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FINITE ELEMENT ANALYSIS OF A NONLINEAR BEAM ON A WINKLER FOUNDATION

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Abstract: In this paper we continue our study of the displacement of an elastic beam resting on a Winkler foundation which reacts only in compression. Here we assume a nonlinear stress-strain law. We use a mixed finite element method with piecewise linear elements to obtain an approximate solution of the fourth order equation. We derive error estimates for the solution we obtain and illustrate our procedure with an example.

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1. Introduction

In this paper we continue our study of the displacement of an elastic beam resting on a tensionless Winkler foundation begun in [1]. This paper extends the work in [1] in two ways: In [1] we modeled the beam using linear elaticity. Here we assume a non-linear stress-strain law. In order to prove existence and uniquenss of the solution of the problem,we must apply a different theorem than the one used in [1]. Also in [1], in the finite element approximation, we used a numerical integration scheme (the trapezoid rule) to compute the integrals. Here we assume that the integrals are computed exactly.

The main purpose of this paper is to obtain approximate solutions to the problem using the finite element method. Since this is a fourth order equation,

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we use a mixed method, which in our case simply means that we introduce the second derivative of the displacement (which is proportional to the bending moment) as a new variable. This enables use to use piecewise linear elements. The use of piecewise linear elements makes it feasible to carry out the integrations exactly in many cases. This simplifies the analysis. The main result of this paper is the error estimate for the finite element solution. We present an iterative method for solving the finite element equations. We illustrate the method on a test problem. The test problem was chosen because the exact solution (found by a shooting method) exhibits regions in which the beam separates from the foundation.

In Section 2, we formulate the problem and prove existence and uniqueness. In Section 3 we will introduce the mixed finite element method for the approximate solution of the problem. In Section 4 we will apply our method to the test problem.

2. Formulation of the Problem

We consider an elastic beam of length $2L$ pressed against an elastic foundation by a load $P(x)$. The elastic foundation is assumed to react in compression only. The governing equation is

$$EIy'''' + \psi^+(y) = P(x), \qquad -L < x < L\,. \tag{2.1}$$

In (2.1) EI is the flexural rigidity of the beam, and y is the deflection of the beam (taken positive in the downward direction) and $\psi(y)$ is the stress which may be a nonlinear function of y. We assume that the ends of the beam are fixed at $y = 0$ but the ends of the beam are free to rotate. Thus the boundary conditions are

$$y(\pm L) = y''(\pm L) = 0\,. \tag{2.2}$$

After rescaling, we can rewrite the problem as

$$u'''' + \Psi^+(u) = F(x), \qquad -L < x < L, \tag{2.3}$$

$$u(\pm L) = u''(\pm L) = 0\,. \tag{2.4}$$

We will consider a more general version of (2.3). Let $\phi : \mathbf{R} \to \mathbf{R}$ be locally Lipschitz continuous and monotone increasing with $\phi(0) = 0$. The problem we will study is

$$u'''' + \phi(u) = F(x)\,, \qquad -L < x < L\,, \tag{2.5}$$

with the boundary conditions (2.4).

We give the weak formulation of the problem. Let $X = H^2(-L,L) \cap H_0^1(-L,L)$ equipped with the norm of H^2. Let X^* be the dual space of X. Then the weak formulation of the problem is: Given $F \in X^*$, find $u \in X$ such that

$$(u'', v'') + (\phi(u), v) = \langle F, v \rangle, \quad \text{for all } v \in X\,. \tag{2.6}$$

In (2.6) round brackets denote the L_2 inner product while the brackets denote the pairing between X and its dual.

To prove existence and uniqueness of solutions of (2.6), we apply the following theorem which is the relevent portion of the theorem of Browder and Minty [2, Theorem 26.A]:

Theorem 1. *Let $A : X \to X^*$ be a strictly monotone, coersive and hemicontinuous operator on the real separable, reflexive Banach space X. Then for any $\mathbf{b} \in X^*$ the equation*

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution.

We define a mapping $A : X \to X^*$ by

$$\langle Au, v \rangle = (u'', v'') + (\phi(u), v), \quad v \in X\,,$$

and check that A satisfies the hypothesis of Theorem 1

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle = \|u_1'' - u_2''\|_{L_2}^2 + (\phi(u_1) - \phi(u_2), u_1 - u_2).$$

By the Poincaré and generalized Poincaré inequalities, on X, $\|u''\|_{L_2}$ is equivalent to $\|u\|_{H^2}$. Thus the mapping A is strictly monotone. In fact, from the monotonicity of ϕ and the assumption that $\phi(0) = 0$ we see that $\langle Au, u \rangle \geq c\|u\|_{L_2}^2$ which implies coercivity. Thus all conditions of Theorem 1 are satisfied and we may conclude

Theorem 2. *Equation (2.6) has a unique solution.*

We now write (2.5), (2.4) as a system. If we let $v = -u''$,(2.5), (2.4) becomes

$$-u'' = v, \quad -v'' + \phi(u) = F, \tag{2.7}$$

$$u(\pm L) = v(\pm L) = 0. \tag{2.8}$$

We will assume that $F \in H^{-1}(-L,L)$. Then the weak formulation of (2.7), (2.8) is: Find $(u,v) \in H_0^1 \times H_0^1$ such that

$$(u', w') = (v, w) \quad \text{for all } w \in H_0^1\,, \tag{2.9a}$$

$$(v', z') + (\phi(u), z) = \langle F, z \rangle \quad \text{for all } z \in H_0^1\,, \tag{2.9b}$$

where now the brackets denote the pairing between H_0^1 and H^{-1}. In the next section we will study the finite element approximation of (2.9).

3. The Finite Element Approximation

In this section we consider the finite element approximation of (2.6) which, since we have transformed the problem into (2.9) is a mixed finite element approximation.

We let $V = H_0^1(-L, L)$ equipped with the Dirichlet norm and inner product

$$D(u, v) = \int_{-L}^{L} u'(t)v'(t)\,\mathrm{d}t, \quad \|u\|_1 = D(u, u)^{1/2}. \tag{3.1}$$

For a positive integer N, let $h = \frac{2L}{N+1}$, $x_i = -L + ih$, $i = 0, \ldots, N + 1$. We define $V_h \subset V$ to be the space of continuous, piecewise linear functions with breakpoints $\{x_i\}_{i=1}^N$. The set $\{\psi_i\}_{i=1}^N$ of "hat functions" form a basis for V_h. ($\psi_i(x_j) = \delta_{ij}$, $i = 1, \ldots, N$, $j = 0, \ldots, N+1$). Our finite element approximation to (u, v) is the solution of the following problem: Find $(u_h, v_h) \in V_h \times V_h$ such that

$$D(u_h, w_h) = (v_h, w_h), \quad \text{for all } w_h \in V_h, \tag{3.2a}$$

$$D(v_h, z_h) + (\phi(u_h), z_h) = \langle F, z_h\rangle \quad \text{for all } z_h \in V_h. \tag{3.2b}$$

Thus the finite dimensional system which must be solved is

$$D(u_h, \psi_j) = (v_h, \psi_j), \quad j = 1, \ldots, N, \tag{3.3a}$$

$$D(v_h, \psi_j) + (\phi(u_h), \psi_j) = \langle F, \psi_j\rangle, \quad j = 1, \ldots, N. \tag{3.3b}$$

We write

$$u_h = \sum_{i=1}^{N} \alpha_i \psi_i, \quad v_h = \sum_{i=1}^{N} \beta_i \psi_i. \tag{3.4}$$

When we insert (3.4) into (3.3) we get a system of nonlinear equations in $\mathbf{R^{2N}}$. We can solve (3.3a) for the β's in terms of the α's. When this is inserted into (3.3b) we get an equation in $\mathbf{R^N}$. By applying the argument of Theorem 2 we see that (3.3) has a unique solution.

We let $\|\cdot\|$ denote the L_2 norm. We wish to estimate $\|u - u_h\|$ and $\|v - v_h\|$. We do not want to assume that $F \in L_2$. Thus we cannot get estimates in the Dirichlet norm. We define

$$e_u = u - u_h, \quad e_v = v - v_h. \tag{3.5}$$

These quantities satisfy

$$D(e_u, w) = (e_v, w), \quad \text{for all } w \in V_h\,, \tag{3.6a}$$

$$D(e_v, z) + (\phi(u) - \phi(u_h), z) = 0 \quad \text{for all } z \in V_h\,. \tag{3.6b}$$

We introduce u^* and v^*, the orthogonal projections of u and v respectively in the Dirichlet norm onto V_h. As is well known, u^* and v^* are simply the piecewise linear interpolants of u and v. We have

$$\|u - u^*\| \le h\|u\|_1, \quad \|v - v^*\| \le h\|v\|_1\,. \tag{3.7}$$

By elliptic regularity $u \in H^3(-L, L) \cap H_0^1(-L, L)$ and so

$$\|u - u^*\| \le h^2\|u\|_{H^2}\,. \tag{3.8}$$

As we show in [1]

$$\|v^* - v_h\|^2 = (v^* - v, v^* - v_h) + D(u^* - u_h, v - v_h)\,. \tag{3.9}$$

The argument then proceeds as in [1] with the alteration that the term denoted by E and representing the error due to numerical integration is now absent. So we obtain

$$\|v^* - v\|^2 + (\phi(u) - \phi(u_h), u - u_h) = (v^* - v, v^* - v_h) - (\phi(u) - \phi(u_h), u^* - u). \tag{3.10}$$

We let $y \in H_0^1(-L, L) \cap H^2(-L, L)$ be the weak solution of

$$y'''' + \frac{\phi(u) - \phi(u_h)}{e_u} y = e_u, \quad y(\pm L) = y''(\pm L) = 0.$$

Then

$$\|e_u\|^2 = -D(y'', e_u) + (\phi(u) - \phi(u_h), y).$$

As in [1] it follows that

$$\|e_u\|^2 = (y'' - p^*, e_v) + D(p^* - y'', u) + D(y - y^*, v) + (\phi(u) - \phi(u_h), y - y^*)\,, \tag{3.11}$$

where y^* is the projection of y on V_h and p^* is the projection of y'' on V_h.

We now use (3.10) and (3.11) to estimate $\|e_u\|$ and $\|e_v\|$. We estimate the terms on the right side of (3.10):

$$|(v^* - v, v^* - v_h)| \le \|v^* - v\|\|v^* - v_h\| \le Ch\|v\|_1\|v^* - v_h\|, \tag{3.12}$$

$$\|\phi(u) - \phi(u_h)\| \le k\|u - u_h\|\,, \tag{3.13}$$

where k is a Lipschitz constant (recall that ϕ is locally Lipschitz and u and u_h are continuous functions). Hence

$$|(\phi(u) - \phi(u_h))| \le Ch^2\|u\|_{H^2}\|e_u\|\,. \tag{3.14}$$

Thus from (3.10), (3.12) and (3.13)

$$\|v^* - v_h\|^2 \le Ch\|v\|_1\|v^* - v_h\| + Ch^2\|u\|_{H^2}\|e_u\|\,.$$

So (using the fact that $x^2 \le ax + b \Rightarrow x \le a + \sqrt{b}$)

$$\|v^* - v_h\| \le Ch(\|v\|_1 + \|u\|_{H^2}^{1/2}\|e_u\|^{1/2}),$$

$$\|e_v\| \le \|v - v^*\| + \|v^* - v_h\| \le Ch(\|v\|_1 + \|u\|_{H^2}^{1/2}\|e_u\|^{1/2}). \tag{3.15}$$

In the preceeding, C is a generic constant independent of h. Now, turning to (3.11), since

$$\|y'' - p^*\| \le Ch^2\|y\|_{H^4} \le Ch^2\|e_u\|,$$

$$|(y'' - p^*, e_v)| \le Ch^2\|e_u\|\,\|e_v\|,$$

$$|D(y'' - p^*, u)| \le Ch\|e_u\|\,\|u\|_1,$$

$$|D(y - y^*, v)| \le Ch\|e_u\|\,\|v\|_1,$$

$$|(\phi(u) - \phi(u_h), y - y^*)| \le Ch^2\|e_u\|^2.$$

So from (3.11)

$$\|e_u\|^2 \le ch^2\|e_u\|\,\|e_v\| + Ch\|e_u\|\,\|u\|_1 + Ch\|e_u\|\,\|v\| + ch^2\|e_u\|^2,$$

so

$$\|e_u\| \le Ch^2\|e_v\| + ch\|u\|_1 + Ch\|v\|_1 + ch^2\|e_u\|.$$

If $Ch^2 < 1$, this becomes

$$\|e_u\| \le ch^2\|e_v\| + Dh, \tag{3.16}$$

where $D = C(\|u\|_1 + \|v\|_1)$. From (3.15)

$$\|e_v\| \le Ah + Bh\|e_u\|^{1/2}. \tag{3.17}$$

If we insert (3.16) into (3.17) we find

$$\|e_u\| \le Ch^2(Ah + Bh\|e_u\|^{1/2}) + Dh,$$

from which it follows that

$$\|e_u\| \le Ch. \tag{3.18}$$

By inserting (3.18) into (3.17) we find

$$\|e_v\| \le Ch. \tag{3.19}$$

Equations (3.18) and (3.19) prove convergence of the finite element approximation.

4. Implimentation

Our test problem will be

$$u'''' + \Psi(u)^+ = F\delta(x) + \sigma, \qquad -l < x < l, \tag{4.1}$$

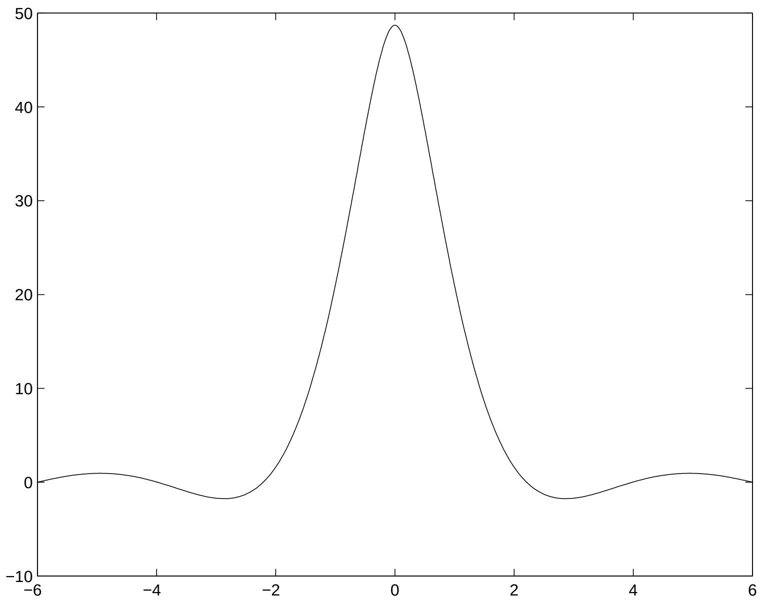

Figure 1: The solution of the test problem

with $L = 6$, $F = 800$, $\sigma = 6$ and

$$\Psi(u) = 4(u + .001u^3)\,. \tag{4.2}$$

An exact solution to (4.1) with the boundary conditions (2.2) was found by a shooting method, using the fact that the this solution must be an even function of x. This solution exhibits two intervals in which the beam separates from the support. The solution is shown in Figure 1.

If we insert (3.4) into (3.3) we obtain

$$\frac{2}{h}\alpha_j - \frac{1}{h}\alpha_{j-1} - \frac{1}{h}\alpha_{j+1} = \frac{2}{3}h\beta_j + \frac{h}{6}\beta_{j-1} + \frac{h}{6}\beta_{j+1}\,, \tag{4.3a}$$

$$\frac{2}{h}\beta_j - \frac{1}{h}\beta_{j-1} - \frac{1}{h}\beta_{j+1} + (\Psi^+(u_h), \psi_j) = \langle F\delta + \sigma, \psi_j\rangle\,. \tag{4.3b}$$

In (4.3), $j = 1, \ldots, N$ with $\alpha_0 = \alpha_{N+1} = \beta_0 = \beta_{N+1} = 0$. We solve (4.3) by an iteration scheme

$$(2+a)\beta_j^{(n+1)} - \beta_{j-1}^{(n+1)} - \beta_{j+1}^{(n+1)} = h^2\sigma - h\psi_j(0) - (\Psi^+(u_j^{(n)}), \psi_j) + a\beta_j^{(n)}\,, \tag{4.4a}$$

$$(2+a)\alpha_j^{(n+1)} - \alpha_{j-1}^{(n+1)} - \alpha_{j+1}^{(n+1)} = \frac{2}{3}h^2\beta_j^{(n+1)} + \frac{h^2}{6}\beta_{j-1}^{(n+1)}$$

$N+1$	h	$\|u-u_h\|$	$\|v-v_h\|$	a	No. of iter.
50	.24	.5923	1.97769	.125	83
100	.12	.1454	..4972	.04	114
200	.06	.0365	.1245	.008	88
400	.03	..0090	.031	.002	89
800	.015	.0023	.0077	.0005	91
1600	.0075	..000648	.0015	.00014	107
3200	.00375	.000161	.00024	.000035	109
6400	.0001875	.0000432	.0000874	.000008	118

Table 1: Results for the sample problem (4.1), (2.4)

$$+\frac{h^2}{6}\beta_{j+1}^{(n+1)} + a\alpha_j^{(n)}\,, \quad (4.4b)$$

where a is a relaxation parameter. The term $(\Psi^+(u_j^{(n)}), \psi_j)$ can be computed exactly and involves only α_{j-1}, α_j and α_{j+1}. If $\alpha^{(k)} = (\alpha_j^{(k)})$ and $\beta^{(k)} = (\beta_j^{(k)})$ we run the iterations (4.4) until

$$\|\alpha^{(k+1)} - \alpha^{(k)}\|_2^2 + \|\beta^{(k+1)} - \beta^{(k)}\|_2^2 < 10^{-6}\,.$$

We chose the relaxation parameter a which seemed to give the fastest convergence. The choice of a affects the rate of convergence dramatically. The iteration scheme will not converge if a is too small. We computed the quantities $\|u_h - u\|$ and $\|v_h - v\|$ by using the *MATLAB* numerical integrator *quadl*. The results are summarized in Table 1.

It appears from Table 1 that u_h converges to u at a rate of h^2 which is not predicted by our analysis. This rate may be explained by the precise nature of the nonlinear term and the forcing term and by the fact that the exact solution is globally $\mathcal{C}^3$ and piecewise $\mathcal{C}^\infty$.

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