International Journal of Pure and Applied Mathematics
Volume 53 No. 4 2009, 601-608

ORBITS AND CONTINUITY OF CERTAIN CLASS OF FUNCTIONS

Kandasamy Muthuvel
Department of Mathematics
University of Wisconsin – Oshkosh
Oshkosh, Wisconsin, 54901-8601, USA
e-mail: muthuvel@uwosh.edu

Abstract: In this paper, we prove that if g is a continuous function that is nonconstant on every nonempty open interval, f is a Darboux function with the dense mapping property, and $g \subseteq \cup_{n \in N} f^n$, then the set of all discontinuity points of f is nowhere dense. Among other things, we prove that in the above statement if we replace "f is a Darboux function with the dense mapping property" by "f is a connectivity function without a fixed point", then f is continuous everywhere.

AMS Subject Classification: 26A15, 26A18, 54C30
Key Words: connectivity function, quasicontinuous function, Darboux function, dense mapping property, second category

1. Introduction

It is shown in [8] that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function that is nonconstant on every nonempty open interval and $f : \mathbb{R} \to \mathbb{R}$ is a Darboux function with the property that $g \subseteq \cup_{k=1}^{k=n} f^k$ for some $n \in N$, then f is continuous. In the above statement if one replaces "$g \subseteq \cup_{k=1}^{k=n} f^k$ for some $n \in N$" by "$g \subseteq \cup_{n \in N} f^n$", then f need not be continuous [9]. In fact, the discontinuous function f given in the example in [9] is both Darboux and quasicontinuous. It is interesting to note that if $g : \mathbb{R} \to \mathbb{R}$ is a Darboux function such that $g \subseteq \cup_{k=1}^{k=n} f_k$ for some $n \in N$, where each $f_k : \mathbb{R} \to \mathbb{R}$ is a continuous function, then g is continuous (see [4], Theorem 2). In this paper, we prove some results concerning con-

Received: May 23, 2009

nectivity functions, quasicontinuous functions, functions having dense mapping property, and Darboux functions. For example, we prove a theorem that directly implies that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function that is nonconstant on every nonempty open interval and $f : \mathbb{R} \to \mathbb{R}$ is a Darboux and quasicontinuous function with the property that$g \subseteq \cup_{n \in N} f^n$, then the set of all discontinuity points of f is nowhere dense. We also show that in the above statement if we replace "$f : \mathbb{R} \to \mathbb{R}$ is a Darboux and quasicontinuous function" by "$f : \mathbb{R} \to \mathbb{R}$ is a connectivity function" then f is continuous everywhere.

Definition 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) f is called quasicontinuous at a real number a if for every open set V containing $f(a)$ and for every open set U containing a, there exists a nonempty open set G such that $G \subseteq U \cap f^{-1}(V)$. f is called quasicontinuous if it is quasicontinuous at every real number.

(b) f is bilaterally quasicontinuous at a if for every $\delta > 0$ and for every open set V containing $f(a)$, there exist nonempty open sets H and W contained in $f^{-1}(V)$ such that $H \subseteq (a - \delta, a)$ and $W \subseteq (a, a + \delta)$.

(c) f is called a connectivity function if its graph is connected in $\mathbb{R}^2$.

(d) f is called is a Darboux function if $f(C)$ is connected whenever C is connected.

(e) f has the dense mapping property if $f(\overline{D}) \subseteq \overline{f(D)}$ whenever D is a subset of $\mathbb{R}$ with $\overline{D}$ connected.

(f) f is said to be n-continuous if there exist sets $X_1, X_2, ..., X_n$ such that $f \restriction X_i$ is continuous for every $1 \le i \le n$,where n is a positive integer. f is said to be strong n-continuous if there exist continuous functions $f_1, f_2, ..., f_n$ from $\mathbb{R}$ into $\mathbb{R}$ such that $f \subseteq \cup_{k=1}^{k=n} f_k$.

It is easy to see that, if f is strong n-continuous, then it is n-continuous.

Definition 2. For a function $f : \mathbb{R} \to \mathbb{R}$, orbit of f at a real number x, denoted by orbit$_f x$, is defined to be $\{f^{n-1}(x) : n \in N\}$, where f^ndenotes the n-th iterate of f and $f^{\,0}(x) = x$.

2. Theorems and Examples

Theorem 1. *Let $g : \mathbb{R} \to \mathbb{R}$be a continuous function that is nonconstant on every nonempty open interval, and let $f : \mathbb{R} \to \mathbb{R}$be a Darboux function. If orbit of $f$at x is closed for every x in an everywhere second category subset E* of $\mathbb{R}$ *and $g \subseteq \cup_{n \in N} f^n$,then f is continuous.*

Proof. Suppose that f is discontinuous at a real number a. Then there exist $\epsilon > 0$ and a sequence (x_n) converging to a such that $f(x_n) \geq f(a) + \epsilon\ \forall n$ or $f(x_n) \leq f(a) - \epsilon\ \forall n$. Consider the case $f(x_n) \geq f(a) + \epsilon\ \forall n$. The other case is similar. Since f is Darboux, for each $y \in (f(a), f(a)+\epsilon)$, there exists t_n between x_n and a such that $f(t_n) = y$. This implies that $t_n \to a$, $g(t_n) \to g(a)$, and, for each $n \in N$, there exists $k \in N$ such that $g(t_n) = f^k(t_n) = f^{k-1}(y)$. Hence, $g(a) \in \overline{\{g(t_n) : n \in N} \subseteq \overline{\{f^{k-1}(y) : k \in N\}} = \overline{\text{orbit}_f\ y} = \{f^{k-1}(y) : k \in N\}$ for each $y \in E \cap (f(a),\ f(a) + \epsilon)$. Consequently, since a countable union of first category sets is first category, there exists a second category subset S of $E \cap (f(a),\ f(a) + \epsilon)$ such that, for some $m \in N$, $g(a) = f^m(s)\ \forall s \in S$. It is easy to see that, for every $s \in S$, there exists $n \geq m + 1$ such that $g^{m+1}(s) = f^n(s) = f^{n-m}(f^m(s)) = f^{n-m}(g(a))$. Again, since a countable union of first category sets is first category, there exist $p \in N$ and a second category subset D of S such that $g^{m+1}(d) = f^{\,p}(g(a))\ \ \forall d \in D$. Every second category set is dense in some interval. So, $\overline{D}$ contains a nonempty open interval. Since g is nowhere constant continuous function, $g(\overline{D})$ contains a nonempty open interval and $g(\overline{D}) \subseteq \overline{g(D)}$. Consequently, $g(D)$ is somewhere dense. By applying this argument a finite number of times, one can easily show that $g^{m+1}(D)$ is somewhere dense, which contradicts that $g^{m+1}(d) = f^{\,p}(g(a))\ \ \forall d \in D$. Thus, f is continuous everywhere. □

Corollary 1. *If $f : \mathbb{R} \to \mathbb{R}$is a Darboux function and every real number is a periodic point of f, then $f$is continuous.*

To see Corollary 1, let $g(x) = x$ in Theorem 1 and note that $\text{orbit}_f x$ is finite.

Corollary 2. *Let $g : \mathbb{R} \to \mathbb{R}$be a continuous function such that every real number is a periodic point of g, and let $f : \mathbb{R} \to \mathbb{R}$be a Darboux function. If $g \subseteq \cup_{n \in N} f^n$, then $f$is continuous.*

Proof. For every real number x, there are positive integers n and m such that $g^n(x) = x$ and $g^n(x) = f^m(x)$. Hence, every real number is a periodic point of f and the corollary follows from Corollary 1. □

Remark 1. In Theorem 1, if Darboux function is replaced by quasicontinuous function, then f*need not be continuous.*

For, let $f(x) = x$ if $0 \leq x \leq 1$ and $f(x) = 1 - x$ otherwise. Let $g(x) = x\ \forall x$. Then $f^2(x) = x\ \forall x$ and orbit ${}_f x$ is closed.

The following theorem shows that in Theorem 1, if Darboux function is replaced by connectivity function without a fixed point, then f is continuous.

Theorem 2. *Let $g : \mathbb{R} \to \mathbb{R}$be a continuous function that is nonconstant on* *every nonempty open interval. If $f : \mathbb{R} \to \mathbb{R}$ is a connectivity function without a fixed point and $g \subseteq \cup_{n\in N} f^n$, then f is continuous.*

Proof. It is well-known that every connectivity function is Darboux. So, by Theorem 1, it is sufficient to show that $\text{orbit}_f x$ is closed for every real number x. Since f is a connectivity function without a fixed point, $f(x) > x\ \forall x \in \mathbb{R}$ or $f(x) < x\ \forall x \in \mathbb{R}$. Consider the case $f(x) > x\ \forall x \in \mathbb{R}$. The other case is similar. $f(x) > x\ \forall x \in \mathbb{R}$ implies that $(f^n(x))$ is an increasing sequence. If the sequence is not bounded above, then $\text{orbit}_f x$ is closed. Suppose that $\{f^n(x) : n \in N\}$ is bounded above. Let $\lim_{n\to\infty} f^n(x) = L$. By the hypothesis, for each x, $g(f^n(x)) = f^{m+n}(x)$ for some $m \in N$. Since $\{g(f^n(x)) : n \in N\} \subseteq \{f^n(x) : n \in N\}$, $(f^n(x))$ is an increasing sequence converging to L, and g is continuous at L, $g(L) = \lim_{n\to\infty} g(f^n(x)) = L$. This implies that $L = g(L) = f^k(L)$ for some $k \in N$, which contradicts that $(f^n(L))$ is an increasing sequence. Thus, $\text{orbit}_f x$ is closed and f is continuous everywhere. □

It is shown in [9] that there exist a continuous function g that is nonconstant on every nonempty open interval, and a discontinuous function f that is both Darboux and quasicontinuous such that $g \subseteq \cup_{n\in N} f^n$. It follows from the following theorem that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function that is nonconstant on every nonempty open interval, $f : \mathbb{R} \to \mathbb{R}$ is both Darboux and quasicontinuous, and $g \subseteq \cup_{n\in N} f^n$, then the set of all discontinuity points of f is nowhere dense.

Theorem 3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function that is nonconstant on every nonempty open interval, and let $f : \mathbb{R} \to \mathbb{R}$ be a Darboux function that has the dense mapping property. If $g \subseteq \cup_{n\in N} f^n$, then the set of all discontinuity points of f is nowhere dense.*

First, we prove the following lemmas.

Lemma 1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function that is nonconstant on every nonempty open interval, and let $f : \mathbb{R} \to \mathbb{R}$ be a Darboux function. If $g \subseteq \cup_{n\in N} f^n$, then:*

(i) $\forall n \in N$, f^n is nonconstant on every nonempty open interval,

(ii) if f is discontinuous at a real number a, then f^n is discontinuous at a for every $n \in N$.

Proof. (i) Suppose that f^m is constant on a nonempty open interval I for some $m \in N$. Let $f^m(x) = k\ \forall x \in I$. For each $x \in I$, $\exists n \geq m$ such that $g^m(x) = f^n(x) = f^{n-m}(f^m(x)) = f^{n-m}(k)$. Hence, $g^m(I) \subseteq \{f^{n-m}(k) : n \geq m\}$. By the hypothesis of g, it is easy to see that $g^m(I)$ contains a nonempty open interval, which contradicts the previous statement.

(ii) Suppose that f is discontinuous at a. As in the proof of Theorem 1, for some $\epsilon > 0$ and $\forall y \in (f(a),\ f(a)+\epsilon)$ or $\forall y \in (f(a)-\epsilon,\ f(a))$, there exists a sequence (t_n) converging to a such that $f(t_n) = y\ \forall n$. Assume, to the contrary, that f^m is continuous at a for some $m > 1$. Since (t_n) converges to a and $f(t_n) = y\ \forall n$, $(f^m(t_n)) = (f^{m-1}(y))$ converges to $f^m(a)$ and hence $f^m(a) = f^{m-1}(y)$. This shows that f^{m-1} is constant on a nonempty open interval, which contradicts part (i). □

Lemma 2. (see [3], Lemma 2.4) *If $f : \mathbb{R} \to \mathbb{R}$positive intege n, f^n is Darboux and has the dense mapping property.*

Proof of Theorem 3. Let D be the set of all discontinuity points of f. Assume, to the contrary, that D is somewhere dense. Then there exists an open interval I such that D is everywhere dense in I. There exist $m \in N$ and a somewhere dense subset S of I such that $f^m(x) = g(x)\ \forall x \in S$. Let $J \subset I$ be an open interval such that S is everywhere dense in J. Let $d \in D \cap J$. By Lemma 1(ii), f^m is discontinuous at d. The composition of Darboux functions is Darboux. So, f^m is Darboux. As in the proof of Theorem 1, there exist $\epsilon > 0$ and a sequence (s_n) converging to d such that $f^m(s_n) > f^m(d) + \epsilon\ \forall n \in N$ or $f^m(s_n) < f^m(d) - \epsilon\ \forall n \in N$. Consider the case $f^m(s_n) > f^m(d) + \epsilon\ \forall n \in N$. The other case is similar. Since f^m is Darboux, and $f^m(s_n) > f^m(d) + \epsilon > f^m(d) + \frac{\epsilon}{4} > f^m(d)\ \ \forall n \in N$, there exist sequences (t_n) and (v_n) converging to d such that $f^m(t_n) = f^m(d) + \epsilon$ and $f^m(v_n) = f^m(d) + \frac{\epsilon}{4}\ \forall n \in N$. By the hypothesis, g is continuous at d. So, $\exists \delta > 0$ such that $|g(x) - g(d)| < \frac{\epsilon}{4}$ whenever $|x - d| < \delta$. Since f^m and g agree on S, $|f^m(x) - g(d)| < \frac{\epsilon}{4}$ whenever $|x - d| < \delta$ and $x \in S$. Hence, $\forall\ x \in S \cap (d-\delta,\ d+\delta)$, $f^m(x) \in (g(d) - \frac{\epsilon}{4},$ $g(d) + \frac{\epsilon}{4})$ and $\overline{f^m(S \cap (d-\delta, d+\delta))} \subseteq [g(d) - \frac{\epsilon}{4},\ g(d) + \frac{\epsilon}{4}]$. Without loss of generality, we may choose δ such that $(d-\delta,\ d+\delta) \subseteq J$. Since $\overline{S \cap (d-\delta, d+\delta)}$ is connected and f^m has the dense mapping property, $f^m(\overline{S \cap (d-\delta, d+\delta)}) \subseteq \overline{f^m(S \cap (d-\delta, d+\delta))} \subseteq [g(d) - \frac{\epsilon}{4},\ g(d) + \frac{\epsilon}{4}]$. Recall that the sequences (t_n) and (v_n) converge to $d \in \overline{S \cap J}$, $f^m(t_n) = f^m(d) + \epsilon$, and $f^m(v_n) = f^m(d) + \frac{\epsilon}{4}\ \forall n \in N$. This implies that $t_n, v_n \in \overline{S \cap (d-\delta, d+\delta)}$ for all but a finitely many n. Hence, $f^m(t_n) = f^m(d) + \epsilon$, $f^m(v_n) = f^m(d) + \frac{\epsilon}{4} \in [g(d) - \frac{\epsilon}{4},\ g(d) + \frac{\epsilon}{4}]$, $|\ f^m(t_n) - f^m(v_n)\ | = \frac{3\epsilon}{4}$, and the length of the interval $[g(d) - \frac{\epsilon}{4},\ g(d) + \frac{\epsilon}{4}]$ is $\frac{\epsilon}{2}$, which is impossible. Thus, D is nowhere dense. □

Corollary 3. Let $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function that is nonconstant on every nonempty open interval, and let $f : \mathbb{R} \to \mathbb{R}$ be a Darboux and quasi-continuous function. If $g \subseteq \cup_{n \in N} f^n$, then the set of all discontinuity points of f is nowhere dense.*

Corollary 3 follows directly from the above theorem and the following facts:

(i) If a function is both Darboux and quasicontinuous, then it is bilaterally quasicontinuous (see [1], Lemma 3.2).

(ii) A function is bilaterally quasicontinuous if and only if it has the dense mapping property (see [6], Theorem 1).

The following example shows that if "Darboux" and "continuous" are interchanged in Theorem 1, the conclusion of Theorem 1 is no longer true.

Example 1. *There exist a discontinuous Darboux function $g : \mathbb{R} \to \mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $g \subseteq \cup_{n\in N} f^n$ and orbits of f and g at x are closed for every real number x.*

Construction. Let $g(x) = |\sin(\frac{1}{x})|$ for $x < 0$, and $g(x) = 2x$ for $x \geq 0$. Let $f(x) = (\frac{1}{2})^{n-1}|\sin(\frac{1}{x})|$ for $x \in [-\frac{1}{n\pi}, -\frac{1}{(n+1)\pi})$, where $n \in N$, and $f(x) = g(x)$ otherwise. It is not hard to see that $f^n(x) = |\sin(\frac{1}{x})|$ for $x \in [-\frac{1}{n\pi}, -\frac{1}{(n+1)\pi})$ and g is a Darboux function that is discontinuous at 0. The remaining conditions are easy to verify. □

It is proved in [4], Theorem 2, that every strongly finitely continuous Darboux function is continuous. The following proposition is a generalization of this result.

Proposition 1. *Let $g : \mathbb{R} \to \mathbb{R}$be a Darboux function such that $g \subseteq \cup_{n\in M} f_n$ for some subset M of N,where each $f_n : \mathbb{R} \to \mathbb{R}$ is a function with the property that $\cup_{n\in M}(f_n)^{-1}(y)$ is closed in $\mathbb{R}$ for each y in an everywhere second category subset S of R. Then $g$is continuous.*

Proof. Suppose that g is discontinuous at a real number a. As shown in the proof of Theorem 3, for some $\epsilon > 0$ and for every $y \in (g(a), g(a) + \epsilon)$ or for every $y \in (g(a) - \epsilon, g(a))$, there exists a sequence (t_n) converging to a such that $g(t_n) = y$. The hypothesis $g \subseteq \cup_{n\in M} f_n$ implies that $\{t_n : n \in N\} \subseteq \cup_{n\in M}(f_n)^{-1}(y)$. Consequently, $a \in \overline{\{t_n : n \in N\}} \subseteq \overline{\cup_{n\in M}(f_n)^{-1}(y)} = \cup_{n\in M}(f_n)^{-1}(y)$ for uncountably many y in S. That is, $f_n(a) = y$ for some $n \in M$, which contradicts that $\{f_n(a) : n \in M\}$ is countable and the set of all such y is uncountable. □

The following corollary is a direct consequence of the above proposition.

Corollary 4. *If $g : \mathbb{R} \to \mathbb{R}$ is a Darboux function such that $g \subseteq \cup_{k=1}^{k=n} f_k$ for some $n \in N$, where each f_k is continuous, then g is continuous (see [4], Theorem 2).*

Corollary 5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a Darboux function and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $g \subseteq \cup_{k=1}^{k=n} f^k$ for some $n \in N$. Then g is continuous.*

Corollary 5 is a special case of Corollary 4.

The following proposition was motivated by Theorem 3 and Corollary 5.

Proposition 2. *Let $g : \mathbb{R} \to \mathbb{R}$be a Darboux function that has the dense mapping property, and let $f : \mathbb{R} \to \mathbb{R}$be a continuous function. If $g \subseteq \cup_{n \in N} f^n$, then the set of all discontinuity points of g is nowhere dense.*

Proof. The proof of this proposition is a minor modification of the proof of Theorem 3. Let D be the set of all discontinuity points of g. Assume, to the contrary, that D is somewhere dense. Then there exists an open interval I such that D is everywhere dense in I. There exist $m \in N$ and a somewhere dense subset S of I such that $f^m(x) = g(x)\ \forall x \in S$. Let $J \subset I$ be an open interval such that S is everywhere dense in J. Let $d \in D \cap J$. Since f^m is continuous at d and g is Darboux that is discontinuous at d, as in the proof of Theorem 3, there exist $\epsilon > 0$ such that $g(d) + \epsilon$, $g(d) + \frac{\epsilon}{4} \in [f^m(d) - \frac{\epsilon}{4}, f^m(d) + \frac{\epsilon}{4}]$, which is impossible. Thus, D is nowhere dense. □

References

[1] J. Borsik, On the points of bilateral quasicontinuity of functions, *Real Analysis Exchange*, **19**, No. 2 (1993-94), 529-536.

[2] K. Ciesielski, *Set Theory for the Working Mathematician*, Cambridge Univ. Press (1997).

[3] P.S. Keller, Chaotic behavior of Newton's method, *Real Analysis Exchange*, **18**, No. 2 (1992-93), 490-507.

[4] M. Marciniak, On finitely continuous Darboux functions and strong finitely continuous functions, *Real Analysis Exchange*, **33**, No. 1 (2008), 15-22.

[5] M. Marciniak, Finitely continuous, Darboux functions, *Real Analysis Exchange*, **26**, No. 1 (2000-01), 417-420.

[6] R.A. Mimna, Omega-limit sets and noncontinuous functions, *Real Analysis Exchange*, **23**, No. 1 (1997-98), 267-273.

[7] T.K.S. Moothathu, Orbits of Darboux-like real functions, *Real Analysis Exchange*, **33**, No. 1 (2008), 143-150.

[8] K. Muthuvel, Continuity of Darboux functions with nice finite iterations, *Real Analysis Exchange*, **32**, No. 2 (2007), 587-595.

[9] K. Muthuvel, A note on iterations of Darboux functions, *International Journal of Pure and Applied Mathematics*, **32**, No. 1 (2006), 61-63.

[10] R.J. Pawlak, On the entropy of Darboux functions, To Appear.