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UNIONS OF ACM SPACE CURVES AND A LINE OR A CONIC

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Abstract: Here we study the postulation of the union of an ACM integral space curve X and a sufficiently general line or conic meeting X at most at 3 general points.

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1. The Statement

Our original motivation was the study of reducible curves $Y \subset \mathbb{P}^3$ containing a complete intersection curve. Some of the lemmas may apply in a more general situation (e.g. for lines through a non-general point). However, to have precise values of the Hilbert functions we used that a line or a conic is a Cartier divisor of any plane containing it, i.e. we need that the ambient projective space has dimension 3. For any reduced projective curve $X \subset \mathbb{P}^r$ let $e(X)$ denote the maximal integer t such that $h^1(X, \mathcal{O}_X(t)) > 0$, i.e. such that $h^2(\mathbb{P}^r, \mathcal{I}_X(t)) > 0$. Here we prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^3$ be an integral ACM curve.*

Set $\eta(X) := h^1(X, \mathcal{O}_X(e(X))$. Fix a general $(P, Q, Q_1) \in X \times X \times X$. Let D_1 be a general line containing P. Set $D_2 := \langle \{P, Q\} \rangle$. Let D_3 (resp. D_4, resp. D_{10}) be a general smooth conic containing P (resp. $\{P, Q\}$, resp. $\{P, Q, Q_1\}$). Let D_5 be a general reducible conic such that $P \in (D_5)_{reg}$. Let D_6 be a general reducible conic such that one of the irreducible components of D_6 contains P

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and the other irreducible component contains Q, but the singular point of D_6 is neither P nor Q. Let D_7 be a general reducible conic such that $P = \mathrm{Sing}(D_7)$. Let D_8 (resp. D_9) be a general line (resp. conic) of $\mathbb{P}^3$. Set $X_i := X \cup D_i$.

(a) $h^1(\mathbb{P}^3, \mathcal{I}_{X_i}(t)) = 0$ for all $i \in \{1, \dots, 10\}$ and all $t \geq e(X) + 1 + \deg(D_i)$.

(b) $h^1(\mathbb{P}^3, \mathcal{I}_{X_1}(e(X) + 2)) = \eta(X) - 1$.

(c) Assume that the general secant line of X is not multisecant. Then

$$h^1(\mathbb{P}^3, \mathcal{I}_{X_2}(e(X) + 2)) = \max\{\eta(X) - 2, 0\}.$$

(d) $h^1(\mathbb{P}^3, \mathcal{I}_{X_8}(e(X) + 2)) = h^1(\mathbb{P}^3, \mathcal{I}_{X_9}(e(X) + 3)) = \eta(X)$.

(e) We have $h^1(\mathbb{P}^3, \mathcal{I}_{X_i}(t)) = h^1(X, \mathcal{O}_X(t + \deg(D_i) + 1) - h^1(X, \mathcal{O}_X(t + \deg(D_i)) - \sharp(D_i \cap S)$ for all integers t such that $2 \leq t \leq e(X) + 1$ and all $i \in \{1, \dots, 9\}$.

The assumption in part (c) is satisfied if X is not strange and in particular if the algebraically closed base field has characteristic zero ([2], Lemma 1.1). For a description of all integral (or smooth) ACM curves in $\mathbb{P}^3$, see [1], p. 453, which contains full references to the mathematicians iwho proved the classification.

2. The Proof

For any closed subscheme Z of any projective space M let $i(Z, M)$ the maximal integer t such that $h^1(M, \mathcal{I}_{Z,M}(t)) > 0$, with the convention $i(Z, M) = -\infty$ if $h^1(M, \mathcal{I}_{Z,M}(t)) = 0$ for all $t \in \mathbb{Z}$. Notice that $i(Z, M) \geq 0$ if Z is zero-dimensional and non-empty. If $M = \mathbb{P}^r$ and Z is a curve X, then set $i(X) := i(X, M)$.

Let $X \subset \mathbb{P}^r$, $r \geq 3$, be an integral ACM curve. Fix a hyperplane $H \subset \mathbb{P}^r$ intersecting transversally X. Set $S := X \cap H$. Since X is ACM, $h^0(H, \mathcal{I}_{S,H}(t)) = h^0(\mathbb{P}^r, \mathcal{I}_X(t)) - h^0(\mathbb{P}^r, \mathcal{I}_X(t-1))$ for all $t \in \mathbb{Z}$, i.e. the Hilbert function $h_{S,H}$ of S in H is the first difference function of the Hilbert function h_X of X in $\mathbb{P}^r$. Hence h_X determines $h_{S,H}$. Since $h^0(\mathbb{P}^r, \mathcal{I}_X) = 0$, the numerical function h_X is uniquely determined by its first difference $h_{S,H}$. Fix $P \in S$. Since H intersects transversally X, $P \in X_{reg}$.

Lemma 1. *Let $X \subset \mathbb{P}^r$, $r \geq 3$, be an integral non-degenerate curve. Fix $P \in X_{reg}$. There is a hyperplane H of $\mathbb{P}^r$ intersecting transversally X and containing P if and only if X is not a strange curve with P as its strange point.*

Proof. Let Φ be the set of all hyperplanes of $\mathbb{P}^r$ containing P. The set Φ is a projective space of dimension $r - 1$. Let H be a general element of Φ. Since

$P \in X_{reg}$ and $\mathrm{Sing}(X)$ is finite, $H \cap \mathrm{Sing}(X) = \emptyset$. For any line $D \subset \mathbb{P}^r$ let D be the set of all hyperplanes of $\mathbb{P}^r$ containing D. If $P \in D$, then $\Phi(D)$ is a hyperplane of Φ. If $P \notin D$, then $\Phi(D) \cap \Phi$ is a codimension 2 linear subspace of Φ. Since $\dim(X_{reg}) = 1$, we see that H contains at least one tangent line to X at some of its smooth points if and only if the general such tangent line contains P. It is easy to check that this condition is equivalent to $P \in T_Q X$ for all $Q \in X_{reg}$, i.e. to the strangeness of X with P as its strange point. □

In the set-up of Lemma 1 no point of X is a strange point of X if either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > \deg(X)$ ([2], Lemma 1.1).

Remark 1. Let $X \subset \mathbb{P}^r$, $r \geq 3$, be an integral non-degenerate curve. Fix $P \in X_{reg}$. Let $E(P, X)$ be the set of all lines $\langle \{P, Q\} \rangle$, for some $Q \in X \backslash \{P\}$. A general $D \in E(P, X)$ contains only two points of X (one of them being P) if and only if the restriction to $X \backslash \{P\}$ of the linear projection $\ell_P : \mathbb{P}^r \backslash \{P\} \to \mathbb{P}^{r-1}$ has separable degree ane. This condition is satisfied for a general $P \in X$ if and only if a general secant line of X is not multisecant. This is always the case if either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > \deg(X)$ (see [2]).

Lemma 2. *Let $A, B \subset \mathbb{P}^r$ be reduced curves without any common irreducible component. Then $h^0(A \cup B, \mathcal{O}_{A \cup B}(t)) = h^0(A, \mathcal{O}_A(t)) + h^0(B, \mathcal{O}_B(t)) - \mathrm{length}(A \cap B)$ for all integers $t \geq \mathrm{length}(A \cap B) - 1$.*

Proof. Look at the Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_{A \cup B}(t) \to \mathcal{O}_A(t) \oplus \mathcal{O}_B(t) \to \mathcal{O}_{A \cap B}(t) \to 0\,. \tag{1}$$

Use that the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(A \cap B, \mathcal{O}_{A \cap B}(t))$ is surjective for all integers $t \geq \mathrm{length}(A \cap B) - 1$. □

Lemma 3. *Let Y, X, A, B curves in $\mathbb{P}^r$ such that $Y \subset X$ and $\dim(A \cap N) \leq 0$.*

(a) $e(A \cup B) \geq \max\{e(A), e(B)\}$ and equality holds if $\mathrm{length}(A \cap B) \leq 1 + \max\{e(A), e(B)\}$.

(b) $e(Y) \leq e(X)$.

(c) Assume X reducible and let $X_1, \dots, X_s$ be the irreducible components of X. Then $e(X) \geq \max\{e(X_i)\}_{1 \leq i \leq s}$ and equality holds if there is an ordering $T_1, \dots, T_s$ of the irreducible components of X such that $\mathrm{length}((T_1 \cup \cdots \cup T_i) \cap T_{i+1}) \leq 1 + \max\{e(X_i)\}_{1 \leq i \leq s}$ for all $i \in \{1, \dots, s-1\}$.

Proof. Part (a) follows from the Mayer-Vietoris exact sequence (1). Part (b) is obvious. Part (c) follows from part (a) using induction on s. □

Lemma 4. *Let $X \subset \mathbb{P}^r$, $r \geq 3$, be an integral non-degenerate curve. Set*

$d := \deg(X)$. Fix $x \in \{d-2, d-1\}$. Let $H \subset \mathbb{P}^r$ be a general hyperplane and set $S := X \cap H$. Fix subsets A, B of S such that $\sharp(A) = \sharp(B) = x$. Then:

(a) $h^i(H, \mathcal{I}_{A,H}(t)) = h^i(H, \mathcal{I}_{B,H}(t))$ for all $i \in \{0, 1\}$ and all $t \in \mathbb{Z}$.

(b) Fix $z \in \mathbb{Z}$. If $h^1(H, \mathcal{I}_S(z)) \geq d - x$, then $h^0(H, \mathcal{I}_{A,H}(z)) = h^0(H, \mathcal{I}_{S,H}(z))$.

Proof. Since the monodromy group of the generic hyperplane section of X is 2)-transitive ([2], Corollary 1.6), and $\sharp(S \backslash A) = \sharp(S \backslash B) \leq 2$, A and B have the same Hilbert function, i.e. part (a) is true. Take z as in part (b). There is $E \subset S$ such that $\sharp(E) = d - h^1((H, \mathcal{I}_S(z))$ and $h^1(H, \mathcal{I}_{E,H}(z)) = 0$, i.e. $h^1(H, \mathcal{I}_{E,H}(z)) = h^0(H, \mathcal{I}_S(z))$. Take any set F such that $E \subseteq F \subset S$ and $\sharp(F) = x$. Apply part (a) to the sets A, F. □

Lemma 5. *Let $X \subset \mathbb{P}^r$ be any reduced curve and $H \subset \mathbb{P}^r$ any hyperplane not containing any irreducible component of X. Set $Z := X \cap H$ (scheme-theoretic intersection). Then $e(X) + 1 \leq i(Z, H) \leq \max\{i(X), e(X) + 1\}$. If X is ACM, then $i(Z, H) = e(X) + 1$.*

Proof. For any integer t there is an exact sequence

$$0 \to \mathcal{I}_X(t-1) \to \mathcal{I}_X(t) \to \mathcal{I}_{Z,H}(t) \to 0\,. \tag{2}$$

Thus $h^1(H, \mathcal{I}_{Z,H}(t)) \leq h^1(\mathbb{P}^r, \mathcal{I}_X(t)) + h^2(\mathbb{P}^r, \mathcal{I}_X(t-1))$. Hence $i(Z, H) \leq \max\{i(X), e(X) + 1\}$. Since the function $h^2(\mathbb{P}^r, \mathcal{I}_X(t))$ is strictly decreasing, until it is 0, i.e. for all $t \leq e(X)$, from (2) we get $i(Z, H) \geq e(X) + 1$. If X is ACM, then $i(X) = -\infty$. Hence the last assertion follows from the first one. □

Lemma 6. *Let $X \subset \mathbb{P}^r$, $r \geq 3$, be an integral and non-degenerate ACM curve. Set $d := \deg(X)$. Let $H \subset \mathbb{P}^r$ be a general hyperplane. Set $S := X \cap H$. Fix $P, Q \in S$ such that $P \neq Q$. Set $S' := S \backslash \{P\}$ and $S'' := S \backslash \{P, Q\}$. We have $h^1(H, \mathcal{I}_{S,H}(t)) = h^1(H, \mathcal{I}_{S',H}(t)) = h^1(H, \mathcal{I}_{S'',H}(t)) = 0$ for all $t \geq e(X) + 2$, $h^1(H, \mathcal{I}_{S,H}(e(X) + 1)) = h^1(X, \mathcal{O}_X(e(X)) > 0$, $h^1(H, \mathcal{I}_{S',H}(e(X) + 1)) = h^1(X, \mathcal{O}_X(e(X)) - 1$ and $h^1(H, \mathcal{I}_{S'',H}(e(X)+1)) = \max\{0, h^1(X, \mathcal{O}_X(e(X)) - 2\}$.*

Proof. All the assertions for S follows from the last sentence Lemma 5. The assertions for S' and S'' at level $e(X) + 1$ follows from Lemma 4 and the corresponding assertions for S. □

Proof of Theorem 1. Lemma 3 gives $h^1(X_i, \mathcal{O}_{X_i}(t)) = h^1(X, \mathcal{O}_X(t))$ (i.e. $h^2(\mathbb{P}^3, \mathcal{I}_{X_i}(t)) = h^2(\mathbb{P}^3, \mathcal{I}_X(t))$ for all integers $t \geq 2$. Let $H \subset \mathbb{P}^3$ be a general plane. Set $S := X \cap H$. Since H and (P, Q, Q_1) are general, without loosing generality we may assume $\{P, Q, Q_1\} \subset H$ and $D_i \subset H$ for all $i \neq \{8, 9\}$ or (in the case $8 \leq i \leq 9$) just $D_i \subset H$. Set $S' := S \backslash \{P\}$ and $S'' := S \backslash \{P, Q\}$. For each $i \in \{1, \ldots, 7\}$ we have $\mathrm{Res}_H(X_i) = X$. Hence for every $t \in \mathbb{Z}$ we have an exact sequence

$$0 \to \mathcal{I}_X(t-1) \to \mathcal{I}_{X_i}(t) \to \mathcal{I}_{S\cup D_i,H}(t) \to 0\,. \qquad (3)$$

Since X is ACM, $h^1(\mathbb{P}^3, \mathcal{I}_X(t-1)) = 0$ for all t. Hence

$$h^1(H, \mathcal{I}_{S\cup D_i,H}(t)) - h^1(X, \mathcal{O}_X(t-1)) \le h^1(\mathbb{P}^3, \mathcal{I}_{X_i}(t)) \le h^1(H, \mathcal{I}_{S\cup D_i,H}(t)).$$

In particular we have $h^1(\mathbb{P}^3, \mathcal{I}_{X_i}(t)) = h^1(H, \mathcal{I}_{S\cup D_i,H}(t))$ if $t \ge e(X)+2$. The generality of each D_i implies $D_i \cap S' = \emptyset$ for all $i\{1,3,5,6\}$ and $D_8 \cap S = D_9 \cap S) = \emptyset$. It is easy to check that the generality of each D_i implies $D_i \cap S'' = \emptyset$ for $i \in \{4,7\}$. If the general secant line of X is not a multisecant line, then $D_2 \cap S'' = \emptyset$. Hence $h^1(H, \mathcal{I}_{S\cup D_1,H}(t)) = h^1(H, \mathcal{I}_{S'}(t-1))$, $h^1(H, \mathcal{I}_{S\cup D_2,H}(t)) = h^1(H, \mathcal{I}_{S''}(t-2))$, $h^1(H, \mathcal{I}_{S\cup D_3,H}(t)) = h^1(H, \mathcal{I}_{S'}(t-2))$, $h^1(H, \mathcal{I}_{S\cup D_4,H}(t)) = h^1(H, \mathcal{I}_{S''}(t-2))$, $h^1(H, \mathcal{I}_{S\cup D_5,H}(t)) = h^1(H, \mathcal{I}_{S'}(t-2))$, $h^1(H, \mathcal{I}_{S\cup D_6,H}(t)) = h^1(H, \mathcal{I}_{S''}(t-2))$ and $h^1(H, \mathcal{I}_{S\cup D_7,H}(t)) = h^1(H, \mathcal{I}_{S'}(t-2))$. Obviously, $\mathcal{I}_{S\cup D_{6+i}}(t)) \cong \mathcal{I}_{S,H}(t-i)$ for $i \in \{1,2\}$. Apply Lemma 6. □

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