \infty$ and $v_2^p \to \infty$. Thus, we ignore the negative powers of S from the solution. Let

$$b_i = r - \frac{1}{2}\sigma_i^2 \quad \text{and} \quad a_i = \frac{1}{2}\sigma_i^2.$$

Then, we have

$$\begin{cases} q_{11}q_{22} &= g_1(\beta)g_2(\beta)\,, \\ q_{11}q_{22} &= (a_1\beta^2 + b_1\beta - q_{11})(a_2\beta^2 + b_2\beta - q_{22})\,, \\ 0 &= (a_1\beta^2 + b_1\beta)(a_2\beta^2 + b_2\beta) - q_{11}(a_2\beta^2 + b_2\beta) - q_{22}(a_1\beta^2 + b_1\beta)\,, \\ 0 &= \beta[a_1a_2\beta^3 + (a_1b_2 + a_2b_1)\beta^2 + (b_1b_2 - q_{11}a_2 - q_{22}a_1)\beta \\ & \quad -(q_{11}b_2 + q_{22}b_1)]\,. \end{cases}$$

There are two cases to be considered.

Case I(A). When $q_{11}(r - \frac{1}{2}\sigma_2^2) + q_{22}(r - \frac{1}{2}\sigma_1^2) \geq 0$, we have $\beta_4 < \beta_3 < \beta_2 = 0 < \beta_1$. Then

$$\begin{cases} V_1^p &= A_1 S^{\beta_1} + A_2\,, \\ V_2^p &= B_1 S^{\beta_1} + B_2\,. \end{cases} \tag{4.20}$$

Now, we solve for $A_1, A_2, C_1, C_2, S_1^p$, and S_2^p. To this end, appropriate boundary conditions are needed. Applying the smooth fit at S_1^p, conditions

$$V_1^p(S_1^p-) = V_1^p(S_1^p+) \quad \text{and} \quad V_1^{p\prime}(S_1^p-) = V_1^{p\prime}(S_1^p+)$$

suggest that

$$\begin{cases} A_1(S_1^p)^{\beta_1} + A_2 & = & S_1^p - B(t)\,, \\ \beta_1 A_1(S_1^p)^{\beta_1} & = & S_1^p\,. \end{cases}$$

Similarly, the smoothness of V_2^p at S_1^p and S_2^p yields

$$\begin{cases} l_1 A_1(S_1^p)^{\beta_1} + l_2 A_2 & = & C_1(S_1^p)^{\gamma_1} + C_2(S_1^p)^{\gamma_2} + \phi(S_1^p)\,, \\ \beta_1 l_1 A_1(S_1^p)^{\beta_1} & = & \gamma_1 C_1(S_1^p)^{\gamma_1} + \gamma_2 C_2(S_1^p)^{\gamma_2} + S_1^p \phi'(S_1^p)\,, \end{cases}$$

and

$$\begin{cases} C_1(S_2^p)^{\gamma_1} + C_2(S_2^p)^{\gamma_2} + \phi(S_2^p) & = & S_2^p - B(t)\,, \\ \gamma_1 C_1(S_2^p)^{\gamma_1} + \gamma_2 C_2(S_2^p)^{\gamma_2} + S_2^p \phi'(S_2^p) & = & S_2^p\,. \end{cases}$$

Combining the above equations and with some algebraic manipulations, we obtain an algebraic equation for S_1^p and S_2^p:

$$\begin{pmatrix} (S_1^p)^{-\gamma_1} & 0 \\ 0 & (S_1^p)^{-\gamma_2} \end{pmatrix} F_1(S_1^p) = \begin{pmatrix} (S_2^p)^{-\gamma_1} & 0 \\ 0 & (S_2^p)^{-\gamma_2} \end{pmatrix} F_2(S_2^p),$$

where

$$F_1(S_1^p) =$$

$$\begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} l_1 & l_2 \\ \beta_1 l_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} S_1^p - B(t) \\ S_1^p \end{pmatrix} - \begin{pmatrix} \phi(S_1^p) \\ S_1^p \phi'(S_1^p) \end{pmatrix} \right]$$

and

$$F_2(S_2^p) = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} S_2^p - B(t) - \phi(S_2^p) \\ S_2^p - S_2^p \phi'(S_2^p) \end{pmatrix}.$$

Finally, the coefficients are given by

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} (S_1^p)^{\beta_1} & 1 \\ \beta_1 (S_1^p)^{\beta_1} & 0 \end{pmatrix}^{-1} \begin{pmatrix} S_1^p - B(t) \\ S_1^p \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} l_1 A_1 \\ l_2 A_2 \end{pmatrix},$$

and

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} (S_2^p)^{\gamma_1} & (S_2^p)^{\gamma_2} \\ \gamma_1 (S_2^p)^{\gamma_1} & \gamma_2 (S_2^p)^{\gamma_2} \end{pmatrix}^{-1} \begin{pmatrix} S_2^p - B(t) - \phi(S_2^p) \\ S_2^p - S_2^p \phi'(S_2^p) \end{pmatrix}.$$

With these coefficients, the value function becomes

$$V_1^p = \begin{cases} A_1 S^{\beta_1} + A_2 & \text{if } S \le S_1^p\,, \\ S - B(t) & \text{if } S > S_1^p\,, \end{cases}$$

$$V_2^p = \begin{cases} B_1 S^{\beta_1} + B_2 & \text{if } S \le S_1^p\,, \\ C_1 S^{\gamma_1} + C_2 S^{\gamma_2} + \phi(S) & \text{if } S_1^p < S \le S_2^p\,, \\ S - B(t) & \text{if } S > S_2^p\,. \end{cases}$$

Case I(B). When $q_{11}(r_2 - \frac{1}{2}\sigma_2^2) + q_{22}(r_1 - \frac{1}{2}\sigma_1^2) < 0$, we have

$\beta_4 < \beta_3 = 0 < \beta_2 < \beta_1$. Then we have

$$\left\{ \begin{array}{lcl} V_1^p & = & A_1 S^{\beta_1} + A_2 S^{\beta_2}\,, \\ V_2^p & = & B_1 S^{\beta_1} + B_2 S^{\beta_2}\,. \end{array} \right. \tag{4.21}$$

Now, we wish to solve for $A_1, A_2, C_1, C_2, S_1^p$, and S_2^p. To this end, appropriate boundary conditions are needed. Applying the smooth fit at S_1^p, conditions $V_1^p(S_1^p-) = V_1^p(S_1^p+)$ and $V_1^{p\prime}(S_1^p-) = V_1^{p\prime}(S_1^p+)$ suggest that

$$\left\{ \begin{array}{lcl} A_1(S_1^p)^{\beta_1} + A_2(S_1^p)^{\beta_2} & = & S_1^p - B(t)\,, \\ \beta_1 A_1(S_1^p)^{\beta_1} + \beta_2 A_2(S_1^p)^{\beta_2} & = & S_1^p\,. \end{array} \right.$$

Similarly, the smoothness of V_2^p at S_1^p and S_2^p yields

$$\left\{ \begin{array}{lcl} l_1 A_1(S_1^p)^{\beta_1} + l_2 A_2(S_1^p)^{\beta_2} & = & C_1(S_1^p)^{\gamma_1} + C_2(S_1^p)^{\gamma_2} + \phi(S_1^p)\,, \\ \beta_1 l_1 A_1(S_1^p)^{\beta_1} + \beta_2 l_2 A_2(S_1^p)^{\beta_2} & = & \gamma_1 C_1(S_1^p)^{\gamma_1} + \gamma_2 C_2(S_1^p)^{\gamma_2} + S_1^p\phi'(S_1^p)\,, \end{array} \right.$$

and

$$\left\{ \begin{array}{lcl} C_1(S_2^p)^{\gamma_1} + C_2(S_2^p)^{\gamma_2} + \phi(S_2^p) & = & S_2^p - B(t)\,, \\ \gamma_1 C_1(S_2^p)^{\gamma_1} + \gamma_2 C_2(S_2^p)^{\gamma_2} + S_2^p\phi'(S_2^p) & = & S_2^p\,. \end{array} \right.$$

Combining the above three equations and with some algebraic manipulation, we obtain an algebraic equation for S_1^p and S_2^p:

$$\begin{pmatrix} (S_1^p)^{-\gamma_1} & 0 \\ 0 & (S_1^p)^{-\gamma_2} \end{pmatrix} F_1(S_1^p) = \begin{pmatrix} (S_2^p)^{-\gamma_1} & 0 \\ 0 & (S_2^p)^{-\gamma_2} \end{pmatrix} F_2(S_2^p),$$

where

$$F_1(S_1^p) = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \times \left[\begin{pmatrix} l_1 & l_2 \\ \beta_1 l_1 & \beta_2 l_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix}^{-1} \begin{pmatrix} S_1^p - B(t) \\ S_1^p \end{pmatrix} - \begin{pmatrix} \phi(S_1^p) \\ S_1^p\phi'(S_1^p) \end{pmatrix} \right],$$

and

$$F_2(S_2^p) = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} S_2^p - B(t) - \phi(S_2^p) \\ S_2^p - S_2^p\phi'(S_2^p) \end{pmatrix}.$$

Finally, the coefficients are given by

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} (S_1^p)^{\beta_1} & (S_1^p)^{\beta_2} \\ \beta_1(S_1^p)^{\beta_1} & \beta_2(S_1^p)^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} S_1^p - B(t) \\ S_1^p \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} l_1 A_1 \\ l_2 A_2 \end{pmatrix},$$

and

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} (S_2^p)^{\gamma_1} & (S_2^p)^{\gamma_2} \\ \gamma_1(S_2^p)^{\gamma_1} & \gamma_2(S_2^p)^{\gamma_2} \end{pmatrix}^{-1} \begin{pmatrix} S_2^p - B(t) - \phi(S_2^p) \\ S_2^p - S_2^p\phi'(S_2^p) \end{pmatrix}.$$

With these coefficients, the value function becomes

$$V_1^p = \begin{cases} A_1 S^{\beta_1} + A_2 S^{\beta_2} & \text{if } S \le S_1^p\,, \\ S - B(t) & \text{if } S > S_1^p\,, \end{cases}$$
$$V_2^p = \begin{cases} B_1 S^{\beta_1} + B_2 S^{\beta_2} & \text{if } S \le S_1^p\,, \\ C_1 S^{\gamma_1} + C_2 S^{\gamma_2} + \phi(S) & \text{if } S_1^p < S \le S_2^p\,, \\ S - B(t) & \text{if } S > S_2^p\,. \end{cases}$$

Case II. $S_1^p > S_2^p$. When $S \in [S_2^p, S_1^p]$: Let $V_i^p := V^p(S, e_i)$, for $i = 1, 2$; $\mathbf{V}^p := (V_1^p, V_2^p)$. Then, by Itô's differentiation rule, $\mathbf{V}^p$ satisfies:

$$\begin{cases} q_{11} V_1^p = rS\frac{\partial V_1^p}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1^p}{\partial S^2} + q_{11}(S - B(t))\,, \\ V_2^p = S - B(t)\,. \end{cases} \tag{4.22}$$

When $S \in [0, S_2^p]$:

$$\begin{cases} q_{11} V_1^p = rS\frac{\partial V_1^p}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1^p}{\partial S^2} + q_{11} V_2^p\,, \\ q_{22} V_2^p = rS\frac{\partial V_2^p}{\partial S} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 V_2^p}{\partial S^2} + q_{22} V_1^p\,. \end{cases} \tag{4.23}$$

When $S \in [S_1^p, \infty]$:

$$V_1^p = V_2^p = S - B(t). \tag{4.24}$$

Now, we consider equation (4.22). The first equation is an inhomogeneous equation whose solution can be written as

$$V_1(S) = C_1 S^{\tilde{\gamma}_1} + C_2 S^{\tilde{\gamma}_2} + \tilde{\phi}(S)\,, \tag{4.25}$$

where $\tilde{\phi}(S)$ is a special solution and $\tilde{\gamma}_1, \tilde{\gamma}_2$ are the two real roots of

$$r\gamma + \frac{1}{2}\sigma_1^2 \gamma(\gamma - 1) = q_{11}. \tag{4.26}$$

In particular, when $q_{11} - r \ne 0$, one can choose

$$\tilde{\phi}(S) = \frac{q_{11} S}{q_{11} - r} - B(t).$$

Next, we consider (4.23). It is analogous to that of $S_1^p < S_2^p$. There are two cases to be considered.

Case II(A). If $q_{11}(r - \frac{1}{2}\sigma_2^2) + q_{22}(r - \frac{1}{2}\sigma_1^2) \ge 0$, we have $\beta_4 < \beta_3 < \beta_2 = 0 < \beta_1$. Then

$$\begin{cases} V_1^p = A_1 S^{\beta_1} + A_2\,, \\ V_2^p = B_1 S^{\beta_1} + B_2\,. \end{cases} \tag{4.27}$$

Now, we solve for A_1, C_1, C_2, S_1^p, and S_2^p. To this end, appropriate boundary conditions are needed. Applying the smooth fit at S_2^p, the conditions

$$V_2^p(S_2^p-) = V_2^p(S_2^p+) \quad \text{and} \quad V_2^{p\prime}(S_2^p-) = V_2^{p\prime}(S_2^p+)$$

suggest that

$$\begin{cases} l_1 A_1 (S_2^p)^{\beta_1} + l_2 A_2 &= S_2^p - B(t)\,, \\ \beta_1 l_1 A_1 (S_2^p)^{\beta_1} &= S_2^p\,. \end{cases}$$

Similarly, the smoothness of V_1^p at S_2^p and S_1^p yields

$$\begin{cases} A_1 (S_2^p)^{\beta_1} + A_2 &= C_1 (S_2^p)^{\tilde{\gamma}_1} + C_2 (S_2^p)^{\tilde{\gamma}_2} + \tilde{\phi}(S_2^p)\,, \\ \beta_1 A_1 (S_2^p)^{\beta_1} &= \tilde{\gamma}_1 C_1 (S_2^p)^{\tilde{\gamma}_1} + \tilde{\gamma}_2 C_2 (S_2^p)^{\tilde{\gamma}_2} + S_2^p \tilde{\phi}'(S_2^p)\,, \end{cases}$$

and

$$\begin{cases} C_1 (S_1^p)^{\tilde{\gamma}_1} + C_2 (S_1^p)^{\tilde{\gamma}_2} + \tilde{\phi}(S_1^p) &= S_1^p - B(t)\,, \\ \tilde{\gamma}_1 C_1 (S_1^p)^{\tilde{\gamma}_1} + \tilde{\gamma}_2 C_2 (S_1^p)^{\tilde{\gamma}_2} + S_1^p \tilde{\phi}'(S_1^p) &= S_1^p\,. \end{cases}$$

Combining the above three equations and with some algebraic manipulations, we obtain an algebraic equation for S_2^p:

$$\begin{pmatrix} (S_1^p)^{-\tilde{\gamma}_1} & 0 \\ 0 & (S_1^p)^{-\tilde{\gamma}_2} \end{pmatrix} \tilde{F}_1(S_1^p) = \begin{pmatrix} (S_2^p)^{-\tilde{\gamma}_1} & 0 \\ 0 & (S_2^p)^{-\tilde{\gamma}_2} \end{pmatrix} \tilde{F}_2(S_2^p),$$

where

$$\tilde{F}_1(S_1^p) = \begin{pmatrix} 1 & 1 \\ \tilde{\gamma}_1 & \tilde{\gamma}_2 \end{pmatrix}^{-1} \begin{pmatrix} S_1^p - B(t) - \tilde{\phi}(S_1^p) \\ S_1^p - S_1^p \tilde{\phi}'(S_1^p) \end{pmatrix},$$

and

$$\tilde{F}_2(S_2^p) =$$

$$\begin{pmatrix} 1 & 1 \\ \tilde{\gamma}_1 & \tilde{\gamma}_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ \beta_1 l_1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} S_2^p - B(t) \\ S_2^p \end{pmatrix} - \begin{pmatrix} \tilde{\phi}(S_2^p) \\ S_2^p \tilde{\phi}'(S_2^p) \end{pmatrix} \right].$$

Finally, the coefficients are given by

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} l_1 (S_2^p)^{\beta_1} & l_2 \\ \beta_1 l_1 (S_2^p)^{\beta_1} & 0 \end{pmatrix}^{-1} \begin{pmatrix} S_2^p - B(t) \\ S_2^p \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} l_1 A_1 \\ l_2 A_2 \end{pmatrix}$$

and

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} (S_1^p)^{\tilde{\gamma}_1} & (S_1^p)^{\tilde{\gamma}_2} \\ \tilde{\gamma}_1 (S_1^p)^{\tilde{\gamma}_1} & \tilde{\gamma}_2 (S_1^p)^{\tilde{\gamma}_2} \end{pmatrix}^{-1} \begin{pmatrix} S_1^p - B(t) - \tilde{\phi}(S_1^p) \\ S_1^p - S_1^p \tilde{\phi}'(S_1^p) \end{pmatrix}.$$

With these coefficients, the value functions are given by

$$V_1^p = \begin{cases} A_1 S^{\beta_1} + A_2 & \text{if } S \le S_2^p\,, \\ C_1 S^{\tilde{\gamma}_1} + C_2 S^{\tilde{\gamma}_2} + \tilde{\phi}(S) & \text{if } S_2^p < S \le S_1^p\,, \\ S - B(t) & \text{if } S > S_1^p\,, \end{cases} \tag{4.28}$$

and

$$V_2^p = \begin{cases} B_1 S^{\beta_1} + B_2 & \text{if } S \le S_2^p\,, \\ S - B(t) & \text{if } S > S_2^p\,. \end{cases} \tag{4.29}$$

Case II(B). If $q_{11}(r_2 - \frac{1}{2}\sigma_2^2) + q_{22}(r_1 - \frac{1}{2}\sigma_1^2) < 0$, we have $\beta_4 < \beta_3 = 0 <$

$\beta_2 < \beta_1$ then

$$\begin{cases} V_1^p &= A_1 S^{\beta_1} + A_2 S^{\beta_2}\,, \\ V_2^p &= B_1 S^{\beta_1} + B_2 S^{\beta_2}\,. \end{cases}$$

This case is similar to Case II(A), so we summarize the results below. The value functions are given by

$$V_1^p = \begin{cases} A_1 S^{\beta_1} + A_2 S^{\beta_2} & \text{if } S \le S_2^p\,, \\ C_1 S^{\tilde{\gamma}_1} + C_2 S^{\tilde{\gamma}_2} + \tilde{\phi}(S) & \text{if } S_2^p < S \le S_1^p\,, \\ S - B(t) & \text{if } S > S_1^p\,, \end{cases} \tag{4.30}$$

and

$$V_2^p = \begin{cases} B_1 S^{\beta_1} + B_2 S^{\beta_2} & \text{if } S \le S_2^p\,, \\ S - B(t) & \text{if } S > S_2^p\,, \end{cases} \tag{4.31}$$

where

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} l_1 (S_2^p)^{\beta_1} & l_2 (S_2^p)^{\beta_2} \\ l_1 \beta_1 (S_2^p)^{\beta_1} & l_2 \beta_2 (S_2^p)^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} S_2^p - B(t) \\ S_2^p \end{pmatrix},$$

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} l_1 A_1 \\ l_2 A_2 \end{pmatrix},$$

and

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} (S_1^p)^{\tilde{\gamma}_1} & (S_1^p)^{\tilde{\gamma}_2} \\ \tilde{\gamma}_1 (S_1^p)^{\tilde{\gamma}_1} & \tilde{\gamma}_2 (S_1^p)^{\tilde{\gamma}_2} \end{pmatrix}^{-1} \begin{pmatrix} S_1^p - B(t) - \tilde{\phi}(S_1^p) \\ S_1^p - S_1^p \tilde{\phi}'(S_1^p). \end{pmatrix}.$$

Case III. $S_1^p = S_2^p = \tilde{S}$. When $q_{11}(r - \frac{1}{2}\sigma_2^2) + q_{22}(r - \frac{1}{2}\sigma_1^2) \ge 0$, we have, for $S \le \tilde{S}$

$$\begin{cases} V_1^p &= A_1 S^{\beta_1} + A_2\,, \\ V_2^p &= B_1 S^{\beta_1} + B_2\,, \end{cases} \tag{4.32}$$

and $V_1^p = V_2^p = S - B$ for $S > \tilde{S}$. The smooth fit scheme leads to the followings:

$$\begin{cases} A_1 \tilde{S}^{\beta_1} + A_2 &= \tilde{S} - B(t)\,, \\ \beta_1 A_1 (\tilde{S})^{\beta_1} &= \tilde{S}\,, \end{cases} \quad \text{and} \quad \begin{cases} B_1 \tilde{S}^{\beta_1} + B_2 &= \tilde{S} - B(t)\,, \\ \beta_1 B_1 (\tilde{S})^{\beta_1} &= \tilde{S}\,, \end{cases} \tag{4.33}$$

and we have $A_1 = B_1$ and $A_2 = B_2$, so $V_1^p = V_2^p$.

Secondly when $q_{11}(r - \frac{1}{2}\sigma_2^2) + q_{22}(r - \frac{1}{2}\sigma_1^2) < 0$, for $S \le \tilde{S}$, we have

$$\begin{cases} V_1^p &= A_1 S^{\beta_1} + A_2 S^{\beta_2}\,, \\ V_2^p &= B_1 S^{\beta_1} + B_2 S^{\beta_2}\,, \end{cases} \tag{4.34}$$

and $V_1^p = V_2^p = S - B(t)$ for $S > \tilde{S}$. The smooth fit scheme leads to

$$\begin{cases} A_1 (\tilde{S})^{\beta_1} + A_2 (\tilde{S})^{\beta_2} &= \tilde{S} - B(t)\,, \\ \beta_1 A_1 (\tilde{S})^{\beta_1} + \beta_2 A_2 (\tilde{S})^{\beta_2} &= \tilde{S}\,, \end{cases} \tag{4.35}$$

and

$$\begin{cases} B_1(\tilde{S})^{\beta_1} + B_2(\tilde{S})^{\beta_2} & = & \tilde{S} - B(t)\,, \\ \beta_1 B_1(\tilde{S})^{\beta_1} + \beta_2 B_2(\tilde{S})^{\beta_2} & = & \tilde{S}\,. \end{cases} \tag{4.36}$$

Then $A_1 = B_1, A_2 = B_2$, so $V_1^p = V_2^p$. Defining $V_1^p = V_2^p = V^p$ then for $S < \tilde{S}$, the first equation in (4.12) reduces to

$$rS\frac{\partial V^p}{\partial S} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V^p}{\partial S^2} = 0$$

for both $i = 1, 2$. Then we have

$$V^p \quad = \quad \begin{cases} A_1 S^{\beta} + A_2 & \text{if } \ S \leq \tilde{S}\,, \\ S - B(t) & \text{if } \ S > \tilde{S}\,, \end{cases} \tag{4.37}$$

where $A_2 = 0$ and β is the positive solution of $r\beta + \frac{1}{2}\sigma_i^2\beta(\beta - 1) = 0$. So if $\beta = 1 - \frac{2r}{\sigma_i^2} > 0$ for $i = 1, 2$. We have

$$V^p \quad = \quad \begin{cases} A_1 S^{\beta} & \text{if } \ S \leq \tilde{S}\,, \\ S - B(t) & \text{if } \ S > \tilde{S}\,. \end{cases} \tag{4.38}$$

The smooth fit scheme leads to

$$\begin{cases} A_1(\tilde{S})^{\beta} & = & \tilde{S} - B(t)\,, \\ \beta A_1(\tilde{S})^{\beta} & = & 1\,. \end{cases} \tag{4.39}$$

The unknowns $\tilde{S}$ and A_1 can then be solved.

6. Numerical Examples

In this section we illustrate the practical implementation of the proposed model via numerical experiments. We compare the prices of the perpetual convertible bond obtained from the Markov regime-switching model, or the proposed model, (Model I) to those arising from the model without switching regimes (Model II). We suppose that the current state of the economy is "low volatility" (i.e. $X_t = e_1$). For the model without switching regimes, we assume that the model parameters are the same as those with switching regimes when the economy is "high volatility" (i.e. state "e_2"). For Model I, the parameters are set to be $q_{11} = q_{22} = 100,\ r = 0.04,\ \sigma_1 = 0.30,\ \sigma_2 = 0.50$. For Model II, we set the parameters to be $q_{11} = q_{22} = 100,\ r = 0.04,\ \sigma_1 = \sigma_2 = 0.50$. Figure 1 depicts the plots of the prices of the perpetual convertible bonds $V^p(S, e_1)$ of Model I and Model II. Whereas, Figure 2 depicts the plots of the prices of the perpetual convertible bonds $V^p(S, e_2)$ of Model I and Model II.

Comparing the results arising from Model I and Model II in Figure 1, we see that Model I seems giving a lower price of the convertible bond than Model II. This is consistent with the intuition that Model I takes into account the possibility that the state of lower volatility Looking at Figure 2, we have a similar observation. Model I seems giving a higher price of the convertible bond since Model I incorporates the state of higher volatility.

Besides valuation, the optimal exercise strategies for the convertible are important as well. We determine the threshold levels of these optimal exercise strategies and report the results as follows. In Model I, for each t, we obtain a threshold levels (S_1^p, S_2^p). Both S_1^p and S_2^p increase with t. When $t = 0$, $(S_1^p, S_2^p) = (1.6882, 1.9280)$; when $t = 14.9$, $(S_1^p, S_2^p) = (1.8546, 2.1354)$. Whereas, in Model II, when $t = 0$, we have $S_1^p > S_2^p$ and $(S_1^p, S_2^p) = (1.8818, 1.6596)$; when $t = 14.9$, $(S_1^p, S_2^p) = (2.1491, 1.8758)$.

It is of practical importance and relevance to take into account structural changes of economic states or regimes in valuing convertible bonds and other structured products. From the recent global financial crisis started from subprime mortgages in the United States, we learned that it is important to reappraise the current practice of valuing, hedging and risk management of structured products. We believe that the model which takes into account structural changes in economic states or regimes can provide us a more realistic and prudent way to value, hedge and managing risks of structured products, in particular, convertible bonds. For example, the proposed regime-switching model for valuing convertible bonds can take into account the possibility that the economic regime or state switch from one state to another. Failure to take into account this feature may lead to under-price structured products or underestimate the risks of these products. So the proposed model provides market practitioners and risk managers with insights into understanding better the risks of convertible bonds and their valuation issues.

7. Concluding Remarks

We developed a valuation model for a perpetual convertible bond when the price dynamics of the underlying share are governed by a continuous-time, Markovian regime-switching model. The valuation problem was formulated as one of valuing a perpetual American option with time-dependent strike price. We employed the regime-switching Esscher transform to determine a price kernel in the incomplete market. A closed-form solution to the optimal stopping problem associated with the valuation was obtained. Numerical examples were provided

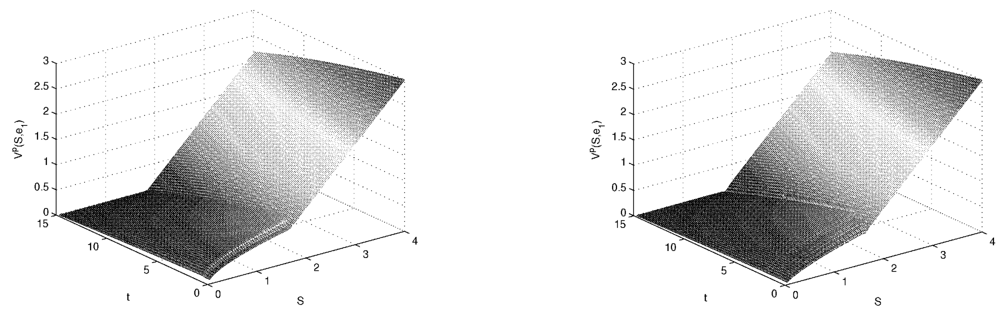

Figure 1: Convertible bond price $V^p(S, e_1)$ Model I (left) and Model II (right)

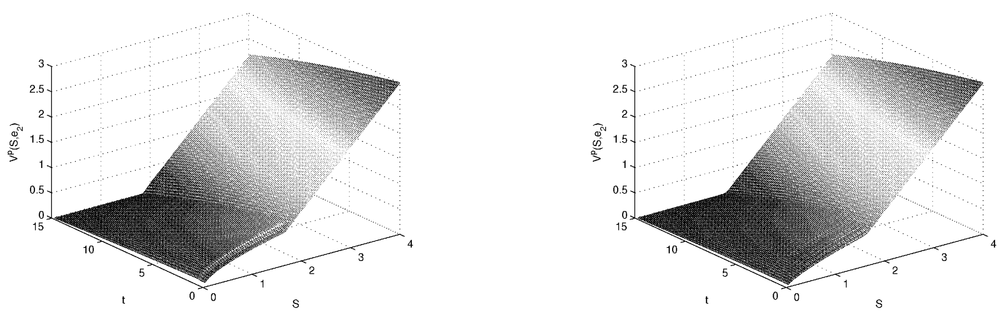

Figure 2: Convertible bond price $V^p(S, e_2)$ Model I (left) and Model II (right)

to illustrate the practical implementation of the proposed model and the effect of switching regimes on the value of the perpetual convertible bond. The proposed model here provides market practitioners and risk managers with insights into developing a prudent model for valuing convertible bonds and understanding better the risks of these important structured products.

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References

[1] M. Brennan, E. Schwartz, Convertible bond: Valuation and optimal strategies for call and conversion, *Journal of Finance*, **5** (1977), 1699-1715.

[2] J. Buffington, R. Elliott, American options with regime switching, *International Journal of Theoretical and Applied Finance*, **5** (2002), 497-514.

[3] F. Delbaen, W. Schachermayer, A general version of the fundamental theorem of asset pricing, *Mathematische Annalen*, **300** (1994), 463-520.

[4] R. Elliott, L. Chan, T. Siu, Option pricing and Esscher transform under regime switching, *Annals of Finance*, **1** (2005), 423-432.

[5] R. Elliott, New finite dimensional filters and smoothers for noisily observed Markov chains, *IEEE Transactions on Information Theory*, **39** (1993), 265-271.

[6] R. Elliott, L. Aggoun, J. Moore, *Hidden Markov Models: Estimation and Control*, Springer-Verlag, New York (1994).

[7] X. Guo, Q. Zhang Closed-form solutions for perpetual american put options with regime switching, *SIAM Journal on Applied Mathematics*, **64** (2004), 2034-2049.

[8] J. Harrison, D. Kreps, Martingales and arbitrage in multiperiod ssecurities, *Markets. Journal of Economic Theory*, **20** (1979), 381-408.

[9] J. Harrison, S.R. Pliska, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications*, **11** (1981), 215-260.

[10] J. Harrison, S.R. Pliska, A stochastic calculus model of continuous trading: Complete markets, *Stochastic Processes and their Applications*, **15** (1983), 313-316.

[11] K. Tsiveriotis, C. Fernandes, Valuing convertible bonds with credit risk, *The Journal of Fixed Income*, **8** (1998), 95-102.

[12] J. Xia, X. Zhou, Stock loans, *Mathematical Finance*, **17** (2007), 307-317.