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A MODIFIED FAMILY OF LAGUERRE ITERATION FUNCTIONS OF QUARTICALLY CONVERGENCE

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Abstract: In this paper, we derive a modified family of iteration functions for finding simple zeros of analytic functions. The family includes, Traub's quartic square root method and, as a limiting cases, the Kiss method, the Halley method and the Newton method. We also present a family of third-derivative-free variants of this method. All the methods of the family are locally and quartically convergence for a simple zero. The asymptotic error constants for this methods of the family and numerical examples are given to show the performance of presented methods.

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1. Introduction

Let $f(z)$ be an analytic function in a region. For any real number $\nu \neq 0, 1$, define the Laguerre iteration function as (see [3, 6, 10]):

$$\hat{z} = z - \frac{\nu f(z)}{f'(z) + \operatorname{sgn}(\nu-1)\sqrt{(\nu-1)^2 f'(z)^2 - \nu(\nu-1) f(z)\, f''(z)}} \,. \qquad (1)$$

When $f(z)$ is a polynomial of degree n, this is the Laguerre method for $\nu = n$ [1, 7]. It is well known that for every $\nu \neq 0$, 1, iteration of (1) converges with order 3 in the neighborhood of a simple zero of f (see [6]).

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We derive a modified one parameter family of iteration functions for finding simple zeros of analytic functions and prove locally quartically convergence in Section 2. We give free from third-derivatives-free variants of this family of iteration functions prove that the method of the family has fourth-order convergence in Section 3. In the last section, we test the Laguerre method by a numerical experiment.

2. Derivation of the Modified Family Including Third-Derivative

We shall use the method in [3, 9] to derive a modified one parameter family of iteration functions. Let $f(z) = \prod_{i=1}^{n}(z-\xi_i)$, $n \geq 4$. By computing the derivation of $\log|f(z)|$, we have

$$\begin{aligned} F_1(z) &= \frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{1}{z-\xi_i}, \\ F_2(z) &= -\Big(\frac{f'(z)}{f(z)}\Big)' = \sum_{i=1}^{n} \frac{1}{(z-\xi_i)^2}, \\ F_3(z) &= \frac{1}{2}\Big(\frac{f'(z)}{f(z)}\Big)'' = \sum_{i=1}^{n} \frac{1}{(z-\xi_i)^3}\,. \end{aligned} \tag{2}$$

Let ξ_n be a zero to be determined and let z be an approximation to ξ_n. Define $\alpha = \frac{1}{z-\xi_n}$ and $\beta = \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{z-\xi_i}$. Then (2) becomes

$$F_k(z) = \alpha^k + (n-1)\beta^k,\ k = 1,2,3. \tag{3}$$

Since $\beta = \frac{F_1(z)-\alpha}{n-1}$,

$$(n-1)^2 F_3(z) = (n-1)^2\alpha^3 + (F_1(z)-\alpha)^3$$

which implies that

$$n(n-2)\alpha^3 + 3F_1(z)\alpha^2 - 3F_1(z)^2\alpha + F_1(z)^3 - (n-1)^2F_3(z) = 0. \tag{4}$$

If $\sum_{i=1}^{n-1}(\frac{1}{z-\xi_i}-\beta)^3$ is sufficiently small when all other zeros are distinct from ξ_n, then by [8]

$$\begin{aligned} &-n(n+1)\alpha^3 + 3(n+1)F_1(z)\,\alpha^2 + 3[(n-1)F_2(z) - 6F_1(z)^2]\,\alpha \\ &\qquad + (n-1)^2F_3(z) - 3(n-1)F_1(z)\,F_2(z) + 2F_1(z)^3 = 0. \end{aligned} \tag{5}$$

Eliminating α^3 in (4) and (5), it yields the quadratic equation

$$\begin{aligned} &(n+1)F_1(z)\,\alpha^2 + \big((n-2)F_2(z) - 3F_1(z)^2\big)\alpha \\ &\qquad + F_1(z)^3 - (n-2)F_1(z)\,F_2(z) - (n-1)F_3(z) = 0. \end{aligned} \tag{6}$$

Solving for ξ_n in (6), then we have

$$\xi_n = z - \frac{2(n+1)F_1(z)}{3F_1(z)^2 - (n-2)F_2(z) \pm \sqrt{R_n}}\,, \tag{7}$$

where

$$R_n = (3F_1(z)^2 - (n-2)F_2(z))^2 - 4(n+1)F_1(z)(F_1(z)^3 - (n-2)F_1(z)\,F_2(z) - (n-1)F_3(z)).$$

We choose the sign in (7) to be maximize the absolute value of denominator of ξ_n as $z \to \xi_n$. Since ξ_n is a simple zero of $f(z)$, we have

$$\frac{F_2(\xi_n)}{F_1(\xi_n)^2} = 1, \quad \frac{F_3(\xi_n)}{F_1(\xi_n)^3} = 1$$

and thus $R_n \to \big((5-n)^2 + 8(n+1)(n-2)\big) = (3(n-1))^2$ as $z \to \xi_n$. Since

$$|3 - (n-2) - 3(n-1)| \le |3 - (n-2) + 3(n-1)| = 2(n+1),\ n \ge 1$$

and the equality holds only for $n = 1$, (7) becomes

$$\xi_n = z - \frac{2(n+1)F_1(z)}{3F_1(z)^2 - (n-2)F_2(z) + \sqrt{R_n}} \tag{8}$$

for $n \neq \pm 1$.

We apply this method to an analytic function $f(z)$ with a simple zero at ξ. Define $u(z) = u = \frac{f(z)}{f'(z)}$ and $A_j(z) = A_j = \frac{f^{(j)}(z)}{j!\,f'(z)}$, $j = 2, 3$. Let ν be a real parameter with $\nu \neq \pm 1$. We define a modified one parameter family of iteration functions by

$$\psi_\nu(z) = z - \frac{2(\nu+1)u}{3 - (\nu-2)(1 - 2A_2\,u) + \operatorname{sgn}(\nu-1)\sqrt{R_\nu}}\,, \tag{9}$$

where

$$R_\nu = 9(\nu-1)^2 - 12(\nu-1)(2\nu-1)A_2\,u + 4(\nu-2)^2(A_2u)^2 + 12(\nu^2-1)A_3\,u^2\,. \tag{10}$$

Theorem 2.1. *Suppose that ξ is a simple zero of $f(z)$. Let $f(z) = (z-\xi)g(z)$ be an analytic function with $g(\xi) \neq 0$. Then for $\nu \neq \pm 1$, $\psi_\nu(z)$ in (9) with (10) converges locally and quartically to ξ, i.e.,*

$$\lim_{z \to \xi} \frac{\psi_\nu(z) - \xi}{(z-\xi)^4} = \frac{2(\nu-2)}{3(\nu-1)}\Big(\frac{g'(\xi)}{g(\xi)}\Big)^3 - \frac{5\nu-7}{6(\nu-1)}\frac{g'(\xi)\,g''(\xi)}{g(\xi)^2} + \frac{g^{(3)}(\xi)}{6\,g(\xi)}\,.$$

Proof. In view of an elementary evaluation of derivatives of $\psi_\nu(z)$, we employ the symbolic computation of the *Maple* package to compute the Taylor expansion of $\psi_\nu(z)$ around $z = \xi$ (see [2] for details). We find after simplifying that

$$\psi_\nu(z) = \xi + \Big[\frac{2(\nu-2)}{3(\nu-1)}\Big(\frac{g'(\xi)}{g(\xi)}\Big)^3 - \frac{5\nu-7}{6(\nu-1)}\frac{g'(\xi)}{g(\xi)}\frac{g''(\xi)}{g(\xi)} + \frac{g^{(3)}(\xi)}{6\,g(\xi)}\Big](z-\xi)^4 + O((z-\xi)^5). \quad \square$$

Example 2.1. When $\nu = 2$ in (9),

$$\psi_2(z) = z - \frac{2u}{1+\sqrt{1-4A_2u+4A_3u^2}}$$

which is Traub's quartic square root method [11].

Example 2.2. Rationalizing (9), we have

$$\psi_\nu(z) = z - \frac{u((5-\nu)+2(\nu-2)A_2\,u - \mathrm{sgn}(\nu-1)\sqrt{R_\nu}}{2[(4-2\nu)+(5\nu-7)A_2\,u-3(\nu-1)A_3u^2]}. \tag{11}$$

From (10), $R_{-1} = 36(1-A_2\,u)^2$ and thus if $\mathrm{Re}(1-A_2\,u) > 0$, taking $\nu \to -1$ in (11), we obtain

$$\hat{z} = z - \frac{u(1-A_2\,u)}{1-2A_2u+A_3u^2}$$

which is the Kiss method [4] of order 4.

Example 2.3. For $\nu = 1$ in (9), $R_{1/2} = (2A_2u)^2$. If $\mathrm{Re}(A_2\,u) > 0$, letting $\nu \to 1^{+0}$ in (11), we obtain

$$\hat{z} = z - u \tag{12}$$

which is the Newton's method. Taking a limit $\nu \to 1^{-0}$ in (11), we obtain

$$\hat{z} = z - \frac{u}{1-A_2u} = z - \frac{f\,f'}{f'^2 - f\,f''/2} \tag{13}$$

which is the Halley's method, see [5].

If $\mathrm{Re}(A_2\,u) < 0$, we obtain the Halley method and Newton method by letting $\nu \to 1^{+0}$ and $\nu \to 1^{-0}$, respectively.

3. Derivation of the Modified Family Free from Third-Derivative

Let $w = z - \frac{f(z)}{f'(z)}$. We consider Taylor expansion of $f(w)$ about z

$$f(w) \simeq f(z) + f'(z)(w-z) + \frac{1}{2}f''(z)(w-z)^2 + \frac{1}{3!}f'''(z)(w-z)^3 \tag{14}$$

which implies $f(w) \simeq \frac{1}{2}\frac{f''(z)f(z)^2}{f'(z)^2} - \frac{1}{3!}\frac{f'''(z)f(z)^3}{f'(z)^3}$. Therefore, we have

$$A_3u^2 \simeq A_2u - \frac{f(w)}{f(z)}. \tag{15}$$

Substituting (15) into (10), we obtain

$$\widehat{\psi}_{\nu}(z) = z - \frac{2(\nu+1)u}{3-(\nu-2)(1-2A_2\,u)+\operatorname{sgn}(\nu-1)\sqrt{\widehat{R}_{\nu}}}\,, \tag{16}$$

where

$$\widehat{R}_{\nu} = \widehat{R}_{\nu}(z) = \big(3(\nu-1)-2(\nu-2)A_2 u\big)^2 - 12(\nu^2-1)\frac{f(w)}{f(z)} \tag{17}$$

and $w = z - \frac{f(z)}{f'(z)}$.

Theorem 3.1. *Suppose that ξ is a simple zero of $f(z)$. Let $f(z) = (z-\xi)g(z)$ be an analytic function with $g(\xi)\neq 0$. Let $f(z) = (z-\xi)\,g(z)$ with $g(\xi)\neq 0$. Then for $\nu \neq \pm 1$, $\widehat{\psi}_{\nu}(z)$ in (16) with (17) converges locally and quartically to ξ, i.e.,*

$$\lim_{z\to\xi}\frac{\widehat{\psi}_{\nu}(z)-\xi}{(z-\xi)^4} = \frac{2(\nu-2)}{3(\nu-1)}\Big(\frac{g'(\xi)}{g(\xi)}\Big)^3 - \frac{5\nu-7}{6(\nu-1)}\frac{g'(\xi)\,g''(\xi)}{g(\xi)^2}.$$

Proof. Let ξ be a simple zero of f. Consider the iteration function $\widehat{\psi}_{\nu}(z)$ in (16), where

$$\widehat{R}_{\nu}(z) = 9(\nu-1)^2 - 12(\nu-1)(\nu-2)A_2\,u + 4(\nu-2)^2(A_2u)^2 - 12(\nu^2-1)\frac{f(w(z))}{f(z)} \tag{18}$$

and $w(z) = z - \frac{f(z)}{f'(z)}$.

In view of an elementary evaluation of derivatives of $\widehat{\psi}_{\nu}(z)$, we employ the symbolic computation of the Maple package to compute the Taylor expansion of $\widehat{\psi}_{\nu}(z)$ around $z=\xi$ (see [2] for details). We find after simplifying that

$$\widehat{\psi}_{\nu}(z) = \xi + \Big[\frac{2(\nu-2)}{3(\nu-1)}\Big(\frac{g'(\xi)}{g(\xi)}\Big)^3 - \frac{5\nu-7}{6(\nu-1)}\frac{g'(\xi)}{g(\xi)}\frac{g''(\xi)}{g(\xi)}\Big](z-\xi)^4 + O((z-\xi)^5). \quad \square$$

4. Numerical Examples

All computations were doing *g++* compiler(gcc version 2.95.2) with *MPPACK* [12] with 60 significant digits. We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|z_{n+1}-z_n| <$

$f(z)$		IT			
		MLM	TDFM	OM	DNM
$f_1, z_0 = 4.1$	$\nu = 10$	3	4	4	5
	$\nu = -7.6$	4	4	3	5
	$\nu = 2$	4	4	4	5
	$\nu = -1$	2	4	Diverge	5
$f_1, z_0 = 10$	$\nu = 10$	5	5	12	12
	$\nu = -7.6$	8	7	5	12
	$\nu = 2$	14	15	14	14
	$\nu = -1$	9	9	9	12
$f_2, z_0 = 1.5i$	$\nu = 10$	4	4	4	4
	$\nu = -7.6$	4	4	4	4
	$\nu = 2$	4	4	4	4
	$\nu = -1$	2	2	Diverge	4
$f_3, z_0 = 1$	$\nu = 10$	4	4	4	4
	$\nu = -7.6$	3	4	4	4
	$\nu = 2$	4	4	4	4
	$\nu = -1$	4	4	4	4
$f_3, z_0 = 2$	$\nu = 10$	4	4	4	4
	$\nu = -7.6$	4	4	4	4
	$\nu = 2$	4	4	4	4
	$\nu = -1$	2	4	Diverge	4

Table 1: Comparison of various fourth-order convergent iterative methods

ϵ, (ii) $|f(z_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, z_{n+1} is taken as the exact root ξ computed. For numerical illustrations we used the fixed stopping criterion $\epsilon = 10^{-60}$. We present some numerical test results for various quartically convergent classical iterative schemes in Table 1. There are compared Osada's modification [8] (OM), double Newton's method [11] (DNM) defined by

$$z_{n+1} = z_n - \frac{f(w_n)}{f'(w_n)}, \tag{19}$$

where $w_n = z_n - f(z_n)/f'(z_n)$ and the methods (9)-(10) (MLM) and (16)-(17) (TDFM) introduced in the present contribution. We used the following test functions and display the approximate zero z_* found up to the 60-th decimal places. We displayed are number of iterations to approximate the zero (IT), the approximate zero z_* and the value $f(z_*)$.

$$f_1(z) = (z-1)^4(z-2)^3(z-3)^2(z-4), \quad z_* = 4.0_470,$$
$$f_2(z) = z^4 + 3z^2 + 2, \quad z_* = \pm i, \quad z_* = \pm 1.41421i,$$

$$f_3(z) = \sin^2(z) - z^2 + 1, \qquad z_* = 1.4044916482153413.$$

5. Conclusion

In this work we presented a new approach to improve the order of convergence of Lagurre's methods and a family of third-derivative-free variants of this method. All the methods of the family are locally and quartically convergent for a simple zero, they are then compared in performance to several other fourth-order methods, and it was observed that they have at least equal performance. Even though we considered only Lagurre's methods, our approach can be applied to any existing iterative method to develop higher order variants of the method.

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