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STRONG SOLUTIONS TO A PHASE FIELD MODEL FOR COMPLEX FLUIDS

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Abstract: This paper is concerned with a phase field model for mixture of two viscous incompressible fluids. The coupled system contains a Navier-Stokes equation and a Cahn-Hilliard equation. Globality and regularity of strong solutions are studied in three dimensions.

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1. Introduction

In this paper, we consider a coupled Navier-Stokes/Cahn-Hilliard equations

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu \operatorname{div} D(\mathbf{u}) = -\lambda \nabla\cdot(\nabla\phi\otimes\nabla\phi) \quad \text{in}\, \Omega\times(0,+\infty), \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in}\, \Omega\times(0,+\infty), \tag{2}$$

$$\phi_t + (\mathbf{u}\cdot\nabla)\phi = -\gamma\Delta(\Delta\phi - f(\phi)) \quad \text{in}\, \Omega\times(0,+\infty). \tag{3}$$

with the initial conditions

$$(\mathbf{u},\phi)|_{t=0} = (\mathbf{u}_0,\phi_0) \quad \text{in}\, \Omega, \tag{4}$$

and the non-slip boundary condition for $\mathbf{u}$ and no-flux condition for ϕ

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$$\mathbf{u} = 0 \quad \text{on}\, \partial\Omega \times (0, \infty), \tag{5}$$

$$\frac{\partial \phi}{\partial n} = \frac{\partial(\Delta\phi)}{\partial n} = 0 \quad \text{on}\, \partial\Omega \times (0, \infty)\,, \tag{6}$$

where $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$.

Equation (1) is the momentum equation. $\mathbf{u}$ denotes the velocity field of the mixture. p is the pressure. $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the stretching tensor. ν is the kinematic viscosity constant. $\nu D(\mathbf{u}) - pI$ is the fluid stress tensor. ϕ represents the "phase" of the fluids. λ is a positive parameter of surface tension (see [10]). This equation is of Navier-Stokes type with an induced elastic stress $\nabla\phi \otimes \nabla\phi$ which is due to the mixing of two different fluids. $(\nabla\phi \otimes \nabla\phi)_{ij} = \frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}$. Equation (2) shows the incompressibility of both fluids. Equation (3) is a phase-field advection-diffusion equation. It is a Cahn-Hilliard equation plus advection. γ is the elastic relaxation time. $f(\phi) = F'(\phi)$ where $F(\phi)$ is the bulk part of mixing energy density. The left-hand side of (3) shows the transport property of the phase function while the right hand side gives the dissipative mechanism. The system (1)-(3) is considered in $\Omega \times (0, +\infty)$, where $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded domain with smooth boundary.

The above system is a hydrodynamical model for mixture of two incompressible viscous fluids. In the physics literature, this model have been used extensively to study phase transition and other critical phenomena (see [6, 7]) of complex fluids. For simplicity, we consider the two fluids with the same density (which is taken to be 1) and the same viscosity constant.

There are some numerical calculations related to this coupled Navier-Stokes and Cahn-Hilliard equations (see [5, 9, 11]). But few papers provided theoretical results. Boyer [2] studied this system in a periodic channel in both two and three dimensions. The coefficients ν and λ in (1)-(3) were replaced by functions with certain restrictions. The existence of global weak solutions was proved. It was also proved the existence of a unique strong solution which is global in two dimensions but local in three dimensions (it exsits only on $[0, T_0]$ with T_0 is a small constant).

In this paper, we suppose $F(\phi)$ satisfies the same assumptions as those in Boyer [2]:

(1) F is of C^3 class, and $F(s) \geq 0$;

(2) $|F'(s)| \leq C(1 + |s|^p)$, , $|F''(s)| \leq C(1 + |s|^{p-1})$, $\forall s \in \mathbb{R}$, where $C \geq 0$, $p \in [1, \infty)$ for $n = 2$ and $p \in [1, 3]$ for $n = 3$;

(3) there exists a constant $C > 0$ such that $F''(s) \geq -C, \forall s \in \mathbb{R}$;

(4) $|F'''(s)| \leq C(1 + |s|^q)$, $\forall s \in \mathbb{R}$, where $C \geq 0$, $q < +\infty$ for $n = 2$ and

$q < 3$ for $n = 3$.

It is easy to check that the famous double-well energy density $F(\phi) = \frac{1}{4}(\phi^2 - 1)^2$ satisfies all the assumptions above. With this smooth $F(\phi)$, we prove that the strong solution is global in three dimensions provided that ν is larger than some constant depending only on initial data and $|\Omega|$. Moreover, we show that the strong solution is a classical solution if initial data have enough regularity.

In a quite recent paper [1], Abels also studied system (1)-(6) in a bounded domain, but the smooth energy density $F(\phi)$ is replaced with a class of singular free energy density. In addition, ν is a function depending on ϕ. The global existence of weak solutions and certain regularity of weak solutions for large times were proved in two and three space dimensions. Moreover, it was obtained that a unique strong solution exists globally in time in two dimensions but locally in three dimensions.

This paper is organized as follows. In Section 2, we introduce mathematical preliminaries and state our main results. Section 3 is devoted to prove the global existence of a strong solution under some condition on the viscosity constant. Finally, in Section 4, we show that a strong solution is in fact a classical solution provided that the initial data have enough regularity.

2. Preliminaries and Main Results

Let Q_T denote $\Omega \times (0,T)$. $\|\cdot\|$ is the norm of $L^2(\Omega)$ and $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$. For a functional space X, $\|\cdot\|_X$ denotes the norm on it. Next we introduce some functional spaces which will be used.

$$
\begin{aligned}
&\mathcal{V} = \{\mathbf{u} \in (C_0^\infty(\Omega))^n; \mathrm{div}\,\mathbf{u} = 0\},\\
&H = \text{the closure of}\,\mathcal{V}\,\text{in}\,L^2(\Omega),\\
&V_s = \text{the closure of}\,\mathcal{V}\,\text{in}\,H^s(\Omega), s \in \mathbb{N}^+, \text{specially, denote}\,V \equiv V_1,\\
&\mathcal{W} = \left\{\phi \in C^\infty(\Omega),\, \frac{\partial \phi}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial(\Delta\phi)}{\partial n}\bigg|_{\partial\Omega} = 0\right\},\\
&\Phi_s = \text{the closure of}\,\mathcal{W}\,\text{in}\,H^s(\Omega).
\end{aligned}
$$

Set $m(\phi) = \frac{1}{|\Omega|}\int_\Omega \phi(x,t)dx$. Integrating (3) over Ω, we deduce

$$
m(\phi(t)) = m(\phi_0). \tag{7}
$$

We introduce the Stokes operator: $S\mathbf{u} = -\Delta\mathbf{u} + \nabla\pi \in H$, $\forall \mathbf{u} \in V_2$ (see [12, 2]). The Stokes operator has the following property:

Lemma 1. (see [12]) *There exists a constant* $C > 0$ *such that for any* $\mathbf{u} \in V_2$, *we have*

$$\|\mathbf{u}\|_{V_2} \leq C\|S\mathbf{u}\|, \quad \|\pi\|_{H^1(\Omega)\backslash\mathbb{R}} \leq C\|S\mathbf{u}\|, \quad \|\pi\|_{L^2(\Omega)\backslash\mathbb{R}} \leq C\|\mathbf{u}\|_V.$$

We also give the notion of weak solutions, strong solutions and classical solutions.

Definition 1. $(\mathbf{u}, \phi)$ is called a weak solution to system (1)-(6) on Q_T $(0 < T < +\infty)$ provided that

$$\mathbf{u} \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{8}$$

$$\phi \in L^2(0,T;\Phi_3) \cap L^\infty(0,T;\Phi_1) \tag{9}$$

and for any $\mathbf{v} \in V$,

$$-\int_0^T \big(\mathbf{u}(t), \mathbf{v}\omega'(t)\big)\, dt + \frac{\nu}{2}\int_0^T (\nabla\mathbf{u}(t), \nabla\mathbf{v}\omega(t))\, dt + \int_0^T ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}\omega(t))\, dt$$
$$= (\mathbf{u}_0, \mathbf{v})\omega(0) - \lambda\int_0^T (\nabla\cdot(\nabla\phi\otimes\nabla\phi), \mathbf{v})\omega(t) dt,$$

and for any $\psi \in \Phi_1$,

$$-\int_0^T (\phi(t), \psi)\omega'(t) dt + \int_0^T ((\mathbf{u}\cdot\nabla)\phi, \psi\omega(t))\, dt$$
$$= (\phi_0, \psi\omega(0)) + \gamma\int_0^T (\nabla(\Delta\phi - f(\phi)), \nabla\psi\omega(t))\, dt,$$

here $\omega(t)$ is any continuous differentiable function on $[0,T]$ with $\omega(T) = 0$.

Definition 2. We say that $(\mathbf{u}, \phi)$ is a strong solution to (1)-(6) on Q_T, if $(\mathbf{u}, \phi)$ is a weak solution on Q_T, and

$$\mathbf{u} \in L^2(0,T;V_2) \cap L^\infty(0,T;V),$$
$$\phi \in L^2(0,T;\Phi_4) \cap L^\infty(0,T;\Phi_2).$$

Definition 3. A pair of functions $(\mathbf{u}, \phi)$ is called a classical global solution to (1)-(6) on Q_T provided that $\mathbf{u} \in C^{2,1}(Q_T) \cap C(\overline{Q_T}), \phi \in C^{4,1}(Q_T) \cap C^{3,0}(\overline{Q_T}), \forall T > 0$, and it satisfies (1)-(6) in the classical sense.

We can follow exactly the same way as in Boyer [2] to get the following two theorems. Although our boundary conditions are slightly different from that in Boyer [2], it does not cause any mathematical difficulty.

Theorem 1. *Assume the initial data* $\mathbf{u}_0 \in H$ *and* $\phi_0 \in \Phi_1$. *Then there exists a global weak solution* $(\mathbf{u}, \phi)$ *to* (1)-(6) *in the sense of Definition 1.*

Theorem 2. *Assume* $\mathbf{u}_0 \in V$ *and* $\phi_0 \in \Phi_2$. *Then*

— *in the 2D case, there exists a unique global strong solution to* (1)-(6).

— *in the 3D case, there exists some* $T_0 > 0$ *depending on* $\mathbf{u}_0$, ϕ_0 *and* $|\Omega|$ *such that there is a unique strong solution to* (1)-(6) *on* Q_{T_0}.

Now we state our main theorem.

Theorem 3. — *In the 2D case, if* $\mathbf{u}_0 \in V_5, \phi_0 \in \Phi_{10}$, *then the strong solution* $(\mathbf{u}, \phi)$ *is a global classical solution.*

— *In the 3D case, if* $\mathbf{u}_0 \in V, \phi_0 \in \Phi_2$ *and* ν *is larger than some constant which depends only on* $\mathbf{u}_0$, ϕ_0 *and* $|\Omega|$, *then the strong solution* $(\mathbf{u}, \phi)$ *is global. If in addition,* $\mathbf{u}_0 \in V_5, \phi_0 \in \Phi_{10}$, *then* $(\mathbf{u}, \phi)$ *is a global classical solution.*

For the 2D case, the key-point to prove the global existence of strong solutions is to use the uniform Gronwall Lemma [2]. However, this method fails in the 3D case because the order of nonlinearity is too high. More precisely, we have a common problem that $\frac{d}{dt}y(t) \leq Cy^{\alpha}(t)$ with $\alpha > 2$ and $\forall t > 0$, $\int_t^{t+r} y(s)ds \leq C$ where $C's$ and r are constants independent of t. It is difficult to get a uniform bound for $y(t)$ in $[0, \infty)$. Here, in order to get the global existence of strong solutions, we first deduce a prior estimate, using dedicate energy estimates. Then we choose a large ν to guarantee that the maximal existence time interval $[0, T)$ is in fact $[0, +\infty)$. The arguments are in a usual way and can apply to similar problems. With the global existence of strong solutions, we shall discuss in the future the asymptotic behavior of system (1)-(6) which physicists are interests in.

Finally, if the initial data have enough regularity, we show that a strong solution is a classical solution. We only give the detailed proof in the 3D case. The 2D case can be proved similarly, using corresponding interpolation inequalities in 2D.

3. Global Strong Solution

Throughout this section, $C's$ are constants depending on ϕ_0, $\mathbf{u}_0$ and $|\Omega|$. Firstly, we give a priori estimates for the solution to (1)-(6).

Lemma 2. *If* $(\mathbf{u}, \phi)$ *is a weak solution to* (1)-(6), *then for any* $t_0 \geq 0$ *and* $\tau > 0$,

$$\sup_{t_0 \leq t \leq t_0+\tau} \left(\|\mathbf{u}(t)\|^2 + \lambda\|\nabla\phi(t)\|^2 + \|\phi(t)\|^2\right) \leq C\left(\|\mathbf{u}_0\|, \|\phi_0\|_{H^1(\Omega)}, |\Omega|\right), \quad (10)$$

$$\int_{t_0}^{t_0+\tau} \left(\nu\|\nabla \mathbf{u}\|^2 + 2\gamma\lambda\|\nabla(\Delta\phi - f(\phi))\|^2\right) dt \le C\left(\|\mathbf{u}_0\|, \|\phi_0\|_{H^1(\Omega)}, |\Omega|\right), \quad (11)$$

$$\int_{t_0}^{t_0+\tau} \|\Delta\phi\|^2 dt \le C\left(\|\mathbf{u}_0\|, \|\phi_0\|_{H^1(\Omega)}, |\Omega|, \tau\right). \quad (12)$$

Proof. We multiply (1) by $\mathbf{u}$ and multiply (3) by $\lambda(-\Delta\phi + f(\phi))$. Adding the two resultants and integrating over Ω yields

$$\frac{d}{dt}\int_\Omega \left[\frac{|\mathbf{u}|^2}{2} + \frac{\lambda}{2}|\nabla\phi|^2 + \lambda F(\phi)\right] dx$$
$$= -\left[\frac{\nu}{2}\|\nabla\mathbf{u}\|^2 + \gamma\lambda\|\nabla(\Delta\phi - f(\phi))\|^2\right] \le 0. \quad (13)$$

Obviously, it follows that for any $t_0 \ge 0$,

$$\|\mathbf{u}(\cdot, t_0)\|^2 + \lambda\|\nabla\phi(\cdot, t_0)\|^2 + 2\lambda\int_\Omega F(\phi(\cdot, t_0))dx$$
$$\le \|\mathbf{u}_0\|^2 + \lambda\|\nabla\phi_0\|^2 + 2\lambda\int_\Omega F(\phi_0)dx. \quad (14)$$

Noting $|F(\phi)| \le C(1 + |\phi|^4)$ and the Sobolev embeddings $H^1(\Omega) \subset L^4(\Omega)$, we obtain

$$\int_\Omega F(\phi_0)dx \le C\|\phi_0\|^4_{H^1(\Omega)} + C|\Omega|. \quad (15)$$

Using Poincaré inequality and (7) yields

$$\|\phi(\cdot, t)\|$$
$$\le C\left(\|\nabla\phi(\cdot, t)\| + \left|\int_\Omega \phi(\cdot, t)dx\right|\right) \le C\left(\|\nabla\phi(\cdot, t)\| + \left|\int_\Omega \phi_0 dx\right|\right). \quad (16)$$

Integrating (13) with respect to t from t_0 to $t_0 + \tau$, we have

$$\sup_{t_0 \le t \le t_0+\tau}\left(\|\mathbf{u}(\cdot, t)\|^2 + \lambda\|\nabla\phi(\cdot, t)\|^2 + 2\lambda\int_\Omega F(\phi)dx\right)$$
$$\le \|\mathbf{u}(t_0)\|^2 + \lambda\|\nabla\phi(t_0)\|^2 + 2\lambda\int_\Omega F(\phi(t_0))dx.$$
$$\int_{t_0}^{t_0+\tau}\left(\nu\|\nabla\mathbf{u}\|^2 + 2\gamma\lambda\|\nabla(\Delta\phi - f(\phi))\|^2\right) dt$$
$$\le \quad \|\mathbf{u}(t_0)\|^2 + \lambda\|\nabla\phi(t_0)\|^2 + 2\lambda\int_\Omega F(\phi(t_0))dx.$$

Combing two equations above with (14), (15), (16), and noting $F \ge 0$, we obtain (10) and (11).

Next we multiply (3) by ϕ and integrate over Ω. Then we have

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2+\gamma\|\Delta\phi\|^2=\gamma\int_\Omega f(\phi)\Delta\phi dx,$$

the right-hand terms can be estimated as follows:

$$\begin{aligned}\left|\gamma\int_\Omega f(\phi)\Delta\phi dx\right| &\le \gamma\int_\Omega \left(1+|\phi|^3\right)|\Delta\phi|dx\\ &\le C\|\phi\|^3_{L^6(\Omega)}\|\Delta\phi\|+C\|\Delta\phi\|\le\frac{\gamma}{2}\|\Delta\phi\|^2+C\|\phi\|^6_{L^6(\Omega)}+C,\\ &\le\frac{\gamma}{2}\|\Delta\phi\|^2+C\|\phi\|^6_{H^1(\Omega)}+C.\end{aligned}$$

Then

$$\frac{d}{dt}\|\phi\|^2+\gamma\|\Delta\phi\|^2\le C\|\phi\|^6_{H^1(\Omega)}+C. \tag{17}$$

Integration of (17) from t_0 to $t_0+\tau$ and utilization of (10) yields (12). □

Next, we give some lemmas which will be used in the proof of Theorem 3.

Lemma 3. (see [13]) *Assume $\Omega\subset\mathbb{R}^n$ is an open bounded domain with a smooth boundary. For every $\beta>0$, $\{\|\Delta\phi\|^2+\beta\|\phi\|^2\}^{\frac{1}{2}}$ and $\{\|\Delta\phi\|^2+\beta(\int_\Omega\phi(x)dx)^2\}^{\frac{1}{2}}$ are norms on $W=\{\varphi\in H^2(\Omega),\frac{\partial\varphi}{\partial n}|_{\partial\Omega}=0\}$ which are equivalent to the H^2-norm. Similarly, $\{\|\Delta^2\phi\|^2+\beta\|\phi\|^2\}^{\frac{1}{2}}$ and $\{\|\Delta^2\phi\|^2+\beta(\int_\Omega\phi(x)dx)^2\}^{\frac{1}{2}}$ are norms on Φ_4 which are equivalent to the H^4-norm.*

Lemma 4. (see [14]) *If $\Omega\subset\mathbb{R}^n$ is a bounded domain with a smooth boundary, $v\in H^1_0(\Omega)$, then*

$$\|v\|_{L^4(\Omega)}\le C\|v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}}\quad\textit{in the case } n=2,$$
$$\|v\|_{L^4(\Omega)}\le C\|v\|^{\frac{1}{4}}\|\nabla v\|^{\frac{3}{4}}\quad\textit{in the case } n=3.$$

Lemma 5. (Agmon's Inequalities, see [13]) *Assume that $\Omega\subset\mathbb{R}^n$ is of class C^n, there exists a constant C depending only on Ω such that*

$$\|v\|_{L^\infty}\le C\|v\|^{\frac{1}{2}}_{H^{(n/2)-1}(\Omega)}\|v\|^{\frac{1}{2}}_{H^{(n/2)+1}(\Omega)},\ \forall v\in H^{(n/2)+1}(\Omega),\textit{ if } n \textit{ is even};$$
$$\|v\|_{L^\infty}\le C\|v\|^{\frac{1}{2}}_{H^{(n-1)/2}(\Omega)}\|v\|^{\frac{1}{2}}_{H^{(n+1)/2}(\Omega)},\ \forall v\in H^{(n+1)/2}(\Omega),\textit{ if } n \textit{ is odd}.$$

Remark 1. With Lemma 5, Poincaré's inequality, Lemma 3 and basic interpolation theory in [8]: $H^s=[H^{p_1},H^{p_2}]_\theta$ where $s=p_1(1-\theta)+p_2\theta$ and $\|v\|_{H^s}\le c(\theta)\|v\|^{1-\theta}_{H^{p_1}}\|v\|^{\theta}_{H^{p_2}}$, specially for $p_1=1,p_2=4,\theta=\frac{1}{3}$ and $\theta=\frac{2}{3}$, we have

$$\|\nabla\phi\|_{L^\infty(\Omega)}=\|\nabla(\phi-m(\phi))\|_{L^\infty(\Omega)}\le C\|\phi-m(\phi)\|^{\frac{1}{2}}_{H^2(\Omega)}\|\phi-m(\phi)\|^{\frac{1}{2}}_{H^3(\Omega)}\le$$

$$C\left[\|\phi-m(\phi)\|_{H^1(\Omega)}^{\frac{2}{3}}\|\phi-m(\phi)\|_{H^4(\Omega)}^{\frac{1}{3}}\right]^{\frac{1}{2}}\left[\|\phi-m(\phi)\|_{H^1(\Omega)}^{\frac{1}{3}}\|\phi-m(\phi)\|_{H^4(\Omega)}^{\frac{2}{3}}\right]^{\frac{1}{2}}$$
$$\leq C\|\phi-m(\phi)\|_{H^1(\Omega)}^{\frac{1}{2}}\|\phi-m(\phi)\|_{H^4(\Omega)}^{\frac{1}{2}}\leq C\|\nabla\phi\|^{\frac{1}{2}}\|\Delta^2\phi\|^{\frac{1}{2}}. \quad (18)$$

Lemma 6. (The Uniform Gronwall Lemma, see [13]) *Let g, h, y be three positive locally integrable functions on $[t_0,+\infty]$ such that y' is locally integrable on $[t_0,+\infty]$, and which satisfy*

$$\frac{dy}{dt}\leq gy+h \quad \text{for} \quad t\geq t_0,$$
$$\int_t^{t+r} g(s)ds\leq a_1,\quad \int_t^{t+r} h(s)ds\leq a_2,\quad \int_t^{t+r} y(s)ds\leq a_3 \quad \text{for} \quad t\geq t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r)\leq\left(\frac{a_3}{r}+a_2\right)\exp(a_1),\quad \forall t\geq t_0.$$

Lemma 7. *If $(\mathbf{u},\phi)$ is a strong solution on Q_T, then $\mathbf{u}_t\in L^2(0,T;L^2(\Omega))$, $\phi_t\in L^2(0,T;L^2(\Omega))$. Moreover, $\mathbf{u}\in C(0,T;V)$ and $\phi\in C(0,T;\Phi_2)$.*

Proof. Multiplying (3) by ϕ_t and integrating on Ω,

$$\frac{\gamma}{2}\frac{d}{dt}\|\Delta\phi\|^2+\|\phi_t\|^2=-((\mathbf{u}\cdot\nabla)\phi,\phi_t)-\gamma(f'(\phi)\nabla\phi,\nabla\phi_t).$$

Let us estimate the right hand terms:

$$|((\mathbf{u}\cdot\nabla)\phi,\phi_t)|\leq\|\phi_t\|\|\mathbf{u}\|_{L^4(\Omega)}\|\nabla\phi\|_{L^4(\Omega)}\leq C\|\phi_t\|\|\mathbf{u}\|_{H^1(\Omega)}\|\nabla\phi\|_{H^1(\Omega)}$$
$$\leq C\|\phi_t\|\|\nabla\mathbf{u}\|\|\Delta\phi\|\leq\varepsilon\|\phi_t\|^2+C(\varepsilon)\|\nabla\mathbf{u}\|^2\|\Delta\phi\|^2.$$

Using assumption (3),

$$-\gamma(f'(\phi)\nabla\phi,\nabla\phi_t)\leq\frac{\gamma C}{2}\frac{d}{dt}\|\nabla\phi\|^2.$$

Taking ε small enough, we conclude that

$$\gamma\frac{d}{dt}\left(\|\Delta\phi\|^2-C\|\nabla\phi\|^2\right)+\|\phi_t\|^2\leq C\|\nabla\mathbf{u}\|^2\|\Delta\phi\|^2. \quad (19)$$

Since $\mathbf{u}\in L^\infty(0,T;V)\cap L^2(0,T;V_2)$ and $\phi\in L^\infty(0,T;\Phi_2)\cap L^2(0,T;\Phi_4)$, integration of (19) gives $\phi_t\in L^2(0,T;L^2(\Omega))$. Similarly, we can have $\mathbf{u}_t\in L^2(0,T;L^2(\Omega))$.

Since $\mathbf{u}\in L^2(0,T;V_2)$, $\mathbf{u}_t\in L^2(0,T;L^2(\Omega))$, $\phi\in L^2(0,T;\Phi_4)$, and $\phi_t\in L^2(0,T;L^2(\Omega))$ by interpolation, $\mathbf{u}\in C(0,T;V)$ and $\phi\in C(0,T;\Phi_2)$. □

Now we begin to prove the globality of strong solutions in Theorem 3 in the 3D case.

It has been proved in Theorem 2 that if $\mathbf{u}_0 \in V$, $\phi_0 \in \Phi_2$, then the strong solution to system (1)-(6) in three dimensions exists locally. To show the global existence of strong solution, it suffices to prove

$$\sup_{0\le t\le T} \|\mathbf{u}(t)\|_V < +\infty, \quad \sup_{0\le t\le T} \|\phi(t)\|_{\Phi_2} < +\infty \tag{20}$$

for any $T > 0$. With Poincaré's inequality, Lemma 3 and (7), we see that (20) is equivalent to

$$\sup_{0\le t\le T} \|\nabla\mathbf{u}(t)\| < +\infty, \quad \sup_{0\le t\le T} \|\Delta\phi(t)\| < +\infty. \tag{21}$$

Set

$$Y(t) = \|\nabla\mathbf{u}(t)\|^2 + \|\Delta\phi(t)\|^2 + 1,$$

then $Y(t)$ satisfies the following a priori estimate.

Lemma 8. *Assume $(\mathbf{u}, \phi)$ is a strong solution to system (1)-(6) on Q_T, then*

$$\frac{d}{dt}Y(t) + \frac{\gamma}{2}\|\Delta^2\phi\|^2 + (\nu - C_1 Y(t) - C_2)\|S\mathbf{u}\|^2 \le C_3 Y(t), \text{ for any } t \in (0,T), \tag{22}$$

where C_1, C_2, C_3 are constants depending only on $\mathbf{u}_0$, ϕ_0 and $|\Omega|$.

Proof. We multiply (1) by $S\mathbf{u}$ and multiply (3) by $\Delta^2\phi$. Addition of the two results and integration over Ω yields

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{u}\|^2 + \|\Delta\phi\|^2) + \frac{\nu}{2}\|S\mathbf{u}\|^2 + \gamma\|\Delta^2\phi\|^2$$
$$= -((\mathbf{u}\cdot\nabla)\mathbf{u}, S\mathbf{u}) - \lambda(\Delta\phi\nabla\phi, S\mathbf{u}) - ((\mathbf{u}\cdot\nabla)\phi, \Delta^2\phi) + \gamma(\Delta f(\phi), \Delta^2\phi). \tag{23}$$

By Lemma 1, we have

$$\|\Delta\mathbf{u}\| = \|-S\mathbf{u} + \nabla\pi\| \le C\|S\mathbf{u}\|. \tag{24}$$

Now we estimate the right hand side of (23), using Lemma 4, (24) and (18).

$$|((\mathbf{u}\cdot\nabla)\mathbf{u}, S\mathbf{u})| \le \|\mathbf{u}\|_{L^4(\Omega)}\|\nabla\mathbf{u}\|_{L^4(\Omega)}\|S\mathbf{u}\|$$
$$\le C\|\mathbf{u}\|^{\frac{1}{4}}\|\nabla\mathbf{u}\|^{\frac{3}{4}}\|\nabla\mathbf{u}\|^{\frac{1}{4}}\|S\mathbf{u}\|^{\frac{7}{4}} = C\left(\|\nabla\mathbf{u}\|^{\frac{3}{4}}\|S\mathbf{u}\|^{\frac{7}{4}}\right)\left(\|\mathbf{u}\|^{\frac{1}{4}}\|\nabla\mathbf{u}\|^{\frac{1}{4}}\right)$$
$$\le C\|\nabla\mathbf{u}\|^{\frac{6}{7}}\|S\mathbf{u}\|^2 + C\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 \le C\left(\|\nabla\mathbf{u}\|^2 + C\right)\|S\mathbf{u}\|^2 + C\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2,$$

$$|(\Delta\phi\nabla\phi, S\mathbf{u})| \le \|\Delta\phi\|\|\nabla\phi\|_{L^\infty(\Omega)}\|S\mathbf{u}\| \le C\|\Delta\phi\|\|\nabla\phi\|^{\frac{1}{2}}\|\Delta^2\phi\|^{\frac{1}{2}}\|S\mathbf{u}\|$$
$$\le \varepsilon\|\Delta^2\phi\|^2 + C(\varepsilon)\|\nabla\phi\|^2 + C(\varepsilon)\|\Delta\phi\|^2\|S\mathbf{u}\|^2,$$

$$|((\mathbf{u}\cdot\nabla)\phi, \Delta^2\phi)| \le \|\mathbf{u}\|\|\nabla\phi\|_{L^\infty(\Omega)}\|\Delta^2\phi\| \le C\|\mathbf{u}\|\|\nabla\phi\|^{\frac{1}{2}}\|\Delta^2\phi\|^{\frac{3}{2}}$$

$$\leq \varepsilon\|\Delta^2\phi\|^2 + C(\varepsilon)\|\mathbf{u}\|^4\|\nabla\phi\|^2.$$

From assumption (2) and assumption (4), we have

$$\begin{aligned}\|f'(\phi)\|_{L^\infty(\Omega)} &\leq C\left(1+\|\phi-m(\phi_0)\|^2_{L^\infty(\Omega)}\right),\\ \|f''(\phi)\|_{L^\infty(\Omega)} &\leq C\left(1+\|\phi-m(\phi_0)\|^q_{L^\infty(\Omega)}\right),\end{aligned}$$

where $q<3$. Thus

$$\begin{aligned}\|\Delta f(\phi)\| &= \|f'(\phi)\Delta\phi + f''(\phi)\nabla\phi\cdot\nabla\phi\| \leq \|f'(\phi)\|_{L^\infty(\Omega)}\|\Delta\phi\|\\ &+ \|f''(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^4(\Omega)} \leq C\left(1+\|\phi-m(\phi_0)\|^2_{L^\infty(\Omega)}\right)\|\Delta\phi\|\\ &+ C\left(1+\|\phi-m(\phi_0)\|^q_{L^\infty(\Omega)}\right)\|\nabla\phi\|^2_{L^4(\Omega)}. \qquad (25)\end{aligned}$$

We need estimate the righthand side terms of (25). By Lemma 5, the interpolation $H^2=[H^1,H^4]_{\frac{1}{3}}$, Poincaré's inequality and Lemma 3, we obtain

$$\begin{aligned}\|\phi-m(\phi_0)\|_{L^\infty(\Omega)} &\leq C\|\phi-m(\phi_0)\|^{\frac{1}{2}}_{H^1(\Omega)}\|\phi-m(\phi_0)\|^{\frac{1}{2}}_{H^2(\Omega)}\\ &\leq C\|\phi-m(\phi_0)\|^{\frac{5}{6}}_{H^1(\Omega)}\|\phi-m(\phi_0)\|^{\frac{1}{6}}_{H^4(\Omega)} \leq C\|\nabla\phi\|^{\frac{5}{6}}\|\Delta^2\phi\|^{\frac{1}{6}}.\end{aligned}$$

Because $H^{\frac{3}{4}}\subset L^4$, $H^{\frac{3}{4}}=[L^2,H^3]_{\frac{1}{4}}$ and $H^2=[H^1,H^4]_{\frac{1}{3}}$, we have

$$\begin{aligned}\|\nabla\phi\|_{L^4(\Omega)} &\leq C\|\nabla\phi\|^{\frac{3}{4}}\|\nabla\phi\|^{\frac{1}{4}}_{H^3(\Omega)} = C\|\nabla\phi\|^{\frac{3}{4}}\|\nabla(\phi-m(\phi_0))\|^{\frac{1}{4}}_{H^3(\Omega)}\\ &\leq C\|\nabla\phi\|^{\frac{3}{4}}\|\phi-m(\phi_0)\|^{\frac{1}{4}}_{H^4(\Omega)} \leq C\|\nabla\phi\|^{\frac{3}{4}}\|\Delta^2\phi\|^{\frac{1}{4}},\end{aligned}$$

$$\begin{aligned}\|\Delta\phi\| &\leq C\|\phi-m(\phi_0)\|_{H^2(\Omega)}\\ &\leq C\|\phi-m(\phi_0)\|^{\frac{2}{3}}_{H^1(\Omega)}\|\phi-m(\phi_0)\|^{\frac{1}{3}}_{H^4(\Omega)} \leq C\|\nabla\phi\|^{\frac{2}{3}}\|\Delta^2\phi\|^{\frac{1}{3}}.\end{aligned}$$

Thus it follows from (25) that

$$\begin{aligned}\|\Delta f(\phi)\| &\leq C\left(1+\|\nabla\phi\|^{\frac{5}{3}}\|\Delta^2\phi\|^{\frac{1}{3}}\right)\|\nabla\phi\|^{\frac{2}{3}}\|\Delta^2\phi\|^{\frac{1}{3}}\\ &+ C\left(1+C\|\nabla\phi\|^{\frac{5q}{6}}\|\Delta^2\phi\|^{\frac{q}{6}}\right)\|\nabla\phi\|^{\frac{3}{2}}\|\Delta^2\phi\|^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}\left|(\Delta f(\phi),\Delta^2\phi)\right| &\leq C\|\Delta f(\phi)\|\|\Delta^2\phi\|\\ &\leq \varepsilon\|\Delta^2\phi\|^2 + C(\varepsilon)\left(\|\nabla\phi\|^2+\|\nabla\phi\|^6+\|\nabla\phi\|^{14}+\|\nabla\phi\|^\alpha\right),\end{aligned}$$

where $\alpha>0$. Finally, using the estimates above and taking ε small enough, we conclude from (23) that

$$\frac{d}{dt}\left(\|\nabla\mathbf{u}\|^2+\|\Delta\phi\|^2\right)+\frac{\gamma}{2}\|\Delta^2\phi\|^2+\left(\nu-C\|\nabla\mathbf{u}\|^2-C\|\Delta\phi\|^2-C\right)\|S\mathbf{u}\|^2$$

$$\leq C\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2 + C\|\Delta\phi\|^2 + C\|\mathbf{u}\|^4\|\nabla\phi\|^2$$
$$+ C\left(\|\nabla\phi\|^2 + \|\nabla\phi\|^6 + \|\nabla\phi\|^{14} + \|\nabla\phi\|^\alpha\right) \leq C\left(\|\nabla\mathbf{u}\|^2 + \|\Delta\phi\|^2\right) + C.$$

Since $Y(t) = \|\nabla\mathbf{u}(t)\|^2 + \|\Delta\phi(t)\|^2 + 1$, we get (22). □

Assume $[0, T)$ is the maximal existence time-interval of the strong solution to system (1)-(6). Suppose $T < +\infty$, if we can prove $Y(T) < +\infty$, then by Theorem 2, there exists some $\delta > 0$ such that the strong solution can be extended to $[0, T+\delta)$. This contradicts to the assumption of T. Thus $T = +\infty$. It remains to prove that

$$Y(T) < +\infty, \ \forall\, T < +\infty. \tag{26}$$

Since $(\mathbf{u}, \phi)$ is also a weak solution to system (1)-(6), we have from Lemma 2 that

$$\int_t^{t+1} Y(t)dt \leq M, \text{ for any } t \geq 0\,, \tag{27}$$

where M is a constant depending on $\mathbf{u}_0$, ϕ_0, $|\Omega|$. In particular, $\int_{T-1}^{T} Y(t)dt \leq M$ (if $T < 1$, we replace $T-1$ with 0). This implies that there exists $\bar{t} \in (T-1, T)$ such that $Y(\bar{t}) \leq M$ since Lemma 7 tells $Y(t) \in C[0, T)$. If ν is so large that $\nu - C_1M - C_2 > 0$, then $\nu - C_1Y(\bar{t}) - C_2 > 0$. Set

$$t_0 := \sup\{t \in [0, T-\bar{t}), \nu - C_1Y(\tau) - C_2 > 0, \ \forall \tau \in [\,\bar{t}, \bar{t}+t)\}.$$

By continuity of $Y(t)$ on $[0, T)$, we get obviously $t_0 \in (0, T-\bar{t}\,]$. Suppose $t_0 < T - \bar{t}$, it follows from $\bar{t} > T-1$ that $t_0 < 1$. Then (22) and (27) imply that

$$Y(t) \leq Y(\bar{t}) + C_3\int_{\bar{t}}^{\bar{t}+t_0} Y(s)ds \leq Y(\bar{t}) + C_3M, \text{ for any } t \in [\bar{t}, \bar{t}+t_0].$$

In particular, $Y(\bar{t}+t_0) \leq Y(\bar{t}) + C_3M$. Because $\nu - C_1Y(\bar{t}+t_0) - C_2 = 0$, a contradiction will arise provided

$$\nu > C_1(M + C_3M) + C_2. \tag{28}$$

Therefore, $t_0 = T - \bar{t}$ and $Y(T) \leq Y(\bar{t}) + C_3M$. Namely, (26) holds and we conclude that the strong solution is global.

4. The Classical Solution

In this section, we shall prove that a strong solution is in fact a classical solution under certain regularity of initial data. If in addition, ν satisfies (28), then the classical solution is global.

Lemma 9. (see [4]) *Let $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, k is a positive integer, $\frac{n+2}{2k} < p < +\infty$, then*

$$W_p^{2k,k}(Q_T) \hookrightarrow C^{\alpha,\alpha/2}(\overline{Q_T}),$$

where $0 < \alpha \le 2k - \frac{n+2}{p}$. The norm on $W_p^{2k,k}(Q_T)$ is

$$\|u\|_{W_p^{2k,k}(Q_T)} = \Sigma_{2r+|\alpha|\le 2k}\|D_t^r D_x^\alpha u\|_{L^q(Q_T)}\,,$$

where D_t^r denotes taking derivatives r times with respect to t, D_x^α denotes taking derivatives α times with respect to x.

According to Lemma 9, $W_3^{4,2}(Q_T) \hookrightarrow C^{2,1}(\overline{Q_T})$. By interpolation

$$L^\infty(0,T;H^1(\Omega)) \subset L^3(Q_T),$$

we see that in order to prove $\mathbf{u} \in C^{2,1}(\overline{Q_T})$ and $\phi \in C^{3,1}(\overline{Q_T})$, it suffices to prove that

$$D_t^r D_x^\alpha \mathbf{u},\ D_t^r D_x^\alpha \nabla\phi \in L^\infty(0,T;H^1(\Omega)), \quad 2r + |\alpha| \le 4. \tag{29}$$

Lemma 10. *Assume $\mathbf{u}_0 \in V_2, \phi_0 \in \Phi_5$, then for any $T > 0$,*

$$\mathbf{u}_t,\ \nabla\phi_t,\ \Delta\mathbf{u},\ \nabla\Delta^2\phi \in L^\infty(0,T;L^2(\Omega)), \quad \nabla\mathbf{u}_t, \nabla\Delta\phi_t \in L^2(0,T;L^2(\Omega)). \tag{30}$$

Proof. Differentiating (3) with respect to t, then multiplying the resultant by $\Delta\phi_t$, and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\phi_t\|^2 + \gamma\|\nabla\Delta\phi_t\|^2 = ((\mathbf{u}_t\cdot\nabla)\phi, \Delta\phi_t) + ((\mathbf{u}\cdot\nabla)\phi_t, \Delta\phi_t) - \gamma\left(\Delta f'(\phi)\phi_t, \Delta\phi_t\right).$$

$$\begin{aligned}
|\left((\mathbf{u}_t\cdot\nabla)\phi, \Delta\phi_t\right)| &\le C\|\mathbf{u}_t\|\|\nabla\phi\|_{L^4(\Omega)}\|\Delta\phi_t\|_{L^4(\Omega)} \le C\|\mathbf{u}_t\|\|\Delta\phi_t\|_{L^4(\Omega)}\\
&\le C\|\mathbf{u}_t\|\left(C\|\nabla\phi_t\|^{\frac18}\|\nabla\Delta\phi_t\|^{\frac78} + C\|\nabla\phi_t\|\right)\\
&\le \varepsilon\|\nabla\Delta\phi_t\|^2 + C(\varepsilon)\|\mathbf{u}_t\|^2 + C(\varepsilon)\|\nabla\phi_t\|^2,\\
|\left((\mathbf{u}\cdot\nabla)\phi_t, \Delta\phi_t\right)| &\le C\|\mathbf{u}\|_{L^4(\Omega)}\|\nabla\phi_t\|\|\Delta\phi_t\|_{L^4(\Omega)} \le C\|\nabla\phi_t\|\|\Delta\phi_t\|_{L^4(\Omega)}\\
&\le C\|\nabla\phi_t\|\left(C\|\nabla\phi_t\|^{\frac18}\|\nabla\Delta\phi_t\|^{\frac78} + C\|\nabla\phi_t\|\right)\\
&\le \varepsilon\|\nabla\Delta\phi_t\|^2 + C(\varepsilon)\|\nabla\phi_t\|^2,\\
|\left(\Delta f'(\phi)\phi_t, \Delta\phi_t\right)| &\le C\left(\|\nabla\phi\|^2_{L^4(\Omega)} + \|\phi\|_{L^\infty(\Omega)}\|\Delta\phi\|\right)\|\phi_t\|_{L^4(\Omega)}\|\Delta\phi_t\|_{L^4(\Omega)}\\
&\le C\|\nabla\phi_t\|\left(C\|\nabla\phi_t\|^{\frac18}\|\nabla\Delta\phi_t\|^{\frac78} + C\|\nabla\phi_t\|\right)\\
&\le \varepsilon\|\nabla\Delta\phi_t\|^2 + C(\varepsilon)\|\nabla\phi_t\|^2,
\end{aligned}$$

where we used Gagliardo-Nirenberg inequality $\|\nabla v\|_{L^4(\Omega)} \le C\|v\|^{\frac18}\|\Delta v\|^{\frac78} +$

$C\|v\|$. Noticing $\int_\Omega \phi_t(x,t)dx = 0$ and using Poincaré's inequality, we have $\|\phi_t\|_{L^4(\Omega)} \le C\|\phi_t\|^{\frac{1}{4}}\|\nabla\phi_t\|^{\frac{3}{4}} \le C\|\nabla\phi_t\|$. Thus

$$\frac{1}{2}\frac{d}{dt}\|\nabla\phi_t\|^2 + \gamma\|\nabla\Delta\phi_t\|^2 \le 3\varepsilon\|\nabla\Delta\phi_t\|^2 + C(\varepsilon)\|\nabla\phi_t\|^2 + C(\varepsilon)\|\mathbf{u}_t\|^2. \tag{31}$$

Differentiating (1) with respect to t, multiplying $\mathbf{u}_t$ on both sides then integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \nu\|\nabla\mathbf{u}_t\|^2 \\
= & -((\mathbf{u}_t\cdot\nabla)\mathbf{u} + \lambda\nabla\phi_t\Delta\phi + \lambda\nabla\phi\Delta\phi_t, \mathbf{u}_t) \\
= & (\mathbf{u}_t\otimes\mathbf{u} + \lambda(\nabla\phi\otimes\nabla\phi)_t, \nabla\mathbf{u}_t) \\
\le & \left(\|\mathbf{u}\|_{L^\infty(\Omega)}\|\mathbf{u}_t\| + \lambda\|\nabla\phi\|_{L^4(\Omega)}\|\nabla\phi_t\|_{L^4(\Omega)}\right)\|\nabla\mathbf{u}_t\| \\
\le & \left(\|\mathbf{u}\|_{V_2}\|\mathbf{u}_t\| + C\|\nabla\phi_t\|_{L^4(\Omega)}\right)\|\nabla\mathbf{u}_t\| \\
\le & \varepsilon\|\nabla\mathbf{u}_t\|^2 + C\|\nabla\phi_t\|^2_{L^4(\Omega)} + C(\varepsilon)\|\mathbf{u}\|^2_{V_2}\|\mathbf{u}_t\|^2 \\
\le & \varepsilon\|\nabla\mathbf{u}_t\|^2 + C\left(C\|\nabla\phi_t\|^{\frac{5}{8}}\|\nabla\Delta\phi_t\|^{\frac{3}{8}} + C\|\nabla\phi_t\|\right)^2 + C(\varepsilon)\|\mathbf{u}\|^2_{V_2}\|\mathbf{u}_t\|^2 \\
\le & \varepsilon\|\nabla\mathbf{u}_t\|^2 + \varepsilon\|\nabla\Delta\phi_t\|^2 + C(\varepsilon)\|\mathbf{u}\|^2_{V_2}\|\mathbf{u}_t\|^2 + C(\varepsilon)\|\nabla\phi_t\|^2.
\end{aligned} \tag{32}$$

Adding (31) and (32), and taking ε small enough, we conclude that

$$\frac{d}{dt}\left(\|\nabla\phi_t\|^2 + \|\mathbf{u}_t\|^2\right) + \nu\|\nabla\mathbf{u}_t\|^2 + \gamma\|\nabla\Delta\phi_t\|^2 \le C(1+\|\mathbf{u}\|^2_{V_2})\left(\|\nabla\phi_t\|^2 + \|\mathbf{u}_t\|^2\right).$$

Noticing that $\mathbf{u} \in L^2(0,T;V_2)$, we easily get $\mathbf{u}_t$, $\nabla\phi_t \in L^\infty(0,T;L^2(\Omega))$ and $\nabla\mathbf{u}_t, \nabla\Delta\phi_t \in L^2(0,T;L^2(\Omega))$.

Multiplying (1) by $\Delta\mathbf{u}$ and integrating over Ω, we obtain

$$\begin{aligned}
\nu\|\Delta\mathbf{u}\|^2 &= (\mathbf{u}_t, \Delta\mathbf{u}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \Delta\mathbf{u}) + \lambda(\nabla\phi\Delta\phi, \Delta\mathbf{u}) \\
&\le \varepsilon\|\Delta\mathbf{u}\|^2 + C(\varepsilon)\|\mathbf{u}_t\|^2 + C(\varepsilon).
\end{aligned}$$

Taking $\varepsilon = \frac{\nu}{2}$ and noticing $\mathbf{u}_t \in L^\infty(0,T;L^2(\Omega))$, we have $\Delta\mathbf{u} \in L^\infty(0,T;L^2(\Omega))$. Similarly, taking ∇ on both side of (3), then multiplying $\nabla\Delta^2\phi$ and integrating over Ω, we get $\nabla\Delta^2\phi \in L^\infty(0,T;L^2(\Omega))$. $\square$

Lemma 11. *Assume $\mathbf{u}_0 \in V_3, \phi_0 \in \Phi_6$, then for any $T \in (0,\infty)$,*

$$\nabla\mathbf{u}_t,\ \Delta\phi_t,\ \nabla\Delta\mathbf{u},\ \Delta^3\phi \in L^\infty(0,T;L^2(\Omega)),\ \ \Delta\mathbf{u}_t,\ \Delta^2\phi_t \in L^2(0,T;L^2(\Omega)). \tag{33}$$

Proof. Taking ∇ on both side of (1), and differentiating with respect to t, then multiplying $\nabla\mathbf{u}_t$ and integrating over Ω, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_t\|^2 + \nu\|\Delta\mathbf{u}_t\|^2 = &-\left(\nabla\left((\mathbf{u}_t\cdot\nabla)\mathbf{u}\right)\right. \\
&\left. +\nabla\left((\mathbf{u}\cdot\nabla)\mathbf{u}_t\right) + \lambda\nabla\left(\nabla\phi_t\Delta\phi\right) + \lambda\nabla\left(\nabla\phi\Delta\phi_t\right), \nabla\mathbf{u}_t\right).
\end{aligned}$$

Differentiating (3) with respect to t, then multiplying $\Delta^2\phi_t$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta\phi_t\|^2+\gamma\|\Delta^2\phi_t\|^2 = -\left(\Delta(\mathbf{u}_t\cdot\nabla\phi)+\Delta(\mathbf{u}\cdot\nabla\phi_t),\Delta\phi_t\right)+\gamma\left(\Delta f'(\phi)\phi_t,\Delta^2\phi_t\right).$$

Using Lemma 10 and interpolation inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_t\|^2+\nu\|\Delta\mathbf{u}_t\|^2\le\varepsilon\|\Delta^2\phi_t\|^2+C(\varepsilon)\|\Delta\phi_t\|^2+C(\varepsilon)\|\nabla\mathbf{u}_t\|^2+C\,, \quad (34)$$

$$\frac{1}{2}\frac{d}{dt}\|\Delta\phi_t\|^2+\gamma\|\Delta^2\phi_t\|^2\le\varepsilon\|\Delta\mathbf{u}_t\|^2+C(\varepsilon)\|\Delta\phi_t\|^2+C\,. \quad (35)$$

Adding (34) and (35), we obtain

$$\frac{d}{dt}\left(\|\nabla\mathbf{u}_t\|^2+\|\Delta\phi_t\|^2\right)+\nu\|\Delta\mathbf{u}_t\|^2+\gamma\|\Delta^2\phi_t\|^2\le C\left(\|\nabla\mathbf{u}_t\|^2+\|\Delta\phi_t\|^2\right)+C.$$

Using Gronwall's inequality, we can obtain $\nabla\mathbf{u}_t$, $\Delta\phi_t\in L^\infty(0,T;L^2(\Omega))$ and $\Delta\mathbf{u}_t$, $\Delta^2\phi_t\in L^2(0,T;L^2(\Omega))$. Similar to Lemma 10,

$$\nabla\Delta\mathbf{u},\ \Delta^3\phi\in L^\infty(0,T;L^2(\Omega)).$$ □

Combing Lemma 10 and Lemma 11, we have

Lemma 12. *Assume $\mathbf{u}_0\in V_3,\phi_0\in\Phi_6$, then for any $T>0$,*

$$\mathbf{u}_t\in L^\infty(0,T;V)\cap L^2(0,T;V_2),\quad \phi_t\in L^\infty(0,T;\Phi_2)\cap L^2(0,T;\Phi_4),$$
$$\mathbf{u}\in L^\infty(0,T;V_3),\quad \phi\in L^\infty(0,T;\Phi_6).$$

Following from higher order estimates which are similar to the above, we obtain

Lemma 13. *Assume $\mathbf{u}_0\in V_5,\phi_0\in\Phi_{10}$, then for any $T>0$,*

$$\mathbf{u}_{tt}\in L^\infty(0,T;V)\cap L^2(0,T;V_2),\quad \phi_{tt}\in L^\infty(0,T;\Phi_2)\cap L^2(0,T;\Phi_4),$$
$$\mathbf{u}\in L^\infty(0,T;V_5),\quad \phi\in L^\infty(0,T;\Phi_{10}).$$

Finally, we get (29). Thus $\mathbf{u}\in C^{2,1}(\overline{Q_T})$, $\phi\in C^{3,1}(\overline{Q_T})$. Moreover, now we have $\phi\in L^\infty(0,T;\Phi_7)$ and $\phi_t\in L^2(0,T;\Phi_4)$, which imply $\phi\in C(0,T;\Phi_6)$. By interpolation $H^6(\Omega)\hookrightarrow C^4(\bar\Omega)$, we can see $\phi\in C(0,T;C^4(\bar\Omega))$. In conclusion, $\phi\in C^{4,1}(\overline{Q_T})$. In fact, the regularity of this solution is stronger than that of classical solutions.

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References

[1] H. Abels, On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities, *Arch. Rational Mech. Anal.*.

[2] F. Boyer, Mathematical study of multi-phase flow under shear through order parameter formulation, *Asymptot. Anal.*, **20** (1999), 175-212.

[3] M.E. Gurtin, D. Polignone, J. Viñals, Two-phase binary fluids and immiscible fluids described by an order parameter, *Math. Models Methods Appl. Sci.*, **6** (1996), 815-831.

[4] L. Gu, *Sencond Order Parabolic Partial Differential Equations*, Xiamen University Press, Xianmen (2002).

[5] D. Jacqmin, Calculation of two-phase Navier-Stokes flows using phase-field modeling, *J. Comput. Phys.*, **155** (1999), 96-127.

[6] K. Kawasaki, T. Ohta, Kinetics of fluctuations for systems undergoing phase transitions-interfacial approach, *Physica A*, **118** (1983), 175-190.

[7] T. Koga, K. Kawasaki, Spinodal decomposition in binary fluids: Effects of hydrodynamic interations, *Phys. Rev. A*, **44** (1991).

[8] J.L. Lions, E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, Dunod, Paris (1968).

[9] C. Liu, J. Shen, A phase field model for the mixture of two incompressible fluids and its approxination by a Fourier-spectral method, *Physica D*, **179** (2003), 211-228.

[10] C. Liu, N.J. Walkington, An Eulerian description of fluids containing visco-hyperelastic particles, *Arch. Rat. Mech. Anal.*, **159** (2001), 229-252.

[11] J.J. Tapia, P.G. Lópoz, Adaptive pseudospectral solution of a diffuse interface model, *J. Comput. Appl. Math.*, **224** (2009), 101-117.

[12] R. Teman, *Navier–Stokes Equations*, Studies in Mathematics and its Applications, **2**, North-Holland Publishing Comp., North-Holland (1977).

[13] R. Teman, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York (1997).

[14] R. Teman, *Navier-Stokes Equations*, North Holland, Amsterdam (1977).