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AN EXTENSION OF METHOD OF QUASILINEARIZATION FOR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract: This paper is devoted to existence and monotone approximation of the unique solution for integro-differential equation including a variety of possible cases by employing the method of generalized quasilinearization. Further, quadratic convergence of the monotone approximation is also discussed.

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Key Words: integro-differential equations, quasilinearization, quadratic convergence, upper and lower solutions

1. Introduction

It is well known that the method of quasilinearization provides a powerful tool for obtaining a sequence of approximate solutions of nonlinear problems involving convex/concave functions. In some cases, it offers a constructive procedure for the solutions, and the iterates serve as upper and lower bounds for those solutions. In recent years, the method of quasilinearization has been general-

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ized, refined and extended in diverse problems. We can refer the reader to the monographes [6], [9], papers [7], [10], [3], [11], [4], [5], [1] and references cited therein.

Integro-differential equations arise frequently as mathematical models in diverse disciplines, for example, the mathematical modeling of biological sciences such as spreading of disease by dispersal of infectious individuals. The theory and the applications of integro-differential equations is now an important area of investigation [2]. In [3], Deo and Knoll investigated the integro-differential equation by the method of quasilinearization developed in [2]. Here, to enlarge the class of functions, we should follow up the idea presented in [5] and attempt to extend the technique of quasilinearization to the following initial value problem

$$x'(t) = f(t, x(t)) + \int_{t_0}^{t} q(t, s, x(s))ds, \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(J \times R, R)$, $q \in C(J \times J \times R, R)$, $t \in J = [t_0, t_0 + T]$, $t_0 \geq 0$, $T > 0$. Consequently, in this paper, the investigations of quasilinearized approximations to (1.1), as well as rapid convergence, become the goals of our investigation.

2. Preliminaries

Firstly, for $y_0(t)$, $z_0(t) \in C^1(J, R)$ with $y_0 \leq z_0$, we denote the following sets

$$\begin{aligned}
\bar{\Omega}_1 &= \{(t, u) : y_0 \leq u \leq z_0,\ t \in J\},\\
\Omega_1 &= \{(t, u, v) : y_0 \leq u, v \leq z_0,\ t \in J\},\\
\bar{\Omega}_2 &= \{(t, s, u) : y_0 \leq u \leq z_0,\ (t, s) \in J \times J\},\\
\Omega_2 &= \{(t, s, u, v) : y_0 \leq u, v \leq z_0,\ (t, s) \in J \times J\}.
\end{aligned}$$

We say $u \in C^1(J, R)$ is a lower solution of problem (1.1), if

$$u'(t) \leq f(t, u) + \int_{t_0}^{t} q(t, s, u(s))ds, \quad t \in J, \quad u(0) \leq x_0.$$

Similarly $v \in C^1(J, R)$ is an upper solution of problem (1.1), if the reverse inequalities hold.

Before we proceed further, we need to list the following known result.

Lemma 2.1. (see [3]) *For problem (1.1), suppose that:*

(A_0) *u, v are lower and upper solutions of problem (1.1) respectively;*

(A_1) *$f \in C(\bar{\Omega}_1, R)$, and $f(t, x) - f(t, y) \leq L(x - y)$, where $t \in J$, $u(t) \leq$*

$y \le x \le v(t)$, and $L \ge 0$;

(A_2) $q \in C(\bar{\Omega}_2, R)$ is monotone nondecreasing in x for each $(t,s) \in J \times J$, and $q(t,s,x) - q(t,s,y) \le N(x-y)$, where $(t,s) \in J \times J$, $u(t) \le y \le x \le v(t)$, and $N \ge 0$.

Then $u(t) \le v(t)$ for $t \in J$, provided that $u(t_0) \le v(t_0)$.

3. Main Results

Suppose that f and q have the splitting $f(t,x) = F(t,x,x)$, and $q(t,s,x) = Q(t,s,x,x)$, where $F \in C(\Omega_1, R)$, $Q \in C(\Omega_2, R)$. Then problem (1.1) shall take the form

$$x'(t) = F(t, x(t), x(t)) + \int_{t_0}^{t} Q(t,s,x(s),x(s))ds, \quad x(t_0) = x_0 \,. \tag{3.1}$$

Theorem 3.1. *For problem (3.1), suppose that:*

(B_0) y_0, $z_0 \in C^1(J,R)$ are lower and upper solutions of problem (3.1), respectively, such that, $y_0(t) \le z_0(t)$ on J;

(B_1) F, F_x, F_y, F_{xx}, F_{xy}, F_{yx}, $F_{yy} \in C(\Omega_1, R)$ and

$$F_{xx}(t,x,y) \ge 0, \; F_{xy}(t,x,y) \le 0, \; F_{yy}(t,x,y) \le 0, \quad \text{for } (t,x,y) \in \Omega_1;$$

(B_2) Q, Q_x, Q_y, Q_{xx}, Q_{xy}, Q_{yx}, $Q_{yy} \in C(\Omega_2, R)$ and

$$Q_{xx}(t,s,x,y) \ge 0, \; Q_{xy}(t,s,x,y) \le 0, \; Q_{yy}(t,x,y) \le 0, \; (t,s,x,y) \in \Omega_2;$$

(B_3) $Q_x(t,s,\eta(s),\nu(s)) - Q_y(t,s,\nu(s),\nu(s)) \ge 0$, for $y_0(t) \le \eta(t) \le \nu(t) \le z_0(t)$, $(t,s) \in J \times J$.

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ which converge uniformly to the unique solution of problem (3.1) on J, and the convergence is quadratic.

Proof. Assumptions (B_0)-(B_2) guarantee that problem (3.1) has a unique solution on Ω_1.

Obviously, (B_1) implies that F_x is nondecreasing in the second variable, F_x is nonincreasing in the third variable and F_y is nonincreasing in the last two variables; (B_2) leads to that Q_x is nondecreasing in the third variable, Q_x is nonincreasing in the last variable; Q_y is nonincreasing in the last two variables. For convenience, we denote these property by (A).

Consider the following linear problems for each $n = 0, 1, 2, 3, \cdots$,

$$y'_{n+1} = F(t, y_n, y_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)](y_{n+1} - y_n)$$

$$+\int_{t_0}^{t}\{Q(t,s,y_n,y_n)+[Q_x(t,s,y_n,z_n)+Q_y(t,s,z_n,z_n)](y_{n+1}-y_n)\}ds,$$

$$y_{n+1}(t_0)=x_0,$$

$$z'_{n+1}=F(t,z_n,z_n)+[F_x(t,y_n,z_n)+F_y(t,z_n,z_n)](z_{n+1}-z_n)$$
$$+\int_{t_0}^{t}\{Q(t,s,z_n,z_n)+[Q_x(t,s,y_n,z_n)+Q_y(t,s,z_n,z_n)](z_{n+1}-z_n)\}ds,$$

$$z_{n+1}(t_0)=x_0.$$

Clearly, each linear problem has a unique solution on J. We wish to show that

$$y_0\le y_1\le\cdots\le y_n\le z_n\le\cdots\le z_1\le z_0 \quad \text{on } J. \tag{3.2}$$

We claim first that $y_0\le y_1$ on J. For this purpose, let $p=y_0-y_1$, and note that $p(t_0)=y_0(t_0)-y_1(t_0)\le 0$. Then

$$\begin{aligned}p'(t)\le{}& F(t,y_0,y_0)+\int_{t_0}^{t}Q(t,s,y_0,y_0)ds-F(t,y_0,y_0)\\&-[F_x(t,y_0,z_0)+F_y(t,z_0,z_0)](y_1-y_0)\\&-\int_{t_0}^{t}\{Q(t,s,y_0,y_0)+[Q_x(t,s,y_0,z_0)+Q_y(t,s,z_0,z_0)](y_1-y_0)\}ds\\={}&[F_x(t,y_0,z_0)+F_y(t,z_0,z_0)]p(t)+\int_{t_0}^{t}[Q_x(t,s,y_0,z_0)+Q_y(t,s,z_0,z_0)]p(s)ds,\\& t\in J.\end{aligned}$$

By Lemma 2.1, in view of (B_3), this implies $y_0\le y_1$ on J.

Similarly, we can show that $z_1\le z_0$ on J.

Next we prove that $y_1\le z_1$ on J. Setting $p=y_1-z_1$, and $p(t_0)=0$. We obtain

$$\begin{aligned}p'(t)={}&F(t,y_0,y_0)+[F_x(t,y_0,z_0)+F_y(t,z_0,z_0)](y_1-y_0)\\&+\int_{t_0}^{t}\{Q(t,s,y_0,y_0)+[Q_x(t,s,y_0,z_0)+Q_y(t,s,z_0,z_0)](y_1-y_0)\}ds\\&-F(t,z_0,z_0)-[F_x(t,y_0,z_0)-F_y(t,z_0,z_0)](z_1-z_0)\\&-\int_{t_0}^{t}\{Q(t,s,z_0,z_0)+[Q_x(t,s,y_0,z_0)+Q_y(t,s,z_0,z_0)](z_1-z_0)\}ds.\end{aligned}$$

From the mean value theorem and property (A), we have

$$
\begin{aligned}
p'(t) \le{}& F(t, y_0, y_0) - F(t, z_0, y_0) + F(t, z_0, y_0) - F(t, z_0, z_0) \\
&+ [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)](y_1 - y_0 - z_1 + z_0) \\
&+ \int_{t_0}^{t} [Q(t, s, y_0, y_0) - Q(t, s, z_0, y_0) + Q(t, s, z_0, y_0) - Q(t, s, z_0, z_0)]ds \\
&+ \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_1 - y_0 - z_1 + z_0)ds \\
={}& [F_x(t, \xi_1, y_0) + F_y(t, z_0, \xi_2) - F_x(t, y_0, z_0) - F_y(t, z_0, z_0)](y_0 - z_0) \\
&+ [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)](y_1 - z_1) \\
&+ \int_{t_0}^{t} [Q_x(t, s, \xi_3, y_0) + Q_y(t, s, z_0, \xi_4) - Q_x(t, s, y_0, z_0) - Q_y(t, s, z_0, z_0)](y_0 - z_0)ds \\
&+ \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_1 - z_1)ds \\
\le{}& [F_x(t, y_0, z_0) - F_x(t, y_0, y_0)](z_0 - y_0) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t) \\
&+ \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) - Q_x(t, s, y_0, y_0)](z_0 - y_0)ds \\
&+ \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)]p(s)ds \le [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t) \\
&+ \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)]p(s)ds, \quad t \in J,
\end{aligned}
$$

where $y_0 \le \xi_1, \xi_2, \xi_3, \xi_4 \le z_0$. Then, we get, by using Lemma 2.1, $p(t) \le 0$ on J and consequently, $y_1 \le z_1$ on J.

As a result, it follows that

$$
y_0 \le y_1 \le z_1 \le z_0, \quad t \in J.
$$

Now we need to show that y_1 and z_1 are lower and upper solutions of problem (3.1), respectively. The mean value theorem and property (A) yield

$$
\begin{aligned}
y_1'(t) ={}& F(t, y_1, y_1) + F(t, y_0, y_0) - F(t, y_1, y_0) + F(t, y_1, y_0) \\
&- F(t, y_1, y_1) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)](y_1 - y_0) \\
&+ \int_{t_0}^{t} [Q(t, s, y_1, y_1) + Q(t, s, y_0, y_0) - Q(t, s, y_1, y_0) \\
&+ Q(t, s, y_1, y_0) - Q(t, s, y_1, y_1)]ds \\
&+ \int_{t_0}^{t} [Q_x(t, s, y_0, z_0) + Q_y(t, s, z_0, z_0)](y_1 - y_0)ds \\
={}& F(t, y_1, y_1) + [F_x(t, \xi_1, y_0) + F_y(t, y_1, \xi_2)](y_0 - y_1)
\end{aligned}
$$

$$
\begin{aligned}
&+ [F_x(t,y_0,z_0) + F_y(t,z_0,z_0)](y_1 - y_0) \\
&+ \int_{t_0}^{t} Q(t,s,y_1,y_1)ds + \int_{t_0}^{t} [Q_x(t,s,\xi_3,y_0) + Q_y(t,s,y_1,\xi_4)](y_0 - y_1)ds \\
&+ \int_{t_0}^{t} [Q_x(t,s,y_0,z_0) + Q_y(t,s,z_0,z_0)](y_1 - y_0)ds \\
&\le F(t,y_1,y_1) + \int_{t_0}^{t} Q(t,s,y_1,y_1)ds \\
&+ [F_x(t,y_0,z_0) - F_x(t,y_0,y_0) + F_y(t,z_0,z_0) - F_y(t,y_1,y_1)](y_1 - y_0) \\
&+ \int_{t_0}^{t} [Q_x(t,s,y_0,z_0) - Q_x(t,s,y_0,y_0) \\
&+ Q_y(t,s,z_0,z_0) - Q_y(t,s,y_1,y_1)](y_1 - y_0)ds \le F(t,y_1,y_1) + \int_{t_0}^{t} Q(t,s,y_1,y_1)ds, \\
&\qquad t \in J,
\end{aligned}
$$

where $y_0 \le \xi_1, \xi_2, \xi_3, \xi_4 \le z_0$.

Similarly, proceeding as before, one can obtain that

$$
z_1'(t) \ge F(t,z_1,z_1) + \int_{t_0}^{t} Q(t,s,z_1,z_1)ds, \qquad t \in J.
$$

The above proves that y_1 and z_1 are lower and upper solutions of problem (3.1).

Suppose that for some $k \ge 1$, we have

$$
y_{k-1} \le y_k \le z_k \le z_{k-1} \quad \text{on} \quad J. \tag{3.3}
$$

Let y_k, z_k be lower and upper solutions of problem (3.1). We shall show that

$$
y_k \le y_{k+1} \le z_{k+1} \le z_k \quad \text{on} \quad J. \tag{3.4}
$$

To do this, consider $p = y_k - y_{k+1}$ on J so that $p(t_0) = 0$. From the fact that y_k is a lower solution of problem (3.1), it is clear that

$$
\begin{aligned}
p'(t) \le\ & F(t,y_k,y_k) + \int_{t_0}^{t} Q(t,s,y_k,y_k)ds \\
&- F(t,y_k,y_k) - [F_x(t,y_k,z_k) + F_y(t,z_k,z_k)](y_{k+1} - y_k) \\
&- \int_{t_0}^{t} \{Q(t,s,y_k,y_k) + [Q_x(t,s,y_k,z_k) + Q_y(t,s,z_k,z_k)](y_{k+1} - y_k)\}ds \\
=\ & [F_x(t,y_k,z_k) + F_y(t,z_k,z_k)]p(t) \\
&+ \int_{t_0}^{t} [Q_x(t,s,y_k,z_k) + Q_y(t,s,z_k,z_k)]p(s)ds, \quad t \in J.
\end{aligned}
$$

This leads to by Lemma 2.1, because of (B_4), that $y_k \le y_{k+1}$ on J. A similar argument holds for $z_{k+1} \le z_k$ on J.

Letting $p = y_{k+1} - z_{k+1}$, again $p(t_0) = 0$, and arguing as before, one can show that

$$p'(t) \le [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t) + \int_{t_0}^{t} [Q_x(t, s, y_k, z_k) + Q_y(t, s, z_k, z_k)]p(s)ds, \quad t \in J,$$

which yields $y_{k+1} \le z_{k+1}$, on J. Thus we have (3.4) and by induction, we see that (3.2) is valid on J. By the standard arguments [6], it can be shown that the sequences $\{y_n(t)\}$ and $\{z_n(t)\}$ converge uniformly and monotonically to $x(t)$, the unique solution of problem (3.1) on J.

Finally, we shall show that the convergence is quadratic. For this purpose, put $p_n = x - y_n$ and $h_n = z_n - x$, where x denotes the unique solution of problems (3.1). Note that $p_n(t_0) = h_n(t_0) = 0$, and $p_n \ge 0$, $h_n \ge 0$. Applying the mean value theorem and property (A), there exist ξ_i, $i = 1, 2, \cdots, 12$, such that $y_n \le \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_9 \le x$, $x \le \xi_6, \xi_{10} \le z_n$, $y_n \le \xi_7, \xi_8, \xi_{11}, \xi_{12} \le z_n$, and

$$\begin{aligned}
p'_{n+1}(t) = {} & F(t, x, x) + \int_{t_0}^{t} Q(t, s, x, x)ds \\
& - F(t, y_n, y_n) - [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)](y_{n+1} - y_n) \\
& - \int_{t_0}^{t} \{Q(t, s, y_n, y_n) + [Q_x(t, s, y_n, z_n) + Q_y(t, s, z_n, z_n)](y_{n+1} - y_n)\}ds \\
= {} & F(t, x, x) - F(t, y_n, x) + F(t, y_n, x) - F(t, y_n, y_n) \\
& - [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)](y_{n+1} - x + x - y_n) \\
& + \int_{t_0}^{t} [Q(t, s, x, x) - Q(t, s, y_n, x) + Q(t, s, y_n, x) - Q(t, s, y_n, z_n)]ds \\
& - \int_{t_0}^{t} [Q_x(t, s, y_n, z_n) + Q_y(t, s, z_n, z_n)](y_{n+1} - x + x - y_n)ds \\
\le {} & [F_x(t, x, x) - F_x(t, y_n, x) + F_x(t, y_n, x) - F_x(t, y_n, z_n) \\
& + F_y(t, y_n, y_n) - F_y(t, z_n, y_n) + F_y(t, z_n, y_n) - F_y(t, z_n, z_n)]p_n(t) \\
& + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]p_{n+1}(t) \\
& + \int_{t_0}^{t} [Q_x(t, s, x, x) - Q_x(t, s, y_n, x) + Q_x(t, s, y_n, x) - Q_x(t, s, y_n, z_n) \\
& + Q_y(t, s, y_n, y_n) - Q_y(t, s, z_n, y_n) + Q_y(t, s, z_n, y_n) - Q_y(t, s, z_n, z_n)]p_n(s)ds \\
& + \int_{t_0}^{t} [Q_x(t, s, y_n, z_n) + Q_y(t, s, z_n, z_n)]p_{n+1}(s)ds
\end{aligned}$$

$$
\begin{aligned}
&= [F_{xx}(t,\xi_5,x)p_n(t) - F_{xy}(t,y_n,\xi_6,)h_n(t) - F_{yx}(t,\xi_7,y_n)(h_n(t)+p_n(t))\\
&\quad - F_{yy}(t,z_n,\xi_8)(h_n(t)+p_n(t))]p_n(t) + [F_x(t,y_n,z_n)+F_y(t,z_n,z_n)]p_{n+1}(t)\\
&\quad + \int_{t_0}^{t}[Q_{xx}(t,s,\xi_9,x)p_n(s) - Q_{xy}(t,s,y_n,\xi_{10},)h_n(s)\\
&\quad - Q_{yx}(t,s,\xi_{11},y_n)(h_n(s)+p_n(s)) - Q_{yy}(t,s,z_n,\xi_{12})(h_n(s)+p_n(s))]p_n(s)ds\\
&\quad + \int_{t_0}^{t}[Q_x(t,s,y_n,z_n)+Q_y(t,s,z_n,z_n)]p_{n+1}(s)ds\\
&\le Mp_{n+1} + \int_{t_0}^{t} Np_{n+1}(s)ds + L_1p_n^2 + L_2h_n^2 + \int_{t_0}^{t}(L_3p_n^2+L_4h_n^2)ds, \quad t\in J,
\end{aligned}
$$

where $|F_{xx}(t,u,v)| \le A_1$, $|F_{xy}(t,u,v)| \le A_2$, $|F_{yy}(t,u,v)| \le A_3$, $|F_x(t,u,v)| \le M_1$, $|F_y(t,u,v)| \le M_2$ on Ω_1 with $L_1 = A_1+2A_2+\frac{3}{2}A_3$, $L_2 = A_2+\frac{1}{2}A_3$, $M = M_1+M_2$; $|Q_{xx}(t,s,u,v)| \le B_1$, $|Q_{xy}(t,s,u,v)| \le B_2$, $|Q_{yy}(t,s,u,v)| \le B_3$, $|Q_x(t,s,u,v)| \le N_1$, $|Q_y(t,s,u,v)| \le N_2$ on Ω_2 with $L_3 = B_1+2B_2+\frac{3}{2}B_3$, $L_4 = B_2+\frac{1}{2}B_3$, $N = N_1+N_2$.

On the same lines in [3], we can arrive at

$$
\max_{t\in J} p_{n+1} \le \frac{2e^{MT}}{\sqrt{M^2+4N}}[(L_1+L_3T)\max_{t\in J} p_n^2(t) + (L_2+L_4T)\max_{t\in J} h_n^2(t)], \quad t\in J,
$$

that is,

$$
\begin{aligned}
\max_{t\in J}|x-y_{n+1}| \le \tfrac{2e^{MT}}{\sqrt{M^2+4N}}[(L_1+L_3T)\max_{t\in J}|x-y_n|^2(t)\\
+ (L_2+L_4T)\max_{t\in J}|z_n-x|^2(t)], \quad t\in J.
\end{aligned}
$$

Analogous to the discussion of $\{z_n\}$, we have

$$
\max_{t\in J} h_{n+1} \le \frac{2e^{MT}}{\sqrt{M^2+4N}}[(K_1+K_3T)\max_{t\in J} p_n^2(t) + (K_2+K_4T)\max_{t\in J} h_n^2(t)], \quad t\in J,
$$

which means

$$
\begin{aligned}
\max_{t\in J}|z_{n+1}-x| \le \tfrac{2e^{MT}}{\sqrt{M^2+4N}}[(K_1+K_3T)\max_{t\in J}|x-y_n|^2(t)\\
+ (K_2+K_4T)\max_{t\in J}|z_n-x|^2(t)], \quad t\in J,
\end{aligned}
$$

where $K_1 = \frac{1}{2}A_1$, $K_2 = \frac{3}{2}A_1+A_2+A_3$, $K_3 = \frac{1}{2}B_1$, $K_4 = \frac{3}{2}B_1+B_2+B_3$. The proof is complete. □

Similarly, we can give various monotone sequences according to the proceeding of Theorem 3.1, as follows.

Theorem 3.2. *For problem (3.1), suppose that:*

(C_0) *y_0, $z_0 \in C^1(J,R)$ are lower and upper solutions of problem (3.1), respectively, with $y_0(t) \le z_0(t)$ on J;*

(C_1) F, F_x, F_y, F_{xx}, F_{xy}, F_{yx}, $F_{yy} \in C(\Omega_1, R)$ and

$$F_{xx}(t,x,y) \geq 0,\ F_{xy}(t,x,y) \geq 0,\ F_{yy}(t,x,y) \leq 0, \quad \text{for } (t,x,y) \in \Omega_1;$$

(C_2) Q, Q_x, Q_y, Q_{xx}, Q_{xy}, Q_{yx}, $Q_{yy} \in C(\Omega_2, R)$ and

$$Q_{xx}(t,s,x,y) \geq 0,\ Q_{xy}(t,s,x,y) \geq 0,\ Q_{yy}(t,x,y) \leq 0, \quad \text{for } (t,s,x,y) \in \Omega_2;$$

(C_3) $Q_x(t,s,\eta(s),\eta(s)) - Q_y(t,s,\eta(s),\nu(s)) \geq 0$, for $y_0(t) \leq \eta(t) \leq \nu(t) \leq z_0(t)$, $(t,s) \in J \times J$.

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$, such that

$$y'_{n+1} = F(t,y_n,y_n) + [F_x(t,y_n,y_n) + F_y(t,y_n,z_n)](y_{n+1} - y_n) + \int_{t_0}^{t} \{Q(t,s,y_n,y_n) + [Q_x(t,s,y_n,y_n) + Q_y(t,s,y_n,z_n)](y_{n+1} - y_n)\} ds,$$

$$y_{n+1}(t_0) = x_0,$$

$$z'_{n+1} = F(t,z_n,z_n) + [F_x(t,y_n,y_n) + F_y(t,y_n,z_n)](z_{n+1} - z_n) + \int_{t_0}^{t} \{Q(t,s,z_n,z_n) + [Q_x(t,s,y_n,y_n) + Q_y(t,s,y_n,z_n)](z_{n+1} - z_n)\} ds,$$

$$z_{n+1}(t_0) = x_0,$$

which converge uniformly to the unique solution of problem (3.1) on J, and the convergence is quadratic.

Theorem 3.3. *For problem (3.1), assume that (B_0)-(B_1) hold. Then conditions (C_2)-(C_3) imply that there exist monotone sequences $\{y_n\}$, $\{z_n\}$, such that*

$$y'_{n+1} = F(t,y_n,y_n) + [F_x(t,y_n,z_n) + F_y(t,z_n,z_n)](y_{n+1} - y_n) + \int_{t_0}^{t} \{Q(t,s,y_n,y_n) + [Q_x(t,s,y_n,y_n) + Q_y(t,s,y_n,z_n)](y_{n+1} - y_n)\} ds,$$

$$y_{n+1}(t_0) = x_0,$$

$$z'_{n+1} = F(t,z_n,z_n) + [F_x(t,y_n,z_n) + F_y(t,z_n,z_n)](z_{n+1} - z_n) + \int_{t_0}^{t} \{Q(t,s,z_n,z_n) + [Q_x(t,s,y_n,y_n) + Q_y(t,s,y_n,z_n)](z_{n+1} - z_n)\} ds,$$

$$z_{n+1}(t_0) = x_0,$$

converge uniformly to the unique solution of problem (3.1) on J, and the convergence is quadratic.

Theorem 3.4. *For problem (3.1), assume that (C_0)-(C_1) hold. Then* *conditions* (B_2)*-*(B_3) *imply that there exist monotone sequences* $\{y_n\}$, $\{z_n\}$, *such that*

$$\begin{aligned} y'_{n+1} &= F(t,y_n,y_n)+[F_x(t,y_n,y_n)+F_y(t,y_n,z_n)](y_{n+1}-y_n)\\ &\quad+\int_{t_0}^{t}\{Q(t,s,y_n,y_n)+[Q_x(t,s,y_n,z_n)+Q_y(t,s,z_n,z_n)](y_{n+1}-y_n)\}ds,\end{aligned}$$

$$y_{n+1}(t_0)=x_0,$$

$$\begin{aligned} z'_{n+1} &= F(t,z_n,z_n)+[F_x(t,y_n,y_n)+F_y(t,y_n,z_n)](z_{n+1}-z_n)\\ &\quad+\int_{t_0}^{t}\{Q(t,s,z_n,z_n)+[Q_x(t,s,y_n,z_n)+Q_y(t,s,z_n,z_n)](z_{n+1}-z_n)\}ds,\end{aligned}$$

$$z_{n+1}(t_0)=x_0,$$

converge uniformly to the unique solution of problem (3.1) on J*, and the convergence is quadratic.*

Remark 3.1. If $q \in C[J\times J\times R, R]$ is monotone nonincreasing in x for each $(t,s)\in J\times J$, then one can also extend the method of quasilinearization to problem (1.1).

Remark 3.2. Assume that $q=0$, then Theorem 3.1 and Theorem 3.2 would include the results of [9], [5] as special cases.

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