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ALTERNATING-DIRECTION FINITE ELEMENT METHOD FOR A CLASS OF NONLINEAR EVOLUTION EQUATIONS

Zhu Hong
Guang Dong Food and Drug Vocational College
Guang Dong, 510520, P.R. CHINA
e-mail: zhuhong1004@126.com

Abstract: We create alternating-direction finite element method for a class of nonlinear evolution equations by using equivalent transformation. The scheme prevents twice cumulating errors by using the common alternating-direction finite element method to approximate u at first, then to approximate u_t. By using the calculation of tensor product, H^1-module estimate and the theory and thechniques of prior estimate we get the best L^2 module error estimate.

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1. Introduction

Alternating-dlrection finite element method can reduce complicated higher dimension problems to lower dimension problems, and has the advantage of having more accuracy degree, needing less storage space and less computing quantity. By using an equivalent transformation, we apply the alternating-direction finite method to a class of 3-dimensional nonlinear evolution equations. This method prevents the twice cumulating erros caused by the common alternating-direction finite element method which approximates u at first then approximates u_t. We also get the best L^2 module error estimate by the use of tensor product, H^1-module estimate and the theory and thechniques of prior estimate.

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We consider the initial and boundary problem of nolinear evolution equation

$$u_{tt} - \Delta u_t - \Delta u = f(u)u_t + g(u), \quad (x,t) \in \Omega \times [0,T], \tag{1a}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_{t0}(x), \quad x \in \Omega, \tag{1b}$$

$$u = u_t = 0, \quad (x,t) \in \partial\Omega \times [0,T]\,, \tag{1c}$$

where $\Omega = [0,1]^3$, $\partial\Omega$ is the boundary of Ω. Initial values are $u_0(x), v_0(x), w_0(x)$ which are sufficiently smooth.

We assume (P)

$$u \in H^1(H^{r+1}) \cap C^2(\Omega \times J) \cap L^\infty(W^1_\infty)\,,$$

$$u_t \in L^\infty(H^{r+1}) \cap L^\infty(W^1_\infty) \cap H^2(L^2)\,,$$

$$u_{tt} \in L^2(H^{r+1})\,,$$

and the condition (H) is satisfied $f(s)$, $g(l)$ are bounded and Lipschitz continuous about s, l.

2. Alternating-Direction Finite Element Scheme

Let $u_t = v$, (1) can be written as

$$v_t - \Delta v - \Delta u + u_t = F(u,v), \qquad (x,t) \in \Omega \times [0,T]\,, \tag{2a}$$

$$u = v = 0, \qquad (x,t) \in \partial\Omega \times [0,T]\,, \tag{2b}$$

$$u(x,0) = \Phi(x),\ v(x,0) = \Psi(x), \qquad x \in \Omega\,, \tag{2c}$$

where, $F(u,v) = [f(u)+1]v + g(u)$. Let $e = u + v$. We have

$$e_t - \Delta e = F(u,v), \quad (x,t) \in \Omega \times [0,T], \tag{3a}$$

$$u_t = v, \tag{3b}$$

$$z = \{p | p, \frac{\partial p}{\partial x_i}, \frac{\partial^2 p}{\partial x_i \partial x_j}(i \neq j), \frac{\partial^3 p}{\partial x_1 \partial x_2 \partial x_3} \in L^2(\Omega)\}.$$

Assume $S_h \subset z$ are finite element spaces of u, v and of order r, e.g.

$$\inf_{z_h \in M_h} \{\sum_{m=0}^{3} \sum_{\substack{i,j,k=0 \\ i+j+k=3}} ||\frac{\partial^m (z - z_h)}{\partial x_1^i \partial x_2^j \partial x_3^k}||\} \leq M h^{r+1} ||z||_{r+1}\,,$$

$$||z_h||_{W^{j,\infty}} \leq M h^{-\frac{3}{2}} ||z_h||_j, \quad j = 0,1, \quad z_h \in S_h\,,$$

$$||z_h||_1 \leq M h^{-1} ||z_h||, \quad z_h \in S_h\,.$$

For $t \in [0,T]$, let $\tilde{s} : [0,T] \longrightarrow S_h$ is the H^1-projection of the solution s to (3a) and satisfies

$$(\nabla(s-\tilde{s}), \nabla z) = 0, \quad z \in S_h\,.$$

From [2], we know that for $0 \le k \le 2$ and $j = 0, 1, 2, 3$

$$||\frac{\partial^k (u-\tilde{u})}{\partial t^k}||_{L^p(H^j)} \le M \sum_{m=0}^{k} ||\frac{\partial^m u}{\partial t^m}||_{L^p(H^{r+1})} h^{r+1-j}\,,$$

where $p = 2$ or ∞.

Assume L is a positive integer satisfing $L\Delta t = T$, $t_n = n\Delta t, n = 0, 1, 2, ...,$ $[\frac{T}{\Delta t}]$, where Δt is the step size in the-time direction.

Let $\partial_t u^n = \frac{u^{n+1}-u^n}{\Delta t}$.

We construct the Galerkin approximation scheme of (3a)

$$(\frac{E^{n+1}-E^n}{\Delta t}, \chi) + (\nabla E^{n+1}, \nabla \chi) = (F(U^n, V^n), \chi), \quad \chi \in S_h\,, \tag{4a}$$

$$(\frac{U^{n+1}-U^n}{\Delta t}, w) = (V^{n+1}, w), \quad w \in S_h\,. \tag{4b}$$

The initial values U_0 and V_0 satisfy the condition

$$(U_0, \chi) = (u_0, \chi), (V_0, \chi) = (v_0, \chi), \quad \chi \in S_h\,. \tag{5}$$

Because $\partial_t E^n = \frac{E^{n+1}-E^n}{\Delta t}$ it can be written as

$$E^{n+1} = \Delta t(\partial_t E^n) + E^n\,. \tag{6}$$

Substituting (6) into (4a), we have

$$(\partial_t E^n, \chi) + \Delta t(\nabla \partial_t E^n, \nabla \chi) = (F(U^n, V^n), \chi) - (\nabla E^n, \nabla \chi), \quad \chi \in S_h\,. \tag{7}$$

If a disturbance term is added to the left side of (7),

$$\sum_{(i,j)} (\Delta t)^2 (\frac{\partial^2(\partial_t E^n)}{\partial x_i \partial x_j}, \frac{\partial^2 \chi}{\partial x_i \partial x_j}) + (\Delta t)^3 (\frac{\partial^3(\partial_t E^n)}{\partial x_1 \partial x_2 \partial x_3}, \frac{\partial^3 \chi}{\partial x_1 \partial x_2 \partial x_3})\,. \tag{8}$$

We have

$$(\partial_t E^n, \chi) + \Delta t(\nabla \partial_t E^n, \nabla \chi) \sum_{(i,j)} (\Delta t)^2 (\frac{\partial^2(\partial_t E^n)}{\partial x_i \partial x_j}, \frac{\partial^2 \chi}{\partial x_i \partial x_j}) +$$
$$(\Delta t)^3 (\frac{\partial^3(\partial_t E^n)}{\partial x_1 \partial x_2 \partial x_3}, \frac{\partial^3 \chi}{\partial x_1 \partial x_2 \partial x_3}) = (F(U^n, V^n), \chi) - (\nabla E^n, \nabla \chi), \quad \chi \in S_h. \tag{9}$$

For u and v, we have the following approximation

$$(1 + \Delta t)U^{n+1} = \Delta t E^{n+1} + U^n\,,$$

$$V^{n+1} = E^{n+1} - U^{n+1}\,.$$

Denote

$$C_1 = (\int_0^1 \xi_i \xi_j dx_1), \quad C_2 = (\int_0^1 \eta_i \eta_j dx_2), \quad C_3 = (\int_0^1 \zeta_i \zeta_j dx_1)\,,$$

$$A_1 = (\int_0^1 \xi_i' \xi_j' dx_1), \quad A_2 = (\int_0^1 \eta_i' \eta_j' dx_2), \quad A_3 = (\int_0^1 \zeta_i' \zeta_j' dx_3)\,.$$

If $E^n = \sum_{p,q,r} \alpha_{pqr}^{n+1} (\xi_p \otimes \eta_q \otimes \zeta_r)$, let $\chi = \xi_p \otimes \eta_q \otimes \zeta_r$ in (9), then

$$(C_{x_1} + \Delta t A_{x_1}) \otimes (C_{x_2} + \Delta t A_{x_2}) \otimes (C_{x_3} + \Delta t A_{x_3})(\alpha^{n+1} - \alpha^n) = \Delta t H^n\,.$$

Here

$$\alpha^n = (\alpha_{111}^n, \alpha_{112}^n, ..., \alpha_{11N_3}^n, \alpha_{121}^n, ..., \alpha_{N_1 N_2 N_3}^n)\,,$$

$$(H^n)_{pqr} = (F^n, (\xi_p \otimes \eta_q \otimes \zeta_r)) - (\nabla E^n, \nabla(\xi_p \otimes \eta_q \otimes \zeta_r))\,.$$

The computing produre of the scheme: First, if the finite element solution $\{U^n, V^n\}$ is known, $\{E^{n+1}\}$ can be obtained from (4a). Then combined with the scheme (4b), we can get $\{U^{n+1}, V^{n+1}\}$. Considering the positive defitness, we come to the conclusion that (4) has uniqne solution.

3. Error Estimate

We now discuss the error estimate of (1). Let $U - \tilde{u} = \sigma, u - \tilde{u} = \pi, V - \tilde{v} = \theta, v - \tilde{v} = \rho$. So

$$\begin{aligned}
&(\frac{\theta^{n+1} - \theta^n}{\Delta t}, \chi) + (\nabla \theta^{n+1}, \nabla \chi) + (\nabla \sigma^{n+1}, \nabla \chi) \\
&+ \sum_{(i,j)} (\Delta t)^2 (\frac{\partial^2 (\partial_t \theta^n + \partial_t \sigma^n)}{\partial x_i \partial x_j}, \frac{\partial^2 \chi}{\partial x_i \partial x_j}) \\
&+ (\Delta t)^3 (\frac{\partial^3 (\partial_t \theta^n + \partial_t \sigma^n)}{\partial x_1 \partial x_2 \partial x_3}, \frac{\partial^3 \chi}{\partial x_1 \partial x_2 \partial x_3}) \\
&= (u_t^{n+1} - \frac{u^{n+1} - u^n}{\Delta t}, \chi) + (\partial_t \pi^n, \chi) + (v_t^{n+1} - \frac{v^{n+1} - v^n}{\Delta t}, \chi) + (\partial_t \rho^n, \chi) \\
&+ \sum_{(i,j)} (\Delta t)^2 [(\frac{\partial^2 (\partial_t \rho^n + \partial_t \pi^n)}{\partial x_i \partial x_j}, \frac{\partial^2 \chi}{\partial x_i \partial x_j}) - (\frac{\partial^2 (\partial_t e^n)}{\partial x_i \partial x_j}, \frac{\partial^2 \chi}{\partial x_i \partial x_j})] \\
&+ (\Delta t)^3 [(\frac{\partial^3 (\partial_t \rho^n + \partial_t \pi^n)}{\partial x_1 \partial x_2 \partial x_3}, \frac{\partial^3 \chi}{\partial x_1 \partial x_2 \partial x_3}) - (\frac{\partial^3 (\partial_t e^n)}{\partial x_1 \partial x_2 \partial x_3}, \frac{\partial^3 \chi}{\partial x_1 \partial x_2 \partial x_3})]
\end{aligned}$$

$$+(F(U^n,V^n)-F(u^{n+1},v^{n+1}),\chi)\,, \qquad (10a)$$

$$(\frac{\sigma^{n+1}-\sigma^n}{\Delta t},w)=(\theta^{n+1}+\partial_t\pi-\rho^{n+1}+(u_t)_{n+1}-\partial_t u^n,w)$$
$$=(\theta^{n+1}+\lambda_n,w)\,. \qquad (10b)$$

In (10a), choose the test function, $\chi=\theta^{n+1}$, its equivalent form is $\chi=\partial_t\sigma-\lambda_n$. Therefore

$$(\nabla\theta^{n+1},\nabla\theta^{n+1})+(\nabla\sigma^{n+1},\nabla(\partial_t\sigma^n-\lambda_n))+(\frac{\sigma^{n+1}-\sigma^n}{\Delta t},\partial_t\sigma^n-\lambda_n)$$
$$\geq||\nabla\theta^{n+1}||^2+||\partial_t\sigma^n||^2+\frac{1}{2\Delta t}[||\nabla\theta^{n+1}||^2-||\nabla\theta^n||^2]$$
$$-(\partial_t\sigma^n,\lambda_n)-(\nabla\sigma^{n+1},\nabla\lambda_n)\,. \qquad (11)$$

The above estimate can be written as

$$\frac{1}{2\Delta t}[||\theta^{n+1}||^2-\theta^n||^2]+||\nabla\sigma^{n+1}||^2+||\partial_t\sigma^n||^2+\frac{1}{2\Delta t}[||\nabla\sigma^{n+1}||^2-\nabla\sigma^n||^2]$$
$$\frac{1}{2\Delta t}\sum_{i,j}(\Delta t)^2[||\frac{\partial^2\theta^{n+1}}{\partial x_i\partial x_j}||^2-||\frac{\partial^2\theta^n}{\partial x_i\partial x_j}||^2]+\sum_{i,j}(\Delta t)^2[||\frac{\partial^2(\partial_t\sigma^n)}{\partial x_i\partial x_j}||^2$$
$$\frac{1}{2\Delta t}(\Delta t)^3[||\frac{\partial^3\theta^{n+1}}{\partial x_1\partial x_2\partial x_3}||^3-||\frac{\partial^2\theta^n}{\partial x_1\partial x_2\partial x_3}||^2]+(\Delta t)^3[||\frac{\partial^2(\partial_t\sigma^n)}{\partial x_1\partial x_2\partial x_3}||^2$$
$$\leq(u_t^{n+1}-\frac{u^{n+1}-u^n}{\Delta t},\theta^{n+1})+(\partial_t\pi^n,\theta^{n+1})+(v_t^{n+1}-\frac{v^{n+1}-v^n}{\Delta t},\theta^{n+1})+$$
$$(\partial_t\rho^n,\theta^{n+1})+\sum_{(i,j)}(\Delta t)^2[(\frac{\partial^2(\partial_t\rho^n+\partial_t\pi^n)}{\partial x_i\partial x_j},\frac{\partial^2\theta^{n+1}}{\partial x_i\partial x_j})-(\frac{\partial^2(\partial_t e^n)}{\partial x_i\partial x_j},\frac{\partial^2\theta^{n+1}}{\partial x_i\partial x_j})]$$
$$+(\Delta t)^3[(\frac{\partial^3(\partial_t\rho^n+\partial_t\pi^n)}{\partial x_1\partial x_2\partial x_3},\frac{\partial^3\theta^{n+1}}{\partial x_1\partial x_2\partial x_3})-(\frac{\partial^3(\partial_t e^n)}{\partial x_1\partial x_2\partial x_3},\frac{\partial^3\theta^{n+1}}{\partial x_1\partial x_2\partial x_3})]$$
$$+(F(U^n,V^n)-F(u^{n+1},v^{n+1}),\theta^{n+1})+(\partial_t\sigma^n,\lambda_n)+(\nabla\sigma^{n+1},\nabla\lambda_n)$$
$$+\sum_{(i,j)}(\Delta t)^2(\frac{\partial^2(\partial_t\sigma^n)}{\partial x_i\partial x_j},\frac{\partial^2(\partial_t\lambda_n)}{\partial x_i\partial x_j})+(\Delta t)^3(\frac{\partial^3(\partial_t\sigma^n)}{\partial x_1\partial x_2\partial x_3},\frac{\partial^3\lambda_n}{\partial x_1\partial x_2\partial x_3})\,. \qquad (12)$$

Estimate the left-hand side of (12) to a smaller one, both sides multiplied by $2\triangle t$, and sum up n from 0 to $L-1$, then we have

$$\|\theta^L\|^2+(\Delta t)\sum_{n=1}^{L-1}\|\nabla\theta^{n+1}\|^2+||\nabla\sigma^L||^2+(\Delta t)\sum_{n=0}^{L-1}||\partial_t\sigma^n||^2+\sum_{(i,j)}(\Delta t)^2||||\frac{\partial^2\theta^L}{\partial x_i\partial x_j}||^2$$

$$+(\Delta t)^3||\frac{\partial^3\theta^L}{\partial x_1\partial x_2\partial x_3}||^2+\Delta t\sum_{n=0}^{L-1}\sum_{(i,j)}(\Delta t)^2||\frac{\partial^2(\partial_t\sigma^n)}{\partial x_i\partial x_j}||^2$$

$$+\Delta t\sum_{n=0}^{L-1}(\Delta t)^3||\frac{\partial^3(\partial_t\sigma^n)}{\partial x_1\partial x_2\partial x_3}||^2\leq[\frac{1}{2}||\theta^0||^2+||\nabla\sigma^0||^2+\sum_{(i,j)}(\Delta t)^2||\frac{\partial^2\sigma^0}{\partial x_i\partial x_j}||^2$$

$$+(\Delta t)^3||\frac{\partial^3\sigma^0}{\partial x_1\partial x_2\partial x_3}||^2]+\sum_{i=1}^{11}T_i. \tag{13}$$

For the first, the second and the third term of the right-hand side of (13), we have

$$\sum_{i=1}^{3}T_i\leq M\Delta t\sum_{n=0}^{L-1}[||(u_t)^{n+1}-\partial_t u^n||^2+||\partial_t\pi^n||^2$$

$$+||(v_t)^{n+1}-\partial_t v^n||^2+||\partial_t\rho^n||^2+||\theta^{n+1}||^2], \tag{14}$$

where

$$||(u_t)^{n+1}-\partial_t u^n||^2\leq M(\Delta t)^2,$$

$$||(v_t)^{n+1}-\partial_t v^n||^2\leq M(\Delta t)^2.$$

For the fifth and the sixth term, we have

$$|T_5|\leq M(\Delta t)^2+(\Delta t)^3\sum_{n=0}^{L-1}[\sum_{(i,j)}||\frac{\partial^2\theta^{n+1}}{\partial x_i\partial x_j}||^2], \tag{15a}$$

$$|T_6|\leq M(\Delta t)^3+(\Delta t)^4\sum_{n=0}^{L-1}[\sum_{(i,j)}||\frac{\partial^2\theta^{n+1}}{\partial x_i\partial x_j}||^2]. \tag{15b}$$

For the fourth and the seventh term, we have

$$|T_4|+|T_7|\leq M[h^{2r+2}+\sum_{n=0}^{L-1}(||\theta^{n+1}||^2+||\sigma^{n+1}||^2)\Delta t]. \tag{16}$$

For the eighth and nighth term, we can see

$$|T_8|+|T_9|\leq M\sum_{n=0}^{L-1}(||\lambda_n||^2+||\partial_t\sigma^{n+1}||^2+||\nabla\lambda_n||^2+||\nabla\sigma^{n+1}||^2)\Delta t. \tag{17}$$

For the tenth and eleven term, it can be seen easily that

$|T_{10}|+|T_{11}|$

$$\leq \sum_{n=0}^{L-1} \sum_{(i,j)} (\Delta t)^2 (\frac{\partial^2(\partial_t \sigma^n)}{\partial x_i \partial x_j}, \frac{\partial^2 \lambda_n}{\partial x_i \partial x_j}) + \sum_{n=0}^{L-1} (\Delta t)^3 (\frac{\partial^3(\partial_t \sigma^n)}{\partial x_1 \partial x_2 \partial x_3}, \frac{\partial^3 \lambda_n}{\partial x_1 \partial x_2 \partial x_3})$$

$$\leq M[(\Delta t)^2 + \sum_{n=0}^{L-1} \sum_{(i,j)} (\Delta t)^3 || \frac{\partial^2 \lambda_n}{\partial x_i \partial x_j} ||^2 + \sum_{n=0}^{L-1} (\Delta t)^4 || \frac{\partial^3 \lambda_n}{\partial x_1 \partial x_2 \partial x_3} ||^2 + (\Delta t)^3] \,. \quad (18)$$

Therefore

$$||\theta^L||^2 + ||\nabla \sigma^L||^2 + \Delta t \sum_{n=0}^{L-1} ||\nabla \theta^{n+1}||^2 + \sum_{(i,j)} (\Delta t)^2 || \frac{\partial^2 \theta^L}{\partial x_i \partial x_j} ||^2 + (\Delta t)^3 || \frac{\partial^3 \theta^L}{\partial x_1 \partial x_2 \partial x_3} ||^2$$

$$\leq M[(\Delta t)^2 + h^{2r+2} + ||\theta^0||^2 + ||\nabla \sigma^0||^2 + \sum_{(i,j)} (\Delta t)^2 || \frac{\partial^2 \theta^0}{\partial x_i \partial x_j} ||^2$$

$$+ (\Delta t)^3 || \frac{\partial^3 \theta^0}{\partial x_1 \partial x_2 \partial x_3} ||^2 + \sum_{n=0}^{L-1} \Delta t (||\theta^{n+1}||^2 + ||\partial_t \sigma^n||^2 + ||\nabla \sigma^{n+1}||^2 + ||\lambda_n||^2)$$

$$+ \sum_{n=0}^{L-1} \sum_{(i,j)} (\Delta t)^3 || \frac{\partial^2 \theta^{n+1}}{\partial x_i \partial x_j} ||^2 + \sum_{n=0}^{L-1} (\Delta t)^4 || \frac{\partial^3 \theta^{n+1}}{\partial x_1 \partial x_2 \partial x_3} ||^2$$

$$+ \sum_{n=0}^{L-1} \sum_{(i,j)} (\Delta t)^3 || \frac{\partial^2 \lambda_n}{\partial x_i \partial x_j} ||^2 + \sum_{n=0}^{L-1} (\Delta t)^4 || \frac{\partial^3 \lambda_n}{\partial x_1 \partial x_2 \partial x_3} ||^2] \,. \quad (19)$$

In (10b), let $w = \sigma^{n+1}$, by the use of Schwarz inequality and the inequality $a(a-b) \geq \frac{1}{2}(a^2 - b^2)$, then we have

$$\frac{||\sigma^{n+1}||^2 - ||\sigma^n||^2}{2\Delta t} \leq M[||\theta^{n+1}||^2 + ||\partial_t \pi^n||^2 + ||\rho^{n+1}||^2$$
$$||(u_t)^{n+1} - \partial_t u^n||^2 + ||\sigma^{n+1}||^2] \,.$$

Because

$$||(u_t)^{n+1} - \partial_t u^n||^2 \leq M(\Delta t)^2 \quad (20)$$

the both sides of the above inequality times $2\Delta t$ and sum up n from 0 to $L-1$, then

$$||\sigma^L||^2 \leq M[(\Delta t)^2 + ||\sigma^0||^2 + \Delta t (\sum_{n=0}^{L-1} ||\theta^n||^2$$
$$+ ||\partial_t \pi^n||^2 + ||\sigma^n||^2 + ||\rho^n||^2)] \,. \quad (21)$$

For an appropriate η (see [1]) becomes to

$$\Delta t \sum_{n=0}^{L-1} ||\partial_t \eta^n||^2 \leq ||\frac{\partial \eta^n}{\partial t}||^2_{L^2(L^2)} .$$

So

$$||\sigma^L||^2 \leq M[(\Delta t)^2 + ||\sigma^0||^2 + ||\frac{\partial \pi}{\partial t}||^2_{L^\infty(L^2)} + ||\frac{\partial \pi}{\partial t}||^2_{L^2(L^2)} + \Delta t \sum_{n=0}^{L-1} (||\theta^n||^2 + ||\sigma^n||^{2)})]$$

$$\leq M[(\Delta t)^2 + ||\sigma^0||^2 + h^{2r+2} + \Delta t \sum_{n=0}^{L-1} (||\theta^n||^2 + ||\sigma^n||^2)] . \tag{22}$$

Considering the above discussion, and by the use of Gronwall inequality, we get

$$||\theta^L||^2 + ||\sigma^L||_1^2 \leq M[(\Delta t)^2 + ||\sigma^0||^2 + h^{2r+2} + ||\theta^0||^2 + ||\nabla \sigma^0||^2$$
$$+ \sum_{(i,j)} (\Delta t)^2 ||\frac{\partial^2 \theta^0}{\partial x_i \partial x_j}||^2 + (\Delta t)^3 ||\frac{\partial^3 \theta^0}{\partial x_1 \partial x_2 \partial x_3}||^2] . \tag{23}$$

We assume that the initial values satisfy

$$||U_0 - \tilde{u}_0||^2 + ||V_0 - \tilde{v}_0||^2 + ||\nabla(U_0 - \tilde{u}_0)||^2$$
$$+ \sum_{(i,j)} (\Delta t)^2 ||\frac{\partial^2 (V_0 - \tilde{v}_0)}{\partial x_i \partial x_j}||^2 + (\Delta t)^3 ||\frac{\partial^3 (V_0 - \tilde{v}_0)}{\partial x_1 \partial x_2 \partial x_3}||^2 \leq M h^{2r+2} . \tag{24}$$

Then we get

$$\|\theta^L\|^2 + \|\sigma^L\|_1^2 \leq M\{(\triangle t)^2 + h^{2r+2}\} . \tag{25}$$

Finally, we come to the following theorem.

Theorem. *Let u, $\{U^n\}_{n=0}^N$ denote the solutions of (2) and (4) respectively, the conditions (P) and (H) are satisfied $r \geq 2$, the initial value satisfy (24). Then there exists a constant M such that the following estimate holds*

$$\max_{0 \leq n \leq L-1} [\|u^n - U^n\|_1 + \|v^n - V^n\|] \leq M\{\triangle t + h^{r+1}\} .$$

Here h is sufficiently small.

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