International Journal of Pure and Applied Mathematics
Volume 53 No. 4 2009, 513-521

ON NONUNIQUE FIXED POINTS FOR FOUR CLASSES OF MAPPINGS

Jinbiao Hao[1], Anshu Jiang[2], Young Chel Kwun[3], Shin Min Kang[4 §]

[1,2]Department of Mathematics
Liaoning Normal University
Dalian, Liaoning, 116029, P.R. CHINA
[1]e-mail: haojinbiao@yahoo.cn
[2]e-mail: anshujiang@163.com

[3]Department of Mathematics
Dong-A University
Pusan, 614-714, KOREA
e-mail: yckwun@daunet.donga.ac.kr

[4]Department of Mathematics
The Research Institute of Natural Science
Gyeongsang National University
Jinju, 660-701, KOREA
e-mail: smkang@gnu.ac.kr

Abstract: The aim of this paper is to establish the existence of nonunique fixed points for four classes of nonlinear mappings in orbitally complete metric spaces. The results presented in this paper extend and improve some known results in the literature.

AMS Subject Classification: 54H25
Key Words: orbitally continuous mapping, orbitally complete metric space, nonunique fixed point

1. Introduction

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}$ denote the set of all positive integers and

Received: May 12, 2009

[§]Correspondence author

$$\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ satisfies (a)-(d)}\},$$

where

(a) $0 < \varphi(t) < t$ for $t > 0$ and $\varphi(0) = 0$;

(b) φ is nondecreasing on $[0, +\infty)$;

(c) $\psi(t) = \frac{t}{t-\varphi(t)}$ is nonincreasing on $(0, +\infty)$;

(d) $\int_0^y \psi(t)dt < +\infty$ for each $y > 0$.

A self mapping T on a metric space (X, d) is called *orbitally continuous* if $\lim_{i\to\infty} T^{n_i}x = u$ implies $\lim_{i\to\infty} TT^{n_i}x = Tu$ for each $x \in X$. A metric space X is said to be T-*orbitally complete* if each Cauchy sequence of the form $\{T^n x\}_{n\in\mathbb{N}}$, $x \in X$, converges in X.

Altman [1] and Watson-Meade-Norris [4] got some fixed point theorems for a generalized contraction T from X into itself satisfying

$$d(Tx, Ty) \le \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$ and some $\varphi \in \Phi$.

Ćirić [2] proved the existence of nonunique fixed points for a mapping T from X into itself satisfying

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \le qd(x, y)\,, \tag{1.2}$$

for all $x, y \in X$, where $q \in [0, 1)$ is a constant.

Liu-Park [3] extended Ćirić's results to a more general case and gave several sufficient conditions, which ensure the existence of nonunique fixed points for a mapping T from X into itself satisfying

$$\begin{aligned} &\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \\ &\le \varphi(\max\{d(x, y), \min\{d(x, Tx), d(y, Ty)\}, \min\{d(x, Ty), d(y, Tx)\}\}) \end{aligned} \tag{1.3}$$

for all $x, y \in X$ and some $\varphi \in \Phi$.

Inspired by the above results, in this paper we introduce and study four classes of nonlinear mappings T from X into itself satisfying, respectively, the following conditions

$$\begin{aligned} &\min\{d(Tx, Ty), d(x, Tx), d(y, Ty), d(x, y)\} - \min\{d(x, Ty), d(y, Tx)\} \\ &\le \varphi\Big(\max\Big\{ d(x, y), \min\{d(x, Tx), d(y, Ty)\}, \min\{d(x, Ty), d(y, Tx)\}, \\ &\min\Big\{\frac{d^2(x, Tx)}{1 + d(x, y)}, \frac{d^2(y, Ty)}{1 + d(x, y)}\Big\}, \\ &\min\Big\{\frac{d^2(x, Tx)}{1 + d(Tx, Ty)}, \frac{d^2(y, Ty)}{1 + d(Tx, Ty)}\Big\}, \min\Big\{\frac{d^2(x, y)}{1 + d(x, Tx)}, \frac{d^2(x, y)}{1 + d(y, Ty)}\Big\}, \end{aligned}$$

$$\min\left\{\frac{d^2(x,Ty)}{1+d(x,y)},\frac{d^2(y,Tx)}{1+d(x,y)}\right\},\min\left\{\frac{d^2(x,Ty)}{1+d(Tx,Ty)},\frac{d^2(y,Tx)}{1+d(Tx,Ty)}\right\},$$
$$\min\left\{\frac{d^2(x,Ty)}{1+d(x,Tx)},\frac{d^2(y,Tx)}{1+d(y,Ty)}\right\}\Big\}\Big) \quad (1.4)$$

for all $x,y\in X$ and some $\varphi\in\Phi$;

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty),d(x,y)\}-\min\{d(x,Ty),d(y,Tx)\}$$
$$<\max\Big\{d(x,y),\min\{d(x,Tx),d(y,Ty)\},\min\{d(x,Ty),d(y,Tx)\},$$
$$\min\left\{\frac{d^2(x,Tx)}{1+d(x,y)},\frac{d^2(y,Ty)}{1+d(x,y)}\right\},\min\left\{\frac{d^2(x,Tx)}{1+d(Tx,Ty)},\frac{d^2(y,Ty)}{1+d(Tx,Ty)}\right\},$$
$$\min\left\{\frac{d^2(x,y)}{1+d(x,Tx)},\frac{d^2(x,y)}{1+d(y,Ty)}\right\},\min\left\{\frac{d^2(x,Ty)}{1+d(x,y)},\frac{d^2(y,Tx)}{1+d(x,y)}\right\},$$
$$\min\left\{\frac{d^2(x,Ty)}{1+d(Tx,Ty)},\frac{d^2(y,Tx)}{1+d(Tx,Ty)}\right\},$$
$$\min\left\{\frac{d^2(x,Ty)}{1+d(x,Tx)},\frac{d^2(y,Tx)}{1+d(y,Ty)}\right\}\Big\} \quad (1.5)$$

for any distinct $x,y\in X$;

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty),d(x,y)\}-\min\{d(x,Ty),d(y,Tx)\}$$
$$\leq\varphi\Big(\max\Big\{d(x,y),\min\{d(x,Tx),d(y,Ty)\},\min\{d(x,Ty),d(y,Tx)\},$$
$$\min\left\{\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)},\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\},$$
$$\min\left\{\frac{d(x,y)d(Tx,Ty)}{1+d(y,Ty)},\frac{d(x,y)d(Tx,Ty)}{1+d(x,Tx)}\right\},$$
$$\min\left\{\frac{d(x,y)d(y,Ty)}{1+d(Tx,Ty)},\frac{d(x,y)d(x,Tx)}{1+d(Tx,Ty)}\right\},$$
$$\min\left\{\frac{d(x,y)d(y,Ty)}{1+d(x,Tx)},\frac{d(x,y)d(x,Tx)}{1+d(y,Ty)}\right\},$$
$$\min\left\{\frac{d(x,Tx)d(Tx,Ty)}{1+d(y,Ty)},\frac{d(y,Ty)d(Tx,Ty)}{1+d(x,Tx)}\right\}\Big\}\Big) \quad (1.6)$$

for all $x,y\in X$ and some $\varphi\in\Phi$;

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty),d(x,y)\}-\min\{d(x,Ty),d(y,Tx)\}$$
$$<\max\Big\{d(x,y),\min\{d(x,Tx),d(y,Ty)\},\min\{d(x,Ty),d(y,Tx)\},$$
$$\min\left\{\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)},\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\},$$

$$\min\Big\{\frac{d(x,y)d(Tx,Ty)}{1+d(y,Ty)},\frac{d(x,y)d(Tx,Ty)}{1+d(x,Tx)}\Big\},$$
$$\min\Big\{\frac{d(x,y)d(y,Ty)}{1+d(Tx,Ty)},\frac{d(x,y)d(x,Tx)}{1+d(Tx,Ty)}\Big\},$$
$$\min\Big\{\frac{d(x,y)d(y,Ty)}{1+d(x,Tx)},\frac{d(x,y)d(x,Tx)}{1+d(y,Ty)}\Big\},$$
$$\min\Big\{\frac{d(x,Tx)d(Tx,Ty)}{1+d(y,Ty)},\frac{d(y,Ty)d(Tx,Ty)}{1+d(x,Tx)}\Big\}\Big\} \quad (1.7)$$

for any distinct $x, y \in X$. Under certain conditions we prove the existence of nonunique fixed points for the four classes of mappings (1.4)-(1.7) in a T-orbitally complete metric space (X, d), respectively. The results obtained in the paper generalize the corresponding results in [1]-[4].

2. Main Results

Now we show several nonunique fixed point theorems for the mappings T from X into itself satisfying (1.4)-(1.7), respectively.

Theorem 2.1. *Let (X, d) be a T-orbitally complete metric space and $T : X \to X$ be an orbitally continuous mapping satisfying* (1.4). *Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T in X.*

Proof. Let $x_0 \in X$ and $x_n = T^n x_0$ for $n \in \mathbb{N}$. Suppose that $x_{k-1} = x_k$ for some $k \in \mathbb{N}$. Then x_{k-1} is a fixed point of T and $x_n \to x_{k-1}$ as $n \to \infty$. We now assume that $x_{k-1} \neq x_k$ for any $k \in \mathbb{N}$. Using (1.4) we get that

$$\begin{aligned}
&\min\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}\\
&\quad=\min\{d(Tx_{n-1},Tx_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},x_n)\}\\
&\qquad\qquad-\min\{d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\}\\
&\quad\le\varphi\Big(\max\Big\{d(x_{n-1},x_n),\min\{d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n)\},\\
&\qquad\min\{d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\},\\
&\qquad\min\Big\{\frac{d^2(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},x_n)},\frac{d^2(x_n,Tx_n)}{1+d(x_{n-1},x_n)}\Big\},\\
&\qquad\min\Big\{\frac{d^2(x_{n-1},Tx_{n-1})}{1+d(Tx_{n-1},Tx_n)},\frac{d^2(x_n,Tx_n)}{1+d(Tx_{n-1},Tx_n)}\Big\},\\
&\qquad\min\Big\{\frac{d^2(x_{n-1},x_n)}{1+d(x_{n-1},Tx_{n-1})},\frac{d^2(x_{n-1},x_n)}{1+d(x_n,Tx_n)}\Big\},
\end{aligned}$$

$$\min\Big\{\frac{d^2(x_{n-1},Tx_n)}{1+d(x_{n-1},x_n)},\frac{d^2(x_n,Tx_{n-1})}{1+d(x_{n-1},x_n)}\Big\},$$
$$\min\Big\{\frac{d^2(x_{n-1},Tx_n)}{1+d(Tx_{n-1},Tx_n)},\frac{d^2(x_n,Tx_{n-1})}{1+d(Tx_{n-1},Tx_n)}\Big\},$$
$$\min\Big\{\frac{d^2(x_{n-1},Tx_n)}{1+d(x_{n-1},Tx_{n-1})},\frac{d^2(x_n,Tx_{n-1})}{1+d(x_n,Tx_n)}\Big\}\Big\}\Big)$$
$$=\varphi\Big(\max\Big\{d(x_{n-1},x_n),\min\{d(x_{n-1},x_n),d(x_n,x_{n+1})\},0,$$
$$\min\Big\{\frac{d^2(x_{n-1},x_n)}{1+d(x_{n-1},x_n)},\frac{d^2(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\Big\},$$
$$\min\Big\{\frac{d^2(x_{n-1},x_n)}{1+d(x_n,x_{n+1})},\frac{d^2(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}\Big\},$$
$$\min\Big\{\frac{d^2(x_{n-1},x_n)}{1+d(x_{n-1},x_n)},\frac{d^2(x_{n-1},x_n)}{1+d(x_n,x_{n+1})}\Big\},0,0,0\Big\}\Big)=\varphi(d(x_{n-1},x_n)),\quad\forall n\in\mathbb{N},$$

which together with (a) implies that

$$d(x_n,x_{n+1})\le\varphi(d(x_{n-1},x_n)),\quad\forall n\in\mathbb{N}.\tag{2.1}$$

Let $t_0=d(x_0,x_1)$ and $t_n=\varphi(t_{n-1})$ for all $n\in\mathbb{N}$. It is easy to show that

$$0<t_n=\varphi(t_{n-1})=\varphi^n(t_0)<t_{n-1},\quad\forall n\in\mathbb{N},$$

which yields that $\{t_n\}_{n\in\mathbb{N}}$ is convergent. Moreover, in the view of (2.1) and (b), we have

$$d(x_n,x_{n+1})\le\varphi(d(x_{n-1},x_n))\le\varphi^n(d(x_0,x_1))=t_n,\quad\forall n\in\mathbb{N}.$$

It follows that for any $n,p\in\mathbb{N}$

$$d(x_n,x_{n+p})\le\sum_{i=n}^{n+p-1}d(x_i,x_{i+1})\le\sum_{i=n}^{n+p-1}t_i=\sum_{i=n}^{n+p-1}\frac{t_i(t_i-t_{i+1})}{t_i-\varphi(t_i)}$$
$$\le\sum_{i=n}^{n+p-1}\int_{t_{i+1}}^{t_i}\frac{t}{t-\varphi(t)}dt=\int_{t_{n+p}}^{t_n}\psi(t)dt,$$

which together with (c) implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is T-orbitally complete, there exists $v\in X$ such that $x_n\to v$ as $n\to\infty$. Hence $Tv=\lim_{n\to\infty}T^{n+1}x_0=v$ by the orbital continuity of T. This completes the proof. □

Taking $\varphi(t)=qt$ in Theorem 2.1, we obtain that

Corollary 2.1. (see Altman [1] and Watson et al [4]) *Let (X,d) be a complete metric space and $T:X\to X$ be a generalized contraction. Then T has a unique fixed point in X.*

Proof. Note that T is a generalized contraction. Hence T is continuous and satisfies (1.4). By Theorem 2.1 it follows that T has a fixed point. The uniqueness of the fixed point is clear. This completes the proof. □

Theorem 2.2. *Let T be an orbitally continuous self mapping of a metric space (X,d) satisfying (1.5). If for some $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ has a cluster point $u \in X$, then u is a fixed point of T.*

Proof. Let $x_n = T^n x_0$ for $n \in \mathbb{N}$. Since u is a cluster point of the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$, it follows that $x_{r-1} \neq x_r$ for any $r \in \mathbb{N}$ and there exists a subsequence $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ such that $\lim_{i\to\infty} x_{n_i} = u$. By (1.5) we have

$$
\begin{aligned}
&\min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}\\
&\quad = \min\{d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, x_n)\}\\
&\qquad - \min\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}\\
&\quad < \max\Big\{d(x_{n-1}, x_n), \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\},\\
&\qquad \min\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\},\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, Tx_{n-1})}{1+d(x_{n-1}, x_n)}, \frac{d^2(x_n, Tx_n)}{1+d(x_{n-1}, x_n)}\Big\},\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, Tx_{n-1})}{1+d(Tx_{n-1}, Tx_n)}, \frac{d^2(x_n, Tx_n)}{1+d(Tx_{n-1}, Tx_n)}\Big\},\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, x_n)}{1+d(x_{n-1}, Tx_{n-1})}, \frac{d^2(x_{n-1}, x_n)}{1+d(x_n, Tx_n)}\Big\},\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, Tx_n)}{1+d(x_{n-1}, x_n)}, \frac{d^2(x_n, Tx_{n-1})}{1+d(x_{n-1}, x_n)}\Big\},\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, Tx_n)}{1+d(Tx_{n-1}, Tx_n)}, \frac{d^2(x_n, Tx_{n-1})}{1+d(Tx_{n-1}, Tx_n)}\Big\},\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, Tx_n)}{1+d(x_{n-1}, Tx_{n-1})}, \frac{d^2(x_n, Tx_{n-1})}{1+d(x_n, Tx_n)}\Big\}\Big\}\\
&\quad = \max\Big\{d(x_{n-1}, x_n), \min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, 0,\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, x_n)}{1+d(x_{n-1}, x_n)}, \frac{d^2(x_n, x_{n+1})}{1+d(x_{n-1}, x_n)}\Big\},\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, x_n)}{1+d(x_n, x_{n+1})}, \frac{d^2(x_n, x_{n+1})}{1+d(x_n, x_{n+1})}\Big\},\\
&\qquad \min\Big\{\frac{d^2(x_{n-1}, x_n)}{1+d(x_{n-1}, x_n)}, \frac{d^2(x_{n-1}, x_n)}{1+d(x_n, x_{n+1})}\Big\}, 0, 0, 0\Big\}
\end{aligned}
$$

$$= d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N},$$

which gives that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}$. Hence the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is convergent. Note that

$$d(u, Tu) = \lim_{i\to\infty} d(x_{n_i}, x_{n_i+1}) = \lim_{i\to\infty} d(x_{n_i+1}, x_{n_i+2}) = d(Tu, T^2u).$$

Suppose that $u \neq Tu$. It follows from (1.5) that

$$\begin{aligned}
&\min\{d(Tu, T^2u), d(u, Tu), d(Tu, T^2u), d(u, Tu)\} - \min\{d(u, T^2u), d(Tu, Tu)\} \\
&< \max\Big\{d(u, Tu), \min\{d(u, Tu), d(Tu, T^2u)\}, \\
&\min\{d(u, T^2u), d(Tu, Tu)\}, \min\Big\{\frac{d^2(u, Tu)}{1 + d(u, Tu)}, \frac{d^2(Tu, T^2u)}{1 + d(u, Tu)}\Big\}, \\
&\min\Big\{\frac{d^2(u, Tu)}{1 + d(Tu, T^2u)}, \frac{d^2(Tu, T^2u)}{1 + d(Tu, T^2u)}\Big\}, \\
&\min\Big\{\frac{d^2(u, Tu)}{1 + d(u, Tu)}, \frac{d^2(u, Tu)}{1 + d(Tu, T^2u)}\Big\}, \\
&\min\Big\{\frac{d^2(u, T^2u)}{1 + d(u, Tu)}, \frac{d^2(Tu, Tu)}{1 + d(u, Tu)}\Big\}, \\
&\min\Big\{\frac{d^2(u, T^2u)}{1 + d(Tu, T^2u)}, \frac{d^2(Tu, Tu)}{1 + d(Tu, T^2u)}\Big\}, \\
&\min\Big\{\frac{d^2(u, T^2u)}{1 + d(u, Tu)}, \frac{d^2(Tu, Tu)}{1 + d(Tu, T^2u)}\Big\}\Big\},
\end{aligned}$$

which implies that

$$d(Tu, T^2u) < d(u, Tu),$$

which is impossible. Hence $u = Tu$. This completes the proof. □

Theorem 2.3. *Let (X, d) be a T-orbitally complete metric space and $T : X \to X$ be an orbitally continuous mapping satisfying* (1.6). *Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T in X.*

Proof. Let $x_0 \in X$ and $x_n = T^n x_0$ for $n \in \mathbb{N}$. Suppose that $x_{k-1} = x_k$ for some $k \in \mathbb{N}$. Then x_{k-1} is a fixed point of T and $x_n \to x_{k-1}$ as $n \to \infty$. Suppose that $x_{k-1} \neq x_k$ for any $k \in \mathbb{N}$. In view of (1.6) we deduce that

$$\begin{aligned}
&\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&= \min\{d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, x_n)\} \\
&\quad - \min\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\
&\le \varphi\Big(\max\Big\{d(x_{n-1}, x_n), \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\},
\end{aligned}$$

$$\min\{d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\},$$
$$\min\left\{\frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{1+d(Tx_{n-1},Tx_n)},\frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{1+d(x_{n-1},x_n)}\right\},$$
$$\min\left\{\frac{d(x_{n-1},x_n)d(Tx_{n-1},Tx_n)}{1+d(x_n,Tx_n)},\frac{d(x_{n-1},x_n)d(Tx_{n-1},Tx_n)}{1+d(x_{n-1},Tx_{n-1})}\right\},$$
$$\min\left\{\frac{d(x_{n-1},x_n)d(x_n,Tx_n)}{1+d(Tx_{n-1},Tx_n)},\frac{d(x_{n-1},x_n)d(x_{n-1},Tx_{n-1})}{1+d(Tx_{n-1},Tx_n)}\right\},$$
$$\min\left\{\frac{d(x_{n-1},x_n)d(x_n,Tx_n)}{1+d(x_{n-1},Tx_{n-1})},\frac{d(x_{n-1},x_n)d(x_{n-1},Tx_{n-1})}{1+d(x_n,Tx_n)}\right\},$$
$$\min\left\{\frac{d(x_{n-1},Tx_{n-1})d(Tx_{n-1},Tx_n)}{1+d(x_n,Tx_n)},\frac{d(x_n,Tx_n)d(Tx_{n-1},Tx_n)}{1+d(x_{n-1},Tx_{n-1})}\right\}\bigg\}\bigg)$$
$$=\varphi\Big(\max\Big\{d(x_{n-1},x_n),\min\{d(x_{n-1},x_n),d(x_n,x_{n+1})\},0,$$
$$\min\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})},\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right\},$$
$$\min\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})},\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right\},$$
$$\min\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})},\frac{d(x_{n-1},x_n)d(x_{n-1},x_n)}{1+d(x_n,x_{n+1})}\right\},$$
$$\min\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)},\frac{d(x_{n-1},x_n)d(x_{n-1},x_n)}{1+d(x_n,x_{n+1})}\right\},$$
$$\min\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})},\frac{d(x_n,x_{n+1})d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right\}\bigg\}\bigg)$$
$$=\varphi(d(x_{n-1},x_n)),\quad\forall n\in\mathbb{N},$$

which together with (a) yields that

$$d(x_n,x_{n+1})\le\varphi(d(x_{n-1},x_n)),\quad\forall n\in\mathbb{N}.\tag{2.2}$$

As in the proof of Theorem 2.1, we infer that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $v\in X$ and v is a fixed point of T. This completes the proof. □

The following example demonstrates that the fixed points in Theorems 2.1 and 2.3 may not be unique.

Example 2.1. Let $X=\{1,2,3\}$ with the usual metric d. Obviously, (X,d) is a complete metric space. Define a mapping $T:X\to X$ by $T1=2$, $T2=2$ and $T3=3$. It is no difficult to check that the conditions of Theorems 2.1 and 2.3 hold. However T has two fixed points 2 and 3 in X.

The proof of the below result is similar to those of Theorems 2.2 and 2.3, and hence is omitted.

Theorem 2.4. *Let T be an orbitally continuous self mapping of a metric space (X, d) satisfying* (1.7). *If for some $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ has a fixed point $u \in X$, then u is a fixed point of T.*

The following example reveals that the fixed points in Theorems 2.2 and 2.4 may not be unique.

Example 2.2. Let $X = [1, +\infty)$ with the usual metric d and define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \sqrt{x} & \text{if } x \text{ is irrational,} \\ x & \text{if } x \text{ is rational.} \end{cases}$$

It is easily verified that the conditions of Theorems 2.2 and 2.4 are satisfied. But T has infinitely many fixed points in X.

Remark 2.1. Theorems 2.1-2.4 are generalizations of Theorems 1 and 3 in [2] and Theorems 2.1 and 2.2 in [3], respectively.

References

[1] M. Altman, An integral test for series and generalized contractions, *Amer. Math. Monthly*, **82** (1975), 827-829.

[2] L.B. Ćirić, On some maps with a nonunique fixed point, *Publ. Inst. Math.*, **17**, No. 31 (1974), 52-58.

[3] Z. Liu, K.P. Park, On two classes of mappings with nonunique fixed points, *Nonlinear Funct. Anal. Appl.*, **6** (2001), 307-311.

[4] B. Watson, B.A. Meade, C.W. Norris, Note on a theorem of Altman, *Indian J. Pure Appl. Math.*, **17**, No. 9 (1986), 1092-1093.