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AN ANALYTICAL SOLUTION FOR A MODEL OF GROWTH

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Abstract: In this paper we derive a closed-form solution for the Ramsey model with CRRA utility, technological diffusion and logistic population law, as defined by Ferrara and Guerrini (2008), for the case where capital's share is equal to the reciprocal of the intertemporal elasticity of substitution.

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1. Introduction

Ferrara and Guerrini [4] have extended the Ramsey growth model with logistic population change introduced by Accinelli and Brida [1] by incorporating technological progress in a way similar to Duczynski [3]. The resulting model has been showed to be described by a four dimensional dynamical system, whose unique non-trivial steady-state equilibrium is saddle-point stable. As well, the stable saddle-path has been proved to be three dimensional, thus enriching the transitional adjustment paths relative to that of the standard Ramsey growth model. Of course, this model has no analytical solution. In this paper, motivated by the work of Smith [6], who derived a closed-form solution for the standard Ramsey model under appropriate assumptions, we show that there is also an explicit solution for this model, when the intertemporal elasticity of substitution is equal to the reciprocal of capital's share of GDP.

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2. Ramsey Model with Technological Diffusion and Logistic Population Law

In this section we give a formulation of the model studied by Ferrara and Guerrini [4]. The economic system may be seen as a closed economy inhabited by many identical agents facing the following optimization problem

$$\max \int_0^\infty \frac{(C_t/L_t)^{1-\sigma}-1}{1-\sigma}\, e^{-\rho t}\, dt, \tag{1}$$

where C_t is consumption, L_t denotes population (labor), $\sigma > 0$ is the inverse of the constant intertemporal elasticity of substitution, $\rho > 0$ is the rate of time preference. The objective maximization is subject to a standard intertemporal budget constraint, which takes the form

$$\dot{K}_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t - \delta K_t, \tag{2}$$

where K_t is capital stock, A_t represents the state of technology, $\alpha \in (0,1)$ is the capital share, $\delta > 0$ is the rate of capital depreciation, a dot over a variable denoting time derivative. The equation of motion for A_t is assumed to be

$$\frac{\dot{A}_t}{A_t} = g + \lambda\, \frac{\tau A_t^L - A_t}{A_t}, \tag{3}$$

where g, λ, τ are positive parameters, A_t^L is the level of technology in the world's technological leader, $A_t/(\tau A_t^L) \le 1$, and $\tau \ge 1$, so that the economy converges to a lower steady-state than the level of the leading country. The term g in (3) reflects domestic forces of the technological innovations (e.g. domestic research and development), while the term $\lambda(\tau A_t^L - A_t)/A_t$ corresponds to technological diffusion from the leading country. The equation of motion for A_t^L is assumed to take the form $\dot{A}_t^L/A_t^L = g$, i.e. $A_t^L = A_0^L e^{gt}$. An assumption similar to (3) was proposed by Nelson and Phelps [6], but in a context with no parameter τ, and with λ assumed to be a positive function of the domestic human-capital intensity. Population is modelled according to the following law

$$\dot{L}_t = L_t(a - bL_t), \tag{4}$$

where a, b are constants such that $a > b$. For simplicity, the initial population has been normalized to one, $L_0 = 1$. Equation (4) is known as the Verhulst equation, and the underlying population model is known as the logistic model. Solving the continuous-time dynamic problem (1) involves using calculus of variations. The current-value Hamiltonian of our optimization problem writes

$$H = \frac{(C_t/L_t)^{1-\sigma}-1}{1-\sigma} + \mu_t \left[K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t - \delta K_t\right],$$

where μ_t is the co-state variable associated to (2). Setting $x_t = \tau A_t^L/A_t$, then by log differentiating this expression, we get $\dot{A}_t/A_t = g - \dot{x}_t/x_t$, so that, from (3), we obtain

$$\dot{x}_t = \lambda x_t(1 - x_t). \tag{5}$$

Notice that $x_0 = \tau A_0^L/A_0 \geq 1$. Next, rewrite variables in terms of variables per effective worker, $k_t = K_t/(A_t L_t)$, $c_t = C_t/(A_t L_t)$, and recall that log differentiation of k_t yields $\dot{k}_t/k_t = \dot{K}_t/K_t - \dot{A}_t/A_t - \dot{L}_t/L_t$. Similarly for c_t. In this way, from the Pontryagin conditions for optimality and the transversality condition to this problem, we can conclude that the economy of our modified Ramsey model is described by

$$\begin{aligned} \dot{k}_t &= k_t^\alpha - \left(\delta + g - \frac{\dot{x}_t}{x_t} + \frac{\dot{L}_t}{L_t}\right) k_t - c_t, \\ \dot{c}_t &= \frac{c_t}{\sigma}\left[\alpha k_t^{\alpha-1} - \rho - \delta - \sigma\left(g - \frac{\dot{x}_t}{x_t}\right) - \frac{\dot{L}_t}{L_t}\right], \end{aligned} \tag{6}$$

together with (4) and (5), as well as the transversality condition

$$\lim_{t\to\infty} e^{-[\rho-(1-\sigma)g]t} c_t^{-\sigma} k_t \, x_t^{-(1-\sigma)} = 0. \tag{7}$$

The above system captures the complete dynamic of the economy. Given initial conditions, this Cauchy problem has a unique solution (k_t, c_t, L_t, x_t) defined on $[0, \infty)$ (see Birkhoff and Rota, [2]). In the following section, the restriction $\sigma = \alpha$, i.e. the reciprocal of the intertemporal elasticity of substitution equals the capital's share, is imposed so that the solution to the previous equations is explicit.

3. Explicit Solutions

The system of equations (4)-(6) is clearly nonlinear. To approach a solution, define the capital-output ratio by $u_t = k_t^{1-\alpha}$, and the consumption-capital ratio by $v_t = c_t/k_t$. Using these transformations, the dynamical system (6) can be rewritten as

$$\begin{aligned} \dot{u}_t &= (1-\alpha)\left[-\left(\delta + g - \frac{\dot{x}_t}{x_t} + \frac{\dot{L}_t}{L_t}\right) - v_t\right] u_t + (1-\alpha), \\ \dot{v}_t &= \left(\frac{\alpha}{\sigma} - 1\right) u_t^{-1} v_t + \left[-\frac{\rho}{\sigma} - \left(\frac{1}{\sigma} - 1\right)\left(\delta + \frac{\dot{L}_t}{L_t}\right)\right] v_t + v_t^2, \end{aligned} \tag{8}$$

with condition (7) which becomes $\lim_{t\to\infty} e^{-[\rho-(1-\sigma)g]t}\ v_t^{-\sigma} u_t\, x_t^{-(1-\sigma)} = 0$. Now, assume that $\sigma = \alpha$. In this case, the term $u_t^{-1} v_t$ disappears from equation (8), so that the system can be recursively solved. In other words, when the restriction $\sigma = \alpha$ is imposed, the economy of our modified Ramsey model can be reduced to the following set of differential equations

$$\dot{u}_t = (1-\alpha)\left[-\left(\delta + g - \frac{\dot{x}_t}{x_t} + \frac{\dot{L}_t}{L_t}\right) - v_t\right] u_t + (1-\alpha), \tag{9}$$

$$\dot{v}_t = \left[-\frac{\rho}{\alpha} - \left(\frac{1}{\alpha} - 1\right)\left(\delta + \frac{\dot{L}_t}{L_t}\right)\right] v_t + v_t^2, \tag{10}$$

$$\dot{L}_t = L_t(a - bL_t), \tag{11}$$

$$\dot{x}_t = \lambda x_t(1 - x_t), \tag{12}$$

plus the transversality condition

$$\lim_{t\to\infty} e^{-[\rho-(1-\alpha)g]t}\ v_t^{-\alpha} u_t\, x_t^{-(1-\alpha)} = 0. \tag{13}$$

We are now going to show that such a system can be solved analytically.

Proposition 1. *For all t, the time path of the consumption-capital ratio is given by*

$$v_t = v_0 e^{[-\frac{\rho}{\alpha}-(\frac{1}{\alpha}-1)\delta]t} L_t^{1-\frac{1}{\alpha}} \left\{1 - v_0 \int_0^t e^{[-\frac{\rho}{\alpha}-(\frac{1}{\alpha}-1)\delta]t} L_t^{1-\frac{1}{\alpha}} dt\right\}^{-1}, \tag{14}$$

where $v_0 = c_0/k_0$.

Proof. Equation (10) is a Bernoulli's differential equation. In order to solve it, we take the substitution $w_t = v_t^{-1}$, and convert this into a linear differential equation in w_t, i.e. $\dot{w}_t = -\phi_t w_t - 1$, where $\phi_t = -\rho/\alpha - (1/\alpha - 1)(\delta + \dot{L}_t/L_t)$. This equation is known to be solved by $w_t = e^{-\int_0^t \phi_t dt}\left(w_0 - \int_0^t e^{\int_0^t \phi_t dt} dt\right)$. Since $\int_0^t \phi_t dt = [-\rho/\alpha - (1/\alpha - 1)\delta]t - (1/\alpha - 1)\ln L_t$, the statement follows by rewriting w_t in terms of v_t.

Proposition 2. *For all t, the time path of the capital-output ratio is*

$$u_t = (g_t h_t)^{1-\alpha}\left[u_0 + (1-\alpha)\int_0^t (g_t h_t)^{-(1-\alpha)} dt\right], \tag{15}$$

where $u_0 = k_0^{1-\alpha}$, *and*

$$g_t = 1 - v_0 \int_0^t e^{[-\frac{\rho}{\alpha} - (\frac{1}{\alpha}-1)\delta]t} L_t^{1-\frac{1}{\alpha}} dt, \quad h_t = x_0^{-1} e^{-(\delta+g)t} x_t L_t.$$

Proof. Equation (9) is a linear differential equation, whose solution is provided by $u_t = e^{-(1-\alpha)\int_0^t \varphi_t dt}\left[u_0 + (1-\alpha)\int_0^t e^{(1-\alpha)\int_0^t \varphi_t dt} dt\right]$, where $\varphi_t = \delta + g - \dot{x}_t/x_t + \dot{L}_t/L_t + v_t$. Since $\int_0^t \varphi_t dt = (\delta+g)t - \ln(x_t/x_0) + \ln L_t + \int_0^t v_t dt$, and the integral $\int_0^t v_t dt = -\int_0^t d\left[\ln\left(1 - v_0 \int_0^t e^{[-\frac{\rho}{\alpha}-(\frac{1}{\alpha}-1)\delta]t} L_t^{1-\frac{1}{\alpha}} dt\right)\right] = \ln g_t^{-1}$, we have that the statement follows immediately. □

Corollary 1. *For all* t, *the time path of capital per effective worker and consumption per effective worker are* $k_t = u_t^{\frac{1}{1-\alpha}}$ *and* $c_t = u_t^{\frac{1}{1-\alpha}} v_t$, *respectively.*

Proposition 3. *For all* t,

$$L_t = \frac{ae^{at}}{a - b + be^{at}}, \quad x_t = \frac{x_0 e^{\lambda t}}{1 - x_0 + x_0 e^{\lambda t}}. \tag{16}$$

Proof. Equations (11), (12) are both Bernoulli's differential equations. The statement is obtained proceeding as in the proof of Proposition 1. □

Lemma 1. *The transversality condition* (13) *will be satisfied if and only if* $\lim_{t\to\infty} g_t = 0$, *i.e. if* $v_0 = \left(\int_0^\infty e^{[-\frac{\rho}{\alpha}-(\frac{1}{\alpha}-1)\delta]t} L_t^{1-\frac{1}{\alpha}} dt\right)^{-1}$.

Proof. Substituting equations (14) and (15) into equation (13), and noting from (16) that $\lim_{t\to\infty} x_t = 1$, we obtain that the condition (13) is equivalent to

$$\lim_{t\to\infty} g_t \left[u_0 + (1-\alpha)\int_0^t (g_t h_t)^{-(1-\alpha)} dt\right] = 0. \tag{17}$$

It is now important to notice that g_t decreases starting from $g_0 = 1$, and $\lim_{t\to\infty} L_t = a/b$. As well, $\int_0^t (g_t h_t)^{-(1-\alpha)} dt$ diverges as t tends to infinity. In fact, $\int_0^t (g_t h_t)^{-(1-\alpha)} dt \geq \int_0^t h_t^{-(1-\alpha)} dt$, with the behaviour of this latter integral being similar to that of the integral $\int_0^t e^{(1-\alpha)(\delta+g)t} dt = \left[e^{(1-\alpha)(\delta+g)t} - 1\right] / (1 - \alpha)(\delta+g) \to +\infty$. The statement is now straightforward in one direction, while the viceversa follows as an application of Hopital's rule, by rewriting the left hand-side of (17) as

$$\lim_{t\to\infty} \frac{g_t}{\left[u_0 + (1-\alpha)\int_0^t (g_t h_t)^{-(1-\alpha)}\, dt\right]^{-1}},$$

where the limit of the numerator is now zero by hypothesis, but it is also zero the limit of the denominator as previously argued. □

Remark 1. From equation (16) we derive that L_t is a monotone increasing function from $L_0 = 1$ to a/b. Consequently, we see that

$$\int_0^t e^{[-\frac{\rho}{\alpha}-(\frac{1}{\alpha}-1)\delta]t} L_t^{1-\frac{1}{\alpha}} dt \geq \left(\frac{a}{b}\right)^{1-\frac{1}{\alpha}} \frac{1 - e^{[-\frac{\rho}{\alpha}-(\frac{1}{\alpha}-1)\delta]t}}{\frac{\rho}{\alpha} + \left(\frac{1}{\alpha}-1\right)\delta}.$$

As t goes to infinity, we get $v_0 < \infty$.

4. Conclusion

In this paper we have considered a modified version of the standard Ramsey growth model obtained by introducing technological diffusion, and assuming a logistic-type population growth law. Within this framework, we have derived an explicit solution of the model when capital's share is equal to the reciprocal of the intertemporal elasticity of substitution.

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