International Journal of Pure and Applied Mathematics
Volume 53 No. 4 2009, 485-487

INTERPOLATION PROBLEMS OVER A FINITE FIELD

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Abstract: Here we show any interpolation problem for general fat points true over $\overline{\mathbb{F}}_q$ has a solution over $\mathbb{F}_q$ for a large q with an effective lower bound for q.

AMS Subject Classification: 14N05
Key Words: interpolation, fat point, finite field

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Fix a finite field $\mathbb{F}_q$ and let $\overline{\mathbb{F}}_q$ denote its algebraic closure. We fix integers $d > 0$, $n > 0$ and consider any interpolation problem $\mathbb{E}$ defined over $\mathbb{F}_q$ for fat points in $\mathbb{P}^n$ with respect to the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$. For instance, we may fix a linear subspace V a closed subscheme $Z \subset \mathbb{P}^n$ and an integer $m > 0$. For any $P \in \mathbb{P}^n$ let mP denote the fat point with multiplicity m (or the m-point) supported by P, i.e. the closed subscheme of $\mathbb{P}^n$ with $(\mathcal{I}_P)^m$ as its ideal sheaf. We have $(mP)_{red} = \{P\}$ and length$(mP) = \binom{n+m-1}{n}$. Fix a linear subspace V of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. We may take as interpolation problem $\mathbb{E}$ the question of the existence of $P \in \mathbb{P}^n$ such that $\dim(H^0(\mathbb{P}^n, \mathcal{I}_{mP}(d)) \cap V) = \max\{\dim(V) - \binom{n+m-1}{n}, 0\}$. We call (V, m) this interpolation problem. This interpolation problem is defined over $\mathbb{F}_q$ if V is induced by a subspace defined over $\mathbb{F}_q$. Assume that this interpolation problem is true over $\overline{\mathbb{F}}_q$, i.e. there is $P \in \mathbb{P}^n(\overline{\mathbb{F}}_q)$ such that $\dim(H^0(\mathbb{P}^n, \mathcal{I}_{mP}(d)) \cap V) = \max\{\dim(V) - \binom{n+m-1}{n}, 0\}$. There is an integer $e > 0$ such that P is defined over $\mathbb{F}_{q^e}$. When we may be sure that there is such point P defined over $\mathbb{F}_q$? In the applications V is usually of the form $V := H^0(\mathbb{P}^n, \mathcal{I}_Z(d))$ with Z a closed subscheme of $\mathbb{P}^n$. If Z is defined

Received: March 4, 2009

over $\mathbb{F}_q$, then V is defined over $\mathbb{F}_q$ and hence (V, m) is defined over $\mathbb{F}_q$. From now on we fix the integer $n > 0$ and set $\mathcal{O} := \mathcal{O}_{\mathbb{P}^n}$. All cohomology groups are computed on $\mathbb{P}^n$.

Theorem 1. *Let (V, m) be a degree d interpolation problem on $\mathbb{P}^n$ defined over $\mathbb{F}_q$ and solvable over $\overline{\mathbb{F}}_q$. Assume $q > \sum_{k=0}^{m-1} \binom{n+k-1}{n-1}(d-k(n+1)/n)$. Then (V, m) has a solution over $\mathbb{F}_q$.*

Proof. Set $v := \dim(V)$. Hence $0 \le v \le \binom{n+d}{n}$. For any integer $m > 0$ let $J^{m-1}(\mathcal{O})(d)$ denote the principal part bundle of order $m-1$ of the line bundle $\mathcal{O}(d)$) ([2], IV.16.10.1 and IV.16.7.3). $J^{m-1}(\mathcal{O})(d)$ is a rank $\binom{n+m-1}{n}$ vector bundle on $\mathbb{P}^n$. We have $J^0(\mathcal{O}(d)) \cong \mathcal{O}(d)$. As in the case of an arbitrary smooth variety for every integer $k \ge 1$ we have an exact sequence of vector bundles on $\mathbb{P}^n$:

$$0 \to S^k(\Omega^1_{\mathbb{P}^n})(d) \to J^k(\mathcal{O}(d)) \to J^{k-1}(\mathcal{O}(d)) \to 0. \tag{1}$$

Since $\mathrm{rank}(S^k(\Omega^1_{\mathbb{P}^n})) = \binom{n+k-1}{n-1}$ and $\mu(\Omega^1_{\mathbb{P}^n}) = (-n-1)/n$, we have

$$\det(S^k(\Omega^1_{\mathbb{P}^n})) \cong \mathcal{O}(x),$$

where $x := -k(n+1) \cdot \binom{n+k-1}{n-1}/n$. Thus $\det(S^k(\Omega^1_{\mathbb{P}^n})(d)) \cong \mathcal{O}(y)$, where $y := \binom{n+k-1}{n-1}(d-k(n+1)/n)$. Hence inductively from(1) we get $\det(J^{m-1}(\mathcal{O}(d)) \cong \mathcal{O}(z)$, where $z := \sum_{k=0}^{m-1} \binom{n+k-1}{n-1}(d-k(n+1)/n)$. The linear subspace V of $H^0(\mathcal{O}(d))$ induces an evaluation map $u : V \otimes \mathcal{O} \to J^{m-1}(\mathcal{O}(d)) \to J^{m-1}(\mathcal{O}((d))$. The interpolation problem (V, m) is solvable over an algebraically closed base field if and only if u has generically maximal rank, i.e. the coherent sheaf $\mathrm{Im}(u)$ has rank $\min\{v, \binom{n+m-1}{n}\}$ at a general point of $\mathbb{P}^n$ and in this case we may take as a solution any point at which $\mathrm{Im}(u)$ is locally free and of rank $\min\{v, \binom{n+m-1}{n}\}$.

(a) Here we assume $v = \binom{m+n-1}{n}$. Recall that $\det(J^{m-1}(\mathcal{O}(d)) \cong \mathcal{O}(z)$, where $z := \sum_{k=0}^{m-1} \binom{n+k-1}{n-1}(d-k(n+1)/n)$. Since $v = \binom{m+n-1}{n}$, u is a map between vector bundles with the same rank. By assumption u is generically injective. Hence it drops rank on a hypersurface of degree z. Since $q > z$, there is at least one $P \in \mathbb{P}^n(\mathbb{F}_q)$ not contained in this hypersurface, i.e. at which u is a local isomorphism, i.e. such that P is a solution of the interpolation problem (V, m).

(b) Here we assume $v < \binom{m+n-1}{n}$. Take a linear subspace $W \subset H^0(\mathcal{O}(d))$ such that $V \subset W$, $\dim(W) = \binom{m+n-1}{m}$ and W is defined over $\mathbb{F}_q$. Part (a) gives $P \in \mathbb{P}^n(\mathbb{F}_q)$ which is a solution for the interpolation problem (W, m). The same point P is a solution for the interpolation problem (V, m).

(c) Here we assume $v > \binom{m+n-1}{n}$. Take a linear subspace $W \subset V$ such that $\dim(W) = \binom{m+n-1}{m}$ and W is defined over $\mathbb{F}_q$. Part (a) gives $P \in \mathbb{P}^n(\mathbb{F}_q)$ which is a solution for the interpolation problem (W, m). The same point P is a solution for the interpolation problem (V, m). □

Corollary 1. *Fix a prime* $p \neq 2, 3$*, an integer* $d \geq 41$*, a* p*-power* q *such that* $q > \sum_{k=0}^{3} \binom{k+2}{2}(d - 4k/3)$ *and integers* $x_i \geq 0$*,* $1 \leq i \leq 4$*, such that* $x_1+x_2+x_3+x_4 \leq (q^4-1)/(q-1)$*. Then there is a union* $Z \subset \mathbb{P}^3$ *of* x_4 *4-points,* x_3 *3-points,* x_2 *2-points and* x_1 *points such that* $Z_{red} \subseteq \mathbb{P}^3(\mathbb{F}_q)$*,* $h^0(\mathcal{I}_Z(d)) = \max\{0, \binom{d+3}{3} - \sum_{i=1}^{4} \binom{i+2}{3} x_i\}$ *and* $h^1(\mathcal{I}_Z(d)) = \max\{0, \sum_{i=1}^{4} \binom{i+2}{3} x_i - \binom{d+3}{3}\}$*.*

Proof. Since $\det(J^i(d)) \cong \mathcal{O}(z_i)$, $z_i := \sum_{i=0}^{i-1} \binom{i+2}{2}(d-4i/3)$ for $i = 1, 2, 3, 4$, we may apply induction on the integer $x_1+\cdots+x_4$, Theorem 1 and [1], Theorem 1. The assumption "$x_1 + x_2 + x_3 + x_4 \leq (q^4 - 1)/(q - 1)$" is used to define Z. Indeed, in the statement of Theorem 1 with $Z = Z' \cup \{mP\}$ in step (c) we could have $h^0(\mathcal{I}_{Z'}(d)) = 0$ and hence in Theorem 1 we would be allowed to take $P \in (Z')_{red}$. This choice is not allowable in the statement of Corollary 1. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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