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POSITIVE $\mathcal{D}$-INVARIANCE CONDITIONS OF POLYHEDRAL SETS FOR LINEAR UNCERTAIN DISCRETE-TIME SYSTEMS

Naohisa Otsuka[1 §], Hiroshi Matsumoto[2], Takuya Soga[3]
[1]Division of Science
School of Science and Engineering
Tokyo Denki University
Hatoyama-Machi, Hiki-Gun, Saitama-Ken, 350-0394, JAPAN
e-mail: otsuka@j.dendai.ac.jp
[2,3]Graduate School of Science and Engineering
Tokyo Denki University
Hatoyama-Machi, Hiki-Gun, Saitama-Ken, 350-0394, JAPAN

Abstract: In this paper, necessary and sufficient conditions for a given polytope and/or polyhedral set which is represented by a set of linear inequalities to be positive $\mathcal{D}$-invariant for uncertain linear systems are investigated. Further, some properties and simulation results for positive $\mathcal{D}$-invariant set are also investigated.

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1. Introduction

A positive invariant set is very important for stability analysis and the design of constrained controllers for linear dynamical systems. Further, positive $\mathcal{D}$-invariant set is also important one in the sense that is insensitive to all disturbances of the systems. In 1988 some algebraic conditions on positive invariance of various types of polyhedral sets for linear discrete-time systems were studied by Bitsoris [1]. After that Blanchini investigated necessary and sufficient

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§Correspondence author

conditions for a given polyhedral region to be positive $\mathcal{D}$-invariant for both discrete-time and continuous-time systems [2]. Further, some usuful results on the positive invariant set have been studied [3]-[6]. On the other hand, from the practical viewpoint, positive invariants set for linear systems whose system matrix contains uncertain parameters have also investigated by Blanchini [7], [8] and Benvenuti and Farina [9], and it was applied to set valued observer by Lin, Zhai and Antsaklis [10].

In this paper, necessary and sufficient conditions for a given polytope set which is represented as a set of linear inequalities to be positive $\mathcal{D}$-invariant for uncertain linear continuous-time systems are given as simplified conditions which can be easily checked. Further, necessary and sufficient conditions for a given polyhedral set to be positive $\mathcal{D}$-invariant for uncertain linear discrete-time systems are also given. Finally, some properties and numerical examples for positive $\mathcal{D}$-invariant set are investigated.

2. Preliminaries

At first, we give some definitions used throughout this study (e.g., see [11],[12]). A set Ω is said to be *convex* if for each pair of points x and y in S it is true that $\lambda x+(1-\lambda)y \in \Omega$ for all $0 \le \lambda \le 1$. The *convex hull* of a set Ω is the intersection of all the convex sets which contain Ω, and it is denoted by $conv\Omega$. The convex hull of a set of finite points in $\mathbf{R}^n$ is called a *polytope* which is represented as

$$\begin{aligned}\mathcal{P} &= conv(v_1^p, \cdots, v_N^p) \\ &:= \left\{ x \in \mathbf{R}^n \mid x = \sum_{i=1}^{N} \alpha_i v_i^p,\ \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \ge 0,\ i = 1, \cdots, N \right\},\end{aligned}$$

where v_i^p $(i = 1, \cdots, N)$ are vertices of $\mathcal{P}$. The set of all vertices of a polytope $\mathcal{P}$ is denoted by vert $\mathcal{P}$. Further, *polyhedral set* is the intersection of a finite number of closed half-spaces. Every polytope $\mathcal{P}$ in $\mathbf{R}^n$ is a bounded polyhedral set. Moreover, every bounded polyhedral set $\mathcal{P}$ in $\mathbf{R}^n$ is a polytope.

Let us denote a polyhedral set $\mathcal{S}$ in $\mathbf{R}^n$ containing the origin in its interior by using closed half space as follows.

$$\begin{aligned}\mathcal{S} &:= \{x \in \mathbf{R}^n \mid g_i^T x \le \theta_i,\ \theta_i > 0, i = 1, \cdots, s\} \\ &= \{x \in \mathbf{R}^n \mid Gx \le \theta\},\end{aligned}$$

where $G := [g_1, \cdots, g_s]^T \in \mathbf{R}^{s\times n}, \theta := [\theta_1, \cdots, \theta_s]^T \in \mathbf{R}^{s\times 1}$, "$\le$" is with respect to componentwise.

A set $\mathcal{W}(\subset \mathbf{R}^n)$ is said to be *convex cone* (that the apex of $\mathcal{W}$ is the origin) if it is convex and for each points x in $\mathcal{W}$ it is true that $\lambda x \in \mathcal{W}$ for all $\lambda \geq 0$. A convex cone set $\mathcal{W}$ is represented by

$$\begin{aligned} \mathcal{W} &= cone(v_1^w, \cdots, v_M^w) \\ &:= \left\{ x \in \mathbf{R}^n \mid \ x = \sum_{i=1}^{M} \beta_i v_i^w, \ \beta_i \geq 0, \ i = 1, \cdots, M \right\}, \end{aligned}$$

where $v_i^w (\neq 0) \in \mathbf{R}^n$ $(i = 1, \cdots, M)$ are called *generating vectors* and the set of all generating vectors is denoted by $gen(\mathcal{W})$. Then, we note that generating vectors for a convex cone $\mathcal{W}$ are not unique.

Theorem 2.1. (see [3], [11]) *Polyhedral set $\mathcal{S}$ is represented as $\mathcal{S} = \mathcal{P} + \mathcal{W}$, for some polytope set $\mathcal{P}$ and convex cone set $\mathcal{W}$.*

For a set $\mathcal{S}$ and positive real number μ a set μS is defined as

$$\mu\mathcal{S} := \{x \in \mathbf{R}^n \mid x = \mu y, \ y \in \mathcal{S}\}.$$

If $\mathcal{S}$ is a polyhedral set as $\mathcal{S} := \{x \in \mathbf{R}^n \mid g_i^T x \leq \theta_i, \ \theta_i > 0, i = 1, \cdots, s\}$, then $\mu\mathcal{S}$ $(\mu > 0)$ is also polyhedral set as

$$\mu\mathcal{S} = \{x \in \mathbf{R}^n \mid g_i^T x \leq \mu\theta_i, \ \theta_i > 0, i = 1, \cdots, s\}.$$

Then, it follows from Theorem 2.1 that $\mu\mathcal{S} = \mu\mathcal{P} + \mu\mathcal{W}$, where $\mu\mathcal{P}$ is a polytope set and $\mu\mathcal{W}$ is a convex cone set.

Consider the following discrete-time and continuous-time systems

$$\Sigma_d \ : \ x(t+1) = Ax(t) + Ed(t),$$

$$\Sigma_c \ : \ \frac{d}{dt}x(t) = Ax(t) + Ed(t),$$

where $x(t) \in \mathcal{X} := \mathbf{R}^n$ is the state, $d(t) \in \mathcal{D} \subset \mathbf{R}^m$ is the disturbance and A and E are constant real matrices of appropriate dimensions. The set $\mathcal{D}$ is a compact and convex set containing the origin.

Definition 2.2. A set $\mathcal{S}$ $(\subset \mathcal{X})$ is said to be a *positive $\mathcal{D}$-invariant* (PDI) for system Σ_d or Σ_c if for every initial state $x(0) \in \mathcal{S}$ and every disturbance $d(t) \in \mathcal{D}$ $(t \geq 0)$, then the state trajectories $x(t) \in \mathcal{S}$ for $t > 0$. In particuler case, $\mathcal{D} = \{0\}$, the positive $\mathcal{D}$-invariant set is said to be positive invariant (PI).

For a set $\mathcal{S}$ the interior of $\mathcal{S}$ is denoted as $int(\mathcal{S})$. Then, we have the following definition.

Definition 2.3. A set $\mathcal{S}$ $(\subset \mathcal{X})$ is said to be *strongly positive $\mathcal{D}$-invariant* (SPDI) for system Σ_d or Σ_c if for every initial state $x(0) \in \mathcal{S}$ and every disturbance $d(t) \in \mathcal{D}$ $(t \geq 0)$, then the state trajectories $x(t) \in int\mathcal{S}$ for $t > 0$.

In particuler case, $\mathcal{D} = \{0\}$, the strongly positive $\mathcal{D}$-invariant set is said to be strongly positive invariant (SPI).

3. Positive $\mathcal{D}$-Invariance Conditions for Poly-Tope Set

Firstly, consider the following uncertain discrete-time and continuous-time systems

$$\Sigma_{ud}: \ x(t+1) = A(q(t))x(t) + E(q(t))d(t),$$

$$\Sigma_{uc}: \ \frac{d}{dt}x(t) = A(q(t))x(t) + E(q(t))d(t),$$

where $A(q(t))$ and $E(q(t))$ have the following structured uncertain parameters

$$A(q(t)) = A_0 + q_1(t)A_1 + \cdots + q_p(t)A_p$$

$$E(q(t)) = E_0 + q_1(t)E_1 + \cdots + q_p(t)E_p$$

where $A_i \in \mathbf{R}^{n\times n}$, $E_i \in \mathbf{R}^{n\times m}$ are known matrices and $x(t) \in \mathcal{X} := \mathbf{R}^n$ is the state vector, $d(t) \in \mathcal{D}(\subset \mathbf{R}^m)$ is the disturbance vector and $q(t) := [q_1(t), \cdots, q_p(t)]^T \in \mathcal{Q}(\subset \mathbf{R}^p)$ is the uncertain vector. Sets $\mathcal{D}$ and $\mathcal{Q}$ are polytope sets which contain origins in their interiors, respectively and let $v_h^q \in vert(\mathcal{Q})(h = 1, \cdots, L)$ and $v_l^d \in vert(\mathcal{D})(l = 1, \cdots, R)$ be vertices of each sets, respectively.

Next, consider the following polytope set $\mathcal{P}(\subset \mathcal{X})$ which contains the origin in its interior.

$$\mathcal{P} := \{x \in \mathbf{R}^n \mid g_i^T x \le \theta_i, \ \theta_i > 0, i = 1, \cdots, s\}.$$

Let v_j^p $(j = 1, \cdots, N)$ be vertices of $\mathcal{P}$. Then the following theorem was given by Blanchini.

Theorem 3.1. (see [7]) *A set $\mathcal{P}$ is positive $\mathcal{D}$-invariant for uncertain discrete-time system Σ_{ud} if and only if for every vertices $v_j^p \in vert(\mathcal{P})$ $(j = 1, \cdots, N)$, $v_h^q \in vert(\mathcal{Q})$ $(h = 1, \cdots, L)$ and $v_l^d \in vert(\mathcal{D})$ $(l = 1, \cdots, R)$ the following conditions are satisfied.*

$$g_i^T\{A(v_h^q)v_j^p + E(v_h^q)v_l^d\} \le \theta_i, \ i = 1, \cdots, s.$$

Now, we show the following two types of numerical examples.

Example 3.2. Consider the following uncertain discrete-time system

$$\Sigma_1 \ : \ x(t+1) = \left(\begin{bmatrix} 0 & 0.16 & 0 \\ 0 & 0 & 0.16 \\ -0.15 & 0.27 & -0.14 \end{bmatrix} + q(t) \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) x(t)$$

$$+\left(\begin{bmatrix} 7 \\ -7 \\ 7 \end{bmatrix} + q(t)\begin{bmatrix} 0.01 \\ -0.01 \\ 0.01 \end{bmatrix}\right) d(t)\,,$$

where $x(t) \in \mathcal{X} := \mathbf{R}^3$ is the state vector, $d(t) \in \mathcal{D} := [-0.1, 0.1]^T \subset \mathbf{R}$ is the disturbance function and $q(t) \in \mathcal{Q} := [-0.1, 0.1]^T \subset \mathbf{R}$ is the uncertain parameter. For the system Σ_1 consider the following polytope set.

$$\mathcal{P} = \left\{ x \in \mathbf{R}^3 \mid \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} x \le \begin{bmatrix} 5 \\ 5 \\ 2 \\ 2 \\ 3 \\ 3 \end{bmatrix} \right\}.$$

Then, since the conditions of Theorem 3.1 are satisfied, we can see the set $\mathcal{P}$ is positive $\mathcal{D}$-invariant for the system Σ_1. In fact, we can see the state trajectory which was started from the initial state in $\mathcal{P}$ still remain in it (see Figure 1).

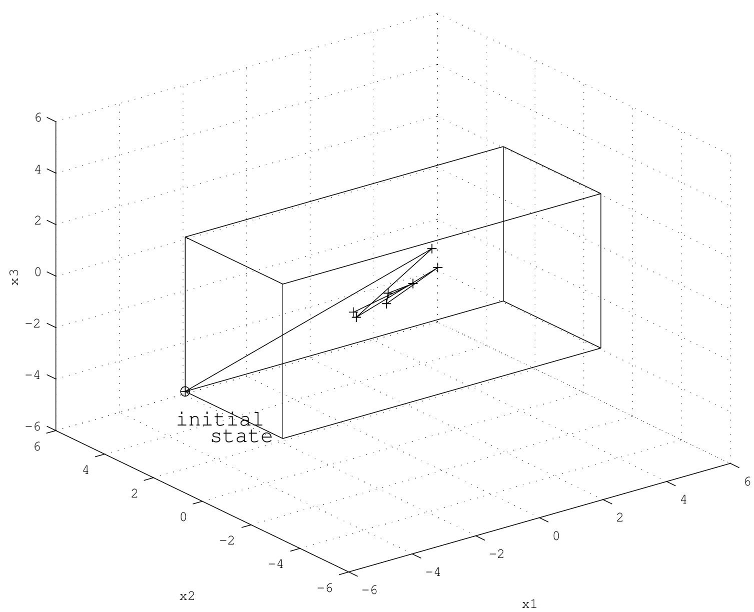

Figure 1: State trajectory $x(t)$ of uncertain system Σ_1

Example 3.3. Consider the following uncertain discrete-time system

$$\Sigma_2 : \; x(t+1) = \left(\begin{bmatrix} 0.24 & -0.32 & 0.20 \\ -0.32 & 0.36 & -0.60 \\ 0.36 & -0.36 & 0.48 \end{bmatrix} + q(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) x(t)$$
$$+ \left(\begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} + q(t) \begin{bmatrix} 0.04 \\ -0.04 \\ 0.04 \end{bmatrix} \right) d(t) ,$$

where $x(t) \in \mathcal{X} := \mathbf{R}^3$ is the state vector, $d(t) \in \mathcal{D} := [-0.2, 0.2]^T \subset \mathbf{R}$ is the disturbance function and $q(t) \in \mathcal{Q} := [-0.3, 0.3]^T \subset \mathbf{R}$ is the uncertain parameter. Then, since the conditions of Theorem 3.1 are not satisfied, we can see the set $\mathcal{P}$ in Example 3.2 is not positive $\mathcal{D}$-invariant for the system Σ_2. In fact, we can see state trajectory which was started from the initial state in $\mathcal{P}$ does not remain in it (see Figure 2).

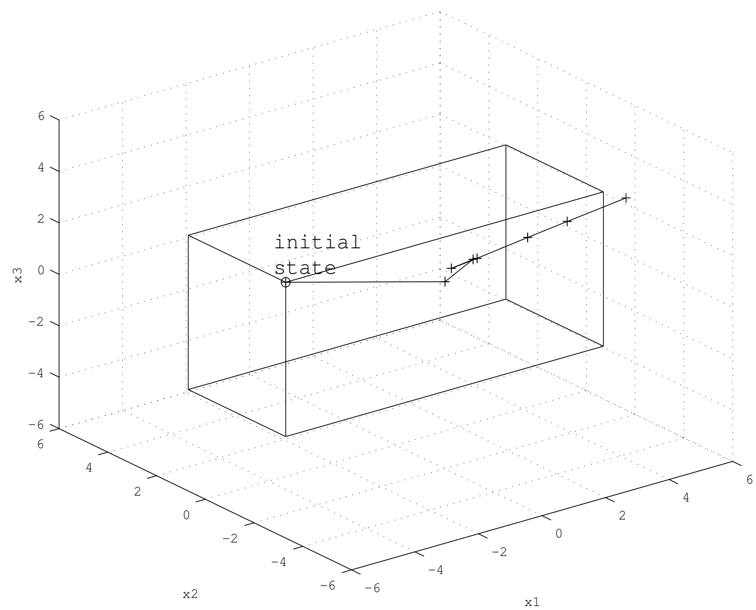

Figure 2: State trajectory $x(t)$ of uncertain system Σ_2

We consider now a continuous-time systems Σ_{uc}. The following theorem was also investigated by Blanchini.

Theorem 3.4. (see [7]) *A set $\mathcal{P}$ is positive $\mathcal{D}$-invariant for uncertain continuous-time systems Σ_{uc} if and only if there exists a τ (> 0) such that for* *every vertices* $v_j^p \in vert(\mathcal{P})$ $(j = 1, \cdots, N)$, $v_h^q \in vert(\mathcal{Q})$ $(h = 1, \cdots, L)$ *and* $v_l^d \in vert(\mathcal{D})$ $(l = 1, \cdots, R)$ *the following conditions are satisfied.*

$$g_i^T\{(I + \tau A(v_h^q))v_j^p + \tau E(v_h^q)v_l^d\} \leq \theta_i, \quad i = 1, \cdots, s.$$

The above theorem say that a set $\mathcal{P}$ is PDI for uncertain continuous-time system Σ_{uc} if and only if $\mathcal{P}$ is PDI for the Euler approximating uncertain discrete-time system which depends on v_h^q :

$$x(t+1) = (I + \tau A(v_h^q))x(t) + \tau E(v_h^q)d(t)$$

for some $\tau > 0$ and $h = 1, \cdots, L$.

Further, the following theorem can be easily obtained from the results of [2] and [7].

Theorem 3.5. *A set $\mathcal{P}$ is positive $\mathcal{D}$-invariant for uncertain continuous-time systems Σ_{uc} if and only if for all (g_i^T, θ_i) satisfying $g_i^T v_j^p = \theta_i$ for each vertex $v_j^p \in$ vert$(\mathcal{P})$ $(j = 1, \cdots, N)$, $v_h^q \in vert(\mathcal{Q})$ $(h = 1, \cdots, L)$ and $v_l^d \in vert(\mathcal{D})$ $(l = 1, \cdots, R)$ the following conditions are satisfied.*

$$g_i^T\{(I + \tau A(v_h^q))v_j^p + \tau E(v_h^q)v_l^d\} \leq \theta_i, \quad i = 1, \cdots, s, \quad \text{for all } \tau (> 0).$$

As the results of the above two theorems depend on parameter τ, it is not easy to check the conditions. However, the following theorem does not include parameter τ is useful result to check the conditions.

Theorem 3.6. *A set $\mathcal{P}$ is positive $\mathcal{D}$-invariant for uncertain continuous-time systems Σ_{uc} if and only if for all (g_i^T, θ_i) satisfying $g_i^T v_j^p = \theta_i$ for each vertex $v_j^p \in$ vert$(\mathcal{P})$ $(j = 1, \cdots, N)$, $v_h^q \in vert(\mathcal{Q})(h = 1, \cdots, L)$ and $v_l^d \in vert(\mathcal{D})(l = 1, \cdots, R)$ the following conditions are satisfied.*

$$g_i^T\{A(v_h^q)v_j^p + E(v_h^q)v_l^d\} \leq 0, \quad i = 1, \cdots, s.$$

Proof. It follows from Theorem 3.5 that

$$\begin{aligned} g_i^T\{(I + \tau A(v_h^q))v_j^p + \tau E(v_h^q)v_l^d\} &= g_i^T v_j^p + \tau g_i^T\{A(v_h^q)v_j^p + E(v_h^q)v_l^d\} \\ &\leq \theta_i \quad \text{for all } \tau\ (> 0). \end{aligned}$$

Noticing that $g_i^T v_j^p = \theta_i$ for each vertex v_j^p and τ is arbitrary positive real number, we can obtain

$$g_i^T\{A(v_h^q)v_j^p + E(v_h^q)v_l^d\} \leq 0,$$

which proves this theorem. □

Example 3.7. Consider the following uncertain continuous time system

$$\dot{x}(t)=\left(\begin{bmatrix} -1.5 & -1 & 0 \\ 0 & -1 & -0.5 \\ 0 & 0 & -0.5 \end{bmatrix}+q(t)\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)x(t)+\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}d(t),$$

where $x(t) \in \mathcal{X} := \mathbf{R}^3$ is the state vector, $d(t) \in \mathcal{D} := [-0.2, 0.2]^T \subset \mathbf{R}$ is the disturbance function and $q(t) \in \mathcal{Q} := [-0.1, 0.1]^T \subset \mathbf{R}$ is the uncertain parameter.

Consider the following polytope $\mathcal{P}$.

$$\mathcal{P} = \left\{ x \in \mathbf{R}^3 \mid \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Then, since the conditions of Theorem 3.6 are satisfied, we can see the set $\mathcal{P}$ is positive $\mathcal{D}$-invariant for the given system (see Figure 3).

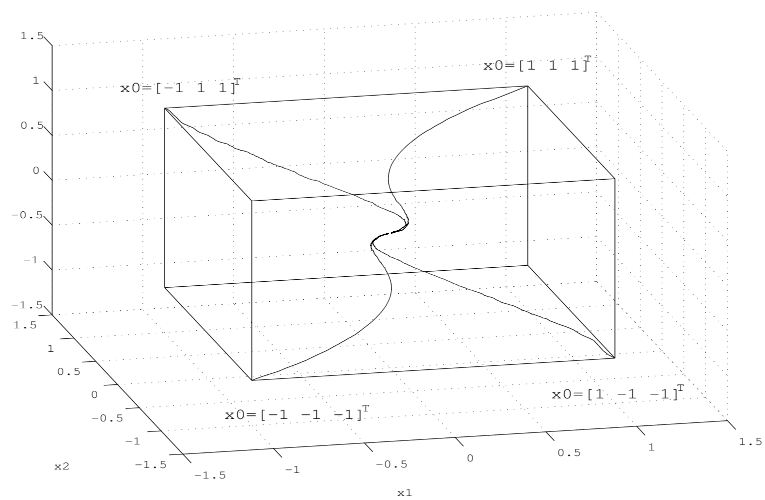

Figure 3: State trajectory $x(t)$ of uncertain system Σ_3

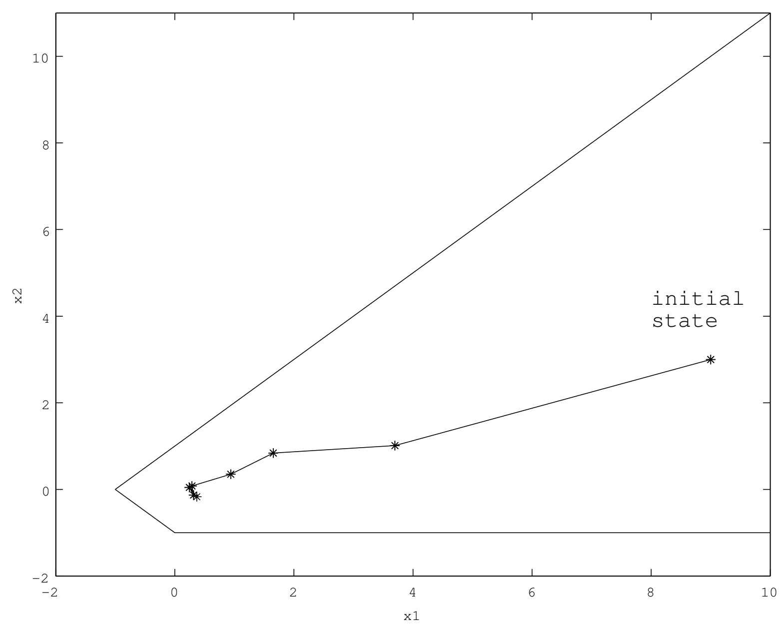

Figure 4: State trajectory $x(t)$ of uncertain systems in polyhedral set S

4. Positive $\mathcal{D}$-Invariance Conditions for Polyhedral Set

4.1. Non-Uncertain Discrete-Time Systems

Consider the following discrete-time system

$$\Sigma_d : \ x(t+1) = Ax(t) + Ed(t)\,,$$

where $x(t) \in \mathcal{X} := \mathbf{R}^n$ is the state, $d(t) \in \mathcal{D} \subset \mathbf{R}^m$ is the disturbance and A and E are constant real matrices of appropriate dimensions. The set $\mathcal{D}$ is a compact and convex containing the origin. Here, we consider the following polyhedral set $\mathcal{S}$ which contains the origin in its interior.

$$\mathcal{S} := \{x \in \mathbf{R}^n \mid g_i^T x \leq \theta_i,\ \theta_i > 0, i = 1, \cdots, s\} = \mathcal{P} + \mathcal{W},$$

where $\mathcal{P} = conv(v_1^p, \cdots, v_N^p)$ is a polytope and $\mathcal{W} = cone(v_1^w, \cdots, v_M^w)$ is a convex cone.

Then, the following theorem can be obtained.

Theorem 4.1. *A set $\mathcal{S}$ of $\mathcal{X}$ is positive $\mathcal{D}$-invariant for system Σ_d if and*

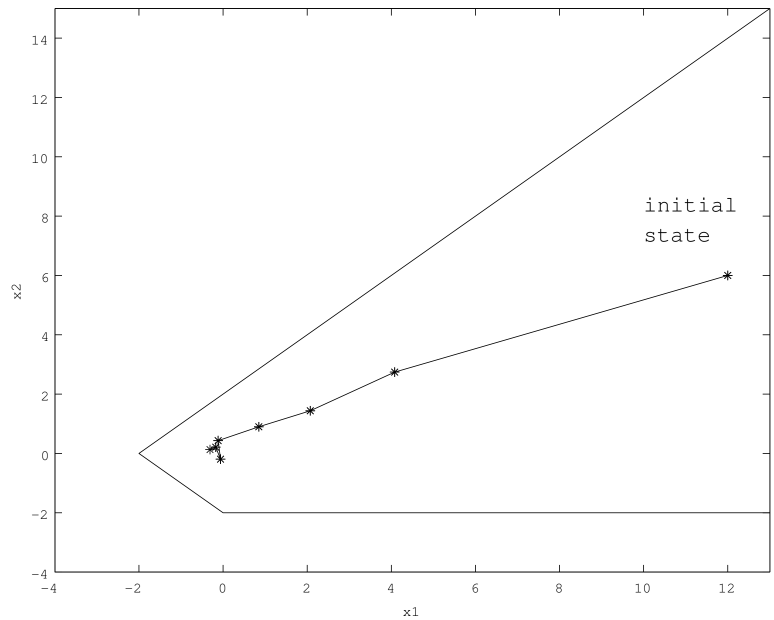

Figure 5: State trajectory $x(t)$ of uncertain systems in polyhedral set $2S$

only if for all vertices v_j^p of $\mathcal{P}$, v_l^d of $\mathcal{D}$ and generating vectors v_k^w of $\mathcal{W}$ the following two conditions are satisfied.

(i) $g_i^T(Av_j^p + Ev_l^d) \leq \theta_i$, for all $i = 1, \cdots, s$.

(ii) $g_i^T(Av_k^w) \leq 0$, for all $i = 1, \cdots, s$.

Proof. (Necessity) Suppose that $\mathcal{S}$ is positive $\mathcal{D}$-invariant. Further, suppose that vertices $v_j^p \in \mathcal{P}$, $v_l^d \in \mathcal{D}$ and generating vector $v_k^w \in cone(\mathcal{W})$. Then, since $v_j^p \in \mathcal{S}$ and $v_l^d \in \mathcal{D}$ it follows from the result of [2] that $Av_j^p + Ev_l^d \in \mathcal{S}$, that is,

$$g_i^T(Av_j^p + Ev_l^d) \leq \theta_i, i = 1, \cdots, s. \tag{1}$$

Next, since $v_k^w \in \mathcal{W}$, for arbitrary $\beta \geq 0$, $\beta v_k^w \in \mathcal{W}$.

Hence, $v_j^p + \beta v_k^w \in \mathcal{S}$. Since $\mathcal{S}$ is positive $\mathcal{D}$-invariant, $A(v_j^p + \beta v_k^w) + Ev_l^d \in \mathcal{S}$, that is,

$$\begin{aligned} g_i^T\{A(v_j^p + \beta v_k^w) + Ev_l^d\} &= g_i^T(Av_j^p) + \beta g_i^T(v_k^w) + g_i^T(Ev_l^d) \\ &\leq \theta_i, \; i = 1, \cdots, s. \end{aligned} \tag{2}$$

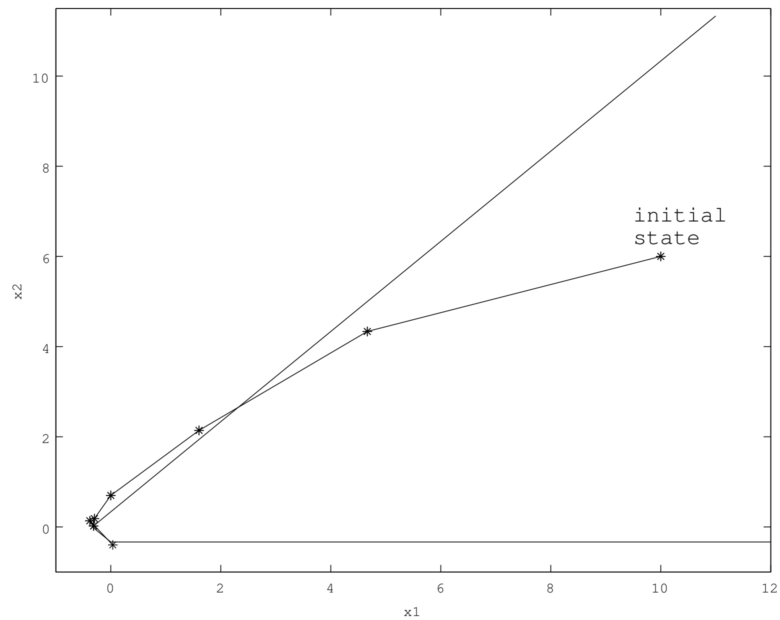

Figure 6: State trajectory $x(t)$ of uncertain systems in polyhedral set $\frac{1}{3}S$

Now, since β is arbitrary positive real number it follows from (1) and (2) that

$$g_i^T(Av_k^w) \leq 0 \ \text{ for all } \ i = 1, \cdots, s. \tag{3}$$

The necessity follows from (1) and (3).

(Sufficiency) Suppose that statements (i) and (ii) hold. In order to prove that a set $\mathcal{S}$ of $\mathcal{X}$ is positive $\mathcal{D}$-invariant it suffices to show that for arbitrary element $x = (\alpha_1 v_1^p + \cdots + \alpha_N v_N^p) + (\beta_1 v_1^w + \cdots + \beta_M v_M^w) \in \mathcal{S}$ and disturbance $d = \delta_1 v_1^d + \cdots + \delta_R v_R^d \in \mathcal{D}$

$$Ax + Ed \in \mathcal{S},$$

$$\text{i.e., } \ g_i^T(Ax + Ed) \leq \theta_i \ \text{ for all } i = 1, \cdots, s,$$

where $\displaystyle\sum_{j=1}^{N} \alpha_j = 1, \ \alpha_j \geq 0, \beta_i \geq 0 \ \ (i = 1, \cdots, M)$ and $\displaystyle\sum_{l=1}^{R} \delta_l = 1, \ \delta_l \geq 0.$

Now, for each $g_i^T, i = 1, \cdots, s$ we have

$$\begin{aligned} g_i^T(Ax + Ed) = g_i^T\{\alpha_1 Av_1^p + \cdots + \alpha_N Av_N^p\} + g_i^T\{\beta_1 Av_1^w \\ + \cdots + \beta_M Av_M^w\} + g_i^T\{\delta_1 Ev_1^d + \cdots + \delta_R Ev_R^d\}. \end{aligned} \tag{4}$$

Further, it follows from (ii) that

$$g_i^T\{\beta_1 Av_1^w + \cdots \beta_M Av_M^w\} \le 0. \tag{5}$$

Since $\sum_{j=1}^{N}\alpha_j = 1,\ \alpha_j \ge 0$ and $\sum_{l=1}^{R}\delta_l = 1,\ \delta_l \ge 0$, we have

$$g_i^T\{\alpha_1 Av_1^p + \cdots + \alpha_N Av_N^p\} \le \max_{l=1,\cdots,N} g_i^T Av_j^p \tag{6}$$

and

$$g_i^T\{\delta_1 Ev_1^d + \cdots + \delta_R Ev_R^d\} \le \max_{l=1,\cdots,R} g_i^T Ev_l^d. \tag{7}$$

It follows from statement (i) that

$$\max_{l=1,\cdots,N} g_i^T Av_j^p + \max_{l=1,\cdots,R} g_i^T Ev_l^d \le \theta_i. \tag{8}$$

Hence from the above inequalities (4)-(8) we have

$$\begin{aligned} g_i^T(Ax+Ed) &= g_i^T\{\alpha_1 Av_1^p + \cdots + \alpha_N Av_N^p\} + g_i^T\{\beta_1 Av_1^w + \cdots + \beta_M Av_M^w\} \\ &\quad + g_i^T\{\delta_1 Ev_1^d + \cdots + \delta_R Ev_R^d\} \\ &\le \max_{l=1,\cdots,N} g_i^T Av_j^p \ + \ 0 \ + \ \max_{l=1,\cdots,R} g_i^T Ev_l^d \le \theta_i, \quad i=1,\cdots,s, \end{aligned}$$

which proves the set $\mathcal{S}$ is positive $\mathcal{D}$-invariant for the system Σ_d. □

The following lemma is used to prove Corollary 4.3.

Lemma 4.2. *If x_0 is in interior of the polyhedral set*

$$\mathcal{S} = \{x \in \mathbf{R}^n \mid g_i^T x \le \theta_i,\ \theta_i > 0, i = 1,\cdots,s\},$$

then there exists a λ $(0 \le \lambda < 1)$ such that $g_i^T x_0 \le \lambda\theta_i,\ i=1,\cdots,s$. □

Corollary 4.3. *A set $\mathcal{S}$ of $\mathcal{X}$ is strongly positive $\mathcal{D}$-invariant for the system Σ_d if and only if for all vertices v_j^p of $\mathcal{P}$, v_l^d of $\mathcal{D}$ and generating vectors v_k^w of $\mathcal{W}$ the following two conditions are satisfied.*

(i) There exists a λ $(0 \le \lambda < 1)$ such that

$$g_i^T(Av_j^p + Ev_l^d) \le \lambda\theta_i \ \text{ for all } \ i=1,\cdots,s.$$

(ii) $g_i^T(Av_k^w) \le 0$ for all $i=1,\cdots,s$.

4.2. Uncertain Discrete-Time Systems

Consider the following uncertain discrete-time systems

$$\Sigma_{ud}:\ x(t+1) = A(q(t))x(t) + E(q(t))d(t)\,,$$

where $A(q(t))$ and $E(q(t))$ have the following structured uncertain parameters

$$A(q(t)) = A_0 + q_1(t)A_1 + \cdots + q_p(t)A_p\,,$$

$$E(q(t)) = E_0 + q_1(t)E_1 + \cdots + q_p(t)E_p\,,$$

where $A_i \in \mathbf{R}^{n\times n}$, $E_i \in \mathbf{R}^{n\times m}$ are known matrices and $x(t) \in \mathbf{R}^n$ is the state vector, $d(t) \in \mathcal{D}(\subset \mathbf{R}^m)$ is the disturbance vector and $q(t) := [q_1, \cdots, q_p(t)]^T \in \mathcal{Q}(\subset \mathbf{R}^p)$ is the uncertain vector. Sets $\mathcal{D}$ and $\mathcal{Q}$ are polytopes which contain origins in their interiors, respectively and let $v_h^q \in vert(\mathcal{Q})(h = 1, \cdots, L)$ and $v_l^d \in vert(\mathcal{D})(l = 1, \cdots, R)$ be vertices, respectively.

Consider the following polyhedral set $\mathcal{S}$ which contains the origin in its interior.

$$\begin{aligned}\mathcal{S} &:= \{x \in \mathbf{R}^n \mid g_i^T x \le \theta_i,\ \theta_i > 0, i = 1, \cdots, s\}\\ &= \mathcal{P} + \mathcal{W},\end{aligned}$$

where $\mathcal{P}=conv(v_1^p, \cdots, v_N^p)$ and $\mathcal{W}=cone(v_1^w, \cdots, v_M^w)$.

Then, the following lemma is used to prove Theorem 4.5.

Lemma 4.4. *A set $\mathcal{S}$ of $\mathcal{X}$ is (strongly) positive $\mathcal{D}$-invariant for the system Σ_{ud} if and only if the set $\mathcal{S}$ is (strongly) positive $\mathcal{D}$-invariant for the following system which depends on each vertex v_h^q of $\mathcal{Q}$.*

$$x(t+1) = A(v_h^q)x(t) + E(v_h^q)d(t).$$

Theorem 4.5. *A set $\mathcal{S}$ of $\mathcal{X}$ is positive $\mathcal{D}$-invariant for the system Σ_{ud} if and only if for all vertices v_j^p of $\mathcal{P}$, v_h^q of $\mathcal{Q}$, v_l^d of $\mathcal{D}$ and generating vectors v_k^w of $\mathcal{W}$ the following two conditions are satisfied.*

(i) $g_i^T(A(v_h^q)v_j^p + E(v_h^q)v_l^d) \le \theta_i$ *for all* $i = 1, \cdots, s$.

(ii) $g_i^T(A(v_h^q)v_k^w) \le 0$ *for all* $i = 1, \cdots, s$.

Proof. *(Necessity)* Suppose that a set $\mathcal{S}$ of $\mathcal{X}$ is positive $\mathcal{D}$-invariant for the system Σ_{ud}. Then, it follows from Lemma 4.4 that the set $\mathcal{S}$ is positive $\mathcal{D}$-invariant for the following system :

$$x(t+1) = A(v_h^q)x(t) + E(v_h^q)d(t)\,,$$

which depends on each vertice v_h^q of $\mathcal{Q}$, which satisfy the conditions (i) and (ii) from Theorem 4.1.

(Sufficiency) The proof follows from Theorem 4.1 and Lemma 4.4. □

Corollary 4.6. *A set $\mathcal{S}$ of $\mathcal{X}$ is strongly positive $\mathcal{D}$-invariant for the system Σ_{ud} if and only if for all vertices v_j^p of $conv(\mathcal{P})$, v_h^q of $\mathcal{Q}$ and v_l^d of $\mathcal{D}$ the following two conditions are satisfied.*

(i) There exists a $\lambda(0 \leq \lambda < 1)$ *such that*

$$g_i^T(A(v_h^q)v_j^p + E(v_h^q)v_l^d) \leq \lambda\theta_i \text{ for all } i = 1, \cdots, s.$$

(ii) $g_i^T(A(v_h^q)v_k^w) \leq 0$ *for all* $i = 1, \cdots, s$.

Proof. The proof follows from Lemma 4.4 and Corollary 4.3.

The following properties on polyhedral set hold.

Theorem 4.7. *If a set* $\mathcal{S}$ *of* $\mathcal{X}$ *is (strongly) positive* $\mathcal{D}$*-invariant for the system* Σ_{ud}*, then for arbitrary* $\mu(\geq 1)$ $\mu\mathcal{S}$ *is also (strongly) positive* $\mu\mathcal{D}$*-invariant for the system* Σ_{ud}*. If a set* $\mathcal{S}$ *of* $\mathcal{X}$ *is (strongly) positive-invariant for the system* Σ_{ud}*, then for arbitrary* $\mu(> 0)$ $\mu\mathcal{S}$ *is also (strongly) positive-invariant for the system* Σ_{ud}*.*

Proof. Firstly, suppose that a set $\mathcal{S}$ of $\mathcal{X}$ is positive $\mathcal{D}$-invariant for the system Σ_{ud}. Then, it follows from Theorem 4.5 that for arbitrary $\mu(\geq 1)$ the following conditions

$$g_i^T(A(v_h^q)\mu v_j^p + E(v_h^q)\mu v_l^d) \leq \mu\theta_i \qquad \text{for all } i = 1, \cdots, s,$$

$$g_i^T(A(v_h^q)\mu v_k^w) \leq 0 \qquad \text{for all } i = 1, \cdots, s,$$

are satisfied. Since $\mu\mathcal{S} = \mu\mathcal{P} + \mu\mathcal{W}$, $\mu v_j^p = vert(\mu\mathcal{P})$, $\mu v_k^w = gen(\mu\mathcal{W})$ and $\mu v_l^d = vert(\mu\mathcal{D})$, it follows again from Theorem 4.5 that $\mu\mathcal{S}$ is positive $\mu\mathcal{D}$-invariant for the system Σ_{ud}. In addition, about strongly positive $\mathcal{D}$-invariance for the system Σ_{ud}, we can similarly prove it by Corollary 4.6.

Next, suppose $\mathcal{D} = \{0\}$. Let a set $\mathcal{S}$ of $\mathcal{X}$ be positive-invariant for the system Σ_{ud}. Then, it follows from Theorem 4.5 that for arbitrary $\mu(> 0)$ the following conditions

$$g_i^T(A(v_h^q)\mu v_j^p) \leq \mu\theta_i \text{ for all } i = 1, \cdots, s,$$

$$g_i^T(A(v_h^q)\mu v_k^w) \leq 0 \text{ for all } i = 1, \cdots, s,$$

are satisfied. Then, it follows from Theorem 4.5 that $\mu\mathcal{S}$ is positive-invariant for the system Σ_{ud}. Similarly, we can easily obtain about strongly positive-invariance. □

From the above theorem $\mu(\geq 1)$-extensions of a positive $\mathcal{D}$-invariant set are also positive $\mu\mathcal{D}$ invariant for the system. However, $\mu(< 1)$-reductions of a positive $\mathcal{D}$-invariant set are not necessarily positive $\mu\mathcal{D}$-invariant for the system. In the case of no disturbances we note both extensions and reductions of a positive invariant set are always positive invariant for the system.

Example 4.8. Consider the following uncertain discrete-time system

$$\Sigma_{ud}: \; x(t+1) = \left(\begin{bmatrix} 0.5 & -0.25 \\ 0.25 & 0 \end{bmatrix} + q(t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) x(t) + \left(\begin{bmatrix} 3 \\ -3 \end{bmatrix} + q(t) \begin{bmatrix} 0.01 \\ -0.01 \end{bmatrix} \right) d(t),$$

where $x(t) \in \mathcal{X} := \mathbf{R}^2$ is the state vector, $d(t) \in \mathcal{D} := [-0.1, 0.1] \subset \mathbf{R}$ is the disturbance function and $q(t) \in \mathcal{Q} := [-0.2, 0.2] \subset \mathbf{R}$ is uncertain parameter. Then, it can be easily shown the following polyhedral set $\mathcal{S} = \mathcal{P} + \mathcal{W}$ satisfy the two conditions (i),(ii) of Theorem 4.5. Hence,$\mathcal{S}$ is positive $\mathcal{D}$-invariant for the system (see Figure 4).

$$\mathcal{S} = \left\{ x \in \mathbf{R}^2 \mid \begin{bmatrix} 0 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \underbrace{conv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{=\mathcal{P}} + \underbrace{cone \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{=\mathcal{W}}.$$

Further, we note that $2\mathcal{S}$ is also positive $2\mathcal{D}$-invariant (which implies positive $\mathcal{D}$-invariant) for the system (see Figure 5), but $\frac{1}{3}\mathcal{S}$ is not positive $\mathcal{D}$-invariant (see Figure 6).

5. Conclusion

In this paper, necessary and sufficient conditions for a given polytope set which is represented as a set of linear inequalities to be positive $\mathcal{D}$-invariant for uncertain linear continuous-time systems were given without parameter τ which was appeared in Euler approximation. Further, necessary and sufficient conditions for a given polyhedral set to be positive $\mathcal{D}$-invariant for uncertain linear discrete-time systems were also given. Finally, some properties and simulation results for positive $\mathcal{D}$-invariant set were investigated.

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