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WEAKLY NONLINEAR BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract: We study weakly nonlinear boundary value problems on time scales. Our analysis is devoted to problems at resonance; that is, problems where the homogeneous linear boundary value problem has a nontrivial solution space. For these boundary value problems we establish conditions for the existence of solutions, and we discuss the dependence of solutions on parameters.

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1. Introduction

This paper is devoted to the study of nonlinear boundary value problems on time scales. We consider problems of the form

$$x^{\Delta}(t) = A(t)x(t) + h(t) + \epsilon f(t, x(t)), \quad t \in [a, b]_{\mathbb{T}}\,, \tag{1}$$

subject to

$$B_1 x(a) + B_2 x(b) = 0\,, \tag{2}$$

where B_1 and B_2 are constant $n \times n$ matrices and ϵ is a "small" real parameter. Throughout the paper we assume that $\mathbb{T}$ is a time scale where $[a, b]_{\mathbb{T}} \subset \mathbb{T}$; h is an rd-continuous function from $\mathbb{T}$ into $\mathbb{R}^n$; A is a regressive rd-continuous $n \times n$

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matrix valued function on $\mathbb{T}$; and f is a continuously differentiable function from $\mathbb{T} \times \mathbb{R}^n$ into $\mathbb{R}^n$. $C_{\text{rd}}[a,b]_{\mathbb{T}}$ will denote the space of rd-continuous $\mathbb{R}^n$-valued maps on $[a,b]_{\mathbb{T}}$, and $C[a,b]_{\mathbb{T}}$ will denote the subspace of $C_{\text{rd}}[a,b]_{\mathbb{T}}$ where the maps are continuous.

The focus of this paper will be the analysis of problems at resonance; that is, those where the boundary value problem

$$x^{\Delta}(t) = A(t)x(t), \quad t \in [a,b]_{\mathbb{T}} \tag{3}$$

subject to

$$B_1 x(a) + B_2 x(b) = 0 \tag{4}$$

has nontrivial solutions. We establish conditions for the existence of solutions, and we provide a qualitative analysis of the dependence of the solution on the parameter ϵ.

Our approach is based on the Lyapunov–Schmidt procedure. This technique has been successfully and widely used in nonlinear analysis [2, 4, 5, 6, 7, 10, 12, 15]. All the aspects of this procedure which are relevant to our discussion are presented here in a self-contained fashion. References are provided for readers interested in other types of applications as well as for those who desire a more general, abstract approach [2, 9, 11, 13]. We show that methods and ideas previously used in the study of continuous and discrete boundary value problems [10, 12, 15, 4, 5, 6, 7] can be directly extended to the analysis of boundary value problems on time scales. In addition to providing a unified approach to the analysis of discrete and continuous dynamic systems, we establish conditions for the solvability of boundary value problems which do not fall within the scope of either discrete or continuous problems.

The literature on time scales is vast. Bohner and Peterson provide a reference for the general theory of dynamic equations on time scales [3]. Comprehensive survey articles, and the references therein, may be helpful for the reader interested in boundary value problems on time scales [3, 1]. We provide references for functional analytic methods for nonresonant boundary value problems [8] and for results regarding the existence of periodic solutions [16].

2. Preliminary

In order to analyze (1)–(2) we will use operators defined on the following spaces.

$$X = \{x \in C[a,b]_{\mathbb{T}} : B_1 x(a) + B_2 x(b) = 0\} \quad \text{and} \quad Y = C_{\text{rd}}[a,b]_{\mathbb{T}}.$$

We will use the supremum norm on the spaces X and Y; that is, for $x \in X \cup Y$

$$||x|| = \sup_{t\in[a,b]_{\mathbb{T}}} |x(t)|,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. With this norm it is clear that X and Y are Banach spaces. We will use the operator norm on matrices, and for $v = (v_1, v_2, \cdots, v_m)$ an element of the product space $V_1 \times V_2 \times \cdots \times V_m$, we will use the norm given by $||v|| = \sum_{i=1}^{m} ||v_i||_i$, where $||\cdot||_i$ denotes the norm on V_i.

We define the linear operator $L : D(L) \to Y$ where $D(L) = X \cap C^1_{\mathrm{rd}}[a, b]_{\mathbb{T}}$ by

$$(Lx)(t) = x^{\Delta}(t) - A(t)x(t), \quad t \in [a, b]_{\mathbb{T}},$$

and the operator $F : X \to Y$ by

$$(Fx)(t) = f(t, x(t)), \quad t \in [a, b]_{\mathbb{T}}.$$

It is evident that x is a solution to (1)–(2) if and only if $Lx = h + \epsilon Fx$. Φ will denote the fundamental matrix solution for $x^{\Delta}(t) = A(t)x(t)$, $t \in [a, b]_{\mathbb{T}}$ that satisfies $\Phi(a) = I$.

Proposition 1. *The solution space for the boundary value problem* (3)–(4) *and the kernel of* $B_1 + B_2\Phi(b)$ *have the same dimension.*

Proof. $x \in \ker(L) \iff$
$x^{\Delta}(t) = A(t)x(t)$ for all $t \in [a, b]_{\mathbb{T}}$ and $B_1x(a) + B_2x(b) = 0 \iff$
There exists a $c \in \mathbb{R}^n$ such that $x(t) = \Phi(t)c$ and $B_1c + B_2\Phi(b)c = 0 \iff$
$x(t) = \Phi(t)c$, where $c \in \ker(B_1 + B_2\Phi(b))$.

Therefore the kernel of L and kernel of $B_1 + B_2\Phi(b)$ have the same dimension. □

Throughout the remainder of the paper we will assume that the dimension of the kernel of L, and hence the dimension of the kernel of $B_1 + B_2\Phi(b)$, is m.

Proposition 2. *The map* $F : X \to Y$ *is continuously Fréchet differentiable and* $DF(\phi) : X \to Y$ *is given by*

$$(DF(\phi))(y)(t) = \left(\frac{\partial f}{\partial x}(t, \phi(t))\right)(y(t)).$$

Proof. First we will show that F is differentiable. Let $\phi \in X$, and let $\Gamma : X \to Y$ be defined by

$$(\Gamma y)(t) = \frac{\partial f}{\partial x}(t, \phi(t))y(t).$$

Let $\epsilon > 0$. By the Mean Value Theorem, for each $t \in [a,b]_{\mathbb{T}}$, we have

$$\begin{aligned}&|f(t,\phi(t)+y(t)) - f(t,\phi(t)) - \frac{\partial f}{\partial x}(t,\phi(t))y(t)| \\ &\leq |y(t)| \sup_{v \in v_t} \left\| \frac{\partial f}{\partial x}(t,v) - \frac{\partial f}{\partial x}(t,\phi(t)) \right\|,\end{aligned}$$

where v_t is the line segment between $\phi(t)$ and $\phi(t)+y(t)$. Since $\frac{\partial f}{\partial x}$ is uniformly continuous on compact sets, there exists a δ such that if $||y|| < \delta$, then

$$|f(t,\phi(t)+y(t)) - f(t,\phi(t)) - \frac{\partial f}{\partial x}(t,\phi(t))y(t)| \leq ||y||\epsilon.$$

Therefore

$$\frac{||F(\phi+y) - F(\phi) - \Gamma y||}{||y||} \leq \epsilon,$$

and thus F is differentiable with

$$(DF(\phi))(y)(t) = \left(\frac{\partial f}{\partial x}(t,\phi(t)) \right) y(t).$$

Now we will show that F is continuously differentiable. Suppose $||\phi - \psi|| < \delta$.

$$\begin{aligned}||DF(\phi) - DF(\psi)|| &= \sup_{||h||=1} ||DF(\phi)h - DF(\psi)h|| \\ &= \sup_{||h||=1} \left\{ \sup_{t \in [a,b]_{\mathbb{T}}} \left\| \frac{\partial f}{\partial x}(t,\phi(t))(h(t)) - \frac{\partial f}{\partial x}(t,\psi(t))(h(t)) \right\| \right\} \\ &\leq \sup_{t \in [a,b]_{\mathbb{T}}} \left\| \frac{\partial f}{\partial x}(t,\phi(t)) - \frac{\partial f}{\partial x}(t,\psi(t)) \right\|.\end{aligned}$$

The continuity of DF follows from the continuity of $\frac{\partial f}{\partial x}$. □

Definition 3. Define $S : [a,b]_{\mathbb{T}} \to \mathbb{R}^{n \times m}$ by

$$S(t) = \Phi(t)D,$$

where the columns of $D \in \mathbb{R}^{n \times m}$ make up a basis for the kernel of $B_1 + B_2\Phi(b)$.

Observation. From Proposition 1 we see that x is an element of the kernel of L if and only if $x(t) = S(t)\alpha$ for some $\alpha \in \mathbb{R}^m$ and for all $t \in [a,b]_{\mathbb{T}}$.

In order to use the Lyapunov–Schmidt procedure we now construct projections onto the kernel and image of L.

Definition 4. Define $P : X \to X$ by

$$(Px)(t) = S(t)(S(a)^T S(a))^{-1} S(a)^T x(a), \quad t \in [a,b]_{\mathbb{T}}.$$

Proposition 5. *P is a projection onto the kernel of L.*

Proof. First we will show that P is well defined by proving that $S(a)^T S(a)$ is invertible. Assume $S(a)^T S(a)c = 0$ for some $c \in \mathbb{R}^m$.

$$\begin{aligned} c^T S(a)^T S(a)c = 0 \quad &\Rightarrow \quad (S(a)c)^T (S(a)c) = 0 \\ &\Rightarrow \quad |S(a)c|^2 = 0 \quad \Rightarrow \quad S(a)c = 0 \\ &\Rightarrow \quad \Phi(a)(d_1 c_1 + \cdots + d_m c_m) = 0, \\ &\quad\ \text{where } c = (c_1, \dots, c_m)^T \text{ and } D = (d_1, \cdots, d_m); \\ &\Rightarrow \quad d_1 c_1 + \cdots + d_m c_m = 0 \\ &\Rightarrow \quad c_i = 0 \text{ for all } i = 1, \cdots, m. \end{aligned}$$

Therefore $S(a)^T S(a)$ is invertible.

Now we show that $P^2 = P$.

$$\begin{aligned} (P(Px))(t) &= S(t)(S(a)^T S(a))^{-1} S(a)^T (Px)(a) \\ &= S(t)(S(a)^T S(a))^{-1} S(a)^T S(a)(S(a)^T S(a))^{-1} S(a)^T x(a) \\ &= (Px)(t), \quad \text{for all } t \in [a, b]_{\mathbb{T}}. \end{aligned}$$

Hence $P^2 = P$.

Finally we will show that $\mathrm{Im}(P) = \ker(L)$. Let $x \in X$; then

$$(Px)(t) = S(t)(S(a)^T S(a))^{-1} S(a)^T x(a) = S(t)\alpha\,,$$

where $\alpha = (S(a)^T S(a))^{-1} S(a)^T x(a)$. Therefore $\mathrm{Im}(P) \subset \ker(L)$.

Let $x \in \ker(L)$; then there exists a $\beta \in \mathbb{R}^n$ that satisfies $x(t) = S(t)\beta$, and

$$\begin{aligned} (Px)(t) &= S(t)(S(a)^T S(a))^{-1} S(a)^T x(a) = S(t)(S(a)^T S(a))^{-1} S(a)^T S(a)\beta \\ &= S(t)\beta = x(t). \end{aligned}$$

Hence $\ker(L) \subset \mathrm{Im}(P)$, and therefore $\mathrm{Im}(P) = \ker(L)$. Since it is clear that P is linear and bounded we have shown that P is a projection onto the kernel of L. □

Definition 6. Let $K \in \mathbb{R}^{n\times m}$ such that the columns of K span the kernel of $(B_1 + B_2\Phi(b))^T$. Define $\Psi : [a, b]_{\mathbb{T}} \to \mathbb{R}^{n\times m}$ by

$$\Psi(t) = [B_2\Phi(b)\Phi^{-1}(\sigma(t))]^T K, \quad t \in [a, b]_{\mathbb{T}}.$$

Proposition 7. *y is in the image of L if and only if* $\int_a^b y^T(\tau)\Psi(\tau)\Delta\tau = 0$.

Proof. By the variation of constant formula (see [3]) and the boundary conditions we have that $y \in \mathrm{Im}(L)$ if and only if there exists $x \in X$ such that

$$B_1 x(a) + B_2 \left(\Phi(b)x(a) + \int_a^b \Phi(b)\Phi^{-1}(\sigma(\tau))y(\tau)\Delta\tau \right) = 0$$

$$\iff \quad B_2 \int_a^b \Phi(b)\Phi^{-1}(\sigma(\tau))y(\tau)\Delta\tau \in \operatorname{Im}(B_1 + B_2\Phi(b))$$

$$\iff \quad \left[\int_a^b B_2\Phi(b)\Phi^{-1}(\sigma(\tau))y(\tau)\Delta\tau\right]^T K = 0,$$

where the columns of K span $\ker((B_1 + B_2\Phi(b))^T)$

$$\iff \quad \int_a^b y^T(\tau)[B_2\Phi(b)\Phi^{-1}(\sigma(\tau))]^T K\Delta\tau = 0.$$

□

Proposition 8. *If the $n \times 2n$ matrix $[B_1|B_2]$ has rank n, then the columns of Ψ are linearly independent.*

Proof. Assume $\Psi(t)c = 0$ for some $c \in \mathbb{R}^m$. Then if Ψ_i is the i-th column of Ψ it must be true that $\sum_{i=1}^{m} \Psi_i(t)c_i = 0$, for all $t \in [a,b]_{\mathbb{T}}$. But, this equivalent to $\sum_{i=1}^{m} [B_2\Phi(b)\Phi^{-1}(\sigma(t))]^T K_i c_i = 0$, where K_i is the i-th column of K. From this it is clear that $(\Phi^{-1}(\sigma(t)))^T\Phi(b)^T \sum_{i=1}^{m} B_2^T K_i c_i = 0$ which implies $\sum_{i=1}^{m} B_2^T K_i c_i = 0$.

Suppose $K_i \in \ker(B_2^T)$ for some $i = 1, \cdots, m$. By the definition of K, $(B_1^T + \Phi^T(b)B_2^T)K_i = 0$. This implies that $K_i \in \ker(B_1^T)$ since $K_i \in \ker\,(B_2^T)$. Therefore $[B_1|B_2]K_i^T = 0$, and since $[B_1|B_2]$ has full rank $K_i = 0$. This is a contradiction since the m columns of K span an m dimensional space. Hence $c_i = 0$ for $i = 1, \cdots, m$ and the columns of Ψ are linearly independent. □

Throughout the rest of the paper we will assume that $[B_1|B_2]$ has rank n.

Definition 9. Define $W : Y \to Y$ by

$$(Wy)(t) = \Psi(t)\left[\int_a^b \Psi^T(\tau)\Psi(\tau)\Delta\tau\right]^{-1} \int_a^b \Psi^T(\tau)y(\tau)\Delta\tau, \quad t \in [a,b]_{\mathbb{T}}.$$

Proposition 10. *$E = I - W$ is a projection onto the image of L.*

Proof. First we will show that W is well defined by proving that $\int_a^b \Psi(\tau)^T\Psi(\tau)\Delta\tau$ is invertible. Let $\alpha \in \mathbb{R}^m$, and assume $\left(\int_a^b \Psi(\tau)^T\Psi(\tau)\Delta\tau\right)\alpha =$

0. We can write $\alpha^T \left(\int\limits_a^b \Psi(\tau)^T \Psi(\tau) \Delta\tau \right) \alpha = 0$ which implies $\int\limits_a^b |\Psi(\tau)\alpha|^2 \Delta\tau = 0$. Therefore $\Psi(t)\alpha = 0$ for all $t \in [a,b]_{\mathbb{T}}$. $\alpha = 0$ since the columns of Ψ are linearly independent, and thus $\int\limits_a^b \Psi(\tau)^T \Psi(\tau) \Delta\tau$ is invertible.

Clearly W is a bounded linear map which implies E is also. If we show $W^2 = W$, then we have shown the same for E.

$$
\begin{aligned}
(W(Wy))(t) &= W \left(\Psi(t) \left[\int_a^b \Psi(\tau)^T \Psi(\tau) \Delta\tau \right]^{-1} \int_a^b \Psi^T(\tau) y(\tau) \Delta\tau \right) \\
&= \Psi(t) \left[\int_a^b \Psi(\tau)^T \Psi(\tau) \Delta\tau \right]^{-1} \int_a^b \Psi^T(\tau) \Psi(\tau) \Delta\tau \\
&\quad * \left[\int_a^b |\Psi(\eta)|^2 \Delta\eta \right]^{-1} \int_a^b \Psi^T(\eta) y(\eta) \Delta\eta \\
&= \Psi(t) \left[\int_a^b |\Psi(\eta)|^2 \Delta\eta \right]^{-1} \int_a^b \Psi^T(\eta) y(\eta) \Delta\eta \\
&= (Wy)(t), \quad t \in [a,b]_{\mathbb{T}}.
\end{aligned}
$$

Finally we will show that $\mathrm{Im}(E) = \mathrm{Im}(L)$. Let $y \in Y$. Clearly $Ey \in \mathrm{Im}(E)$.

$$
\begin{aligned}
&\int_a^b \Psi^T(\tau)(Ey)(\tau)\Delta\tau = \int_a^b \Psi^T(\tau)(y - Wy)(\tau)\Delta\tau \\
&= \int_a^b \Psi^T(\tau) y(\tau) \Delta\tau - \int_a^b \Psi^T(\tau)(Wy)(\tau)\Delta\tau \\
&= \int_a^b \Psi^T(\tau) y(\tau) \Delta\tau - \int_a^b \Psi^T(\tau)\Psi(\tau)\Delta\tau \left[\int_a^b |\Psi(\eta)|^2 \Delta\eta \right]^{-1} \int_a^b \Psi^T(\eta) y(\eta) \Delta\eta \\
&= 0.
\end{aligned}
$$

Hence $Ey \in \mathrm{Im}(L)$ and $\mathrm{Im}(E) \subset \mathrm{Im}(L)$. Now let $y \in \mathrm{Im}(L)$. Since

$\int_a^b \Psi^T(\tau)y(\tau)\Delta\tau = 0$ we can write

$$(Ey)(t) = y(t) - \Psi(t)\left[\int_a^b \Psi(\tau)^T\Psi(\tau)\Delta\tau\right]^{-1}\int_a^b \Psi^T(\tau)y(\tau)\Delta\tau = y(t),$$

for $t \in [a,b]_{\mathbb{T}}$. Therefore $y \in \mathrm{Im}(E)$ and $\mathrm{Im}(L) \subset \mathrm{Im}(E)$. □

The characterization of the kernel of L and the image of L used in projections is essentially that which appears in [14, 10, 6, 7]. We provide details for the readers convenience. Utilizing the fact that P and E are projections, we write

$$X = \mathrm{Im}(P) \oplus \mathrm{Im}(I-P)$$

and

$$Y = \mathrm{Im}(I-E) \oplus \mathrm{Im}(E).$$

Proposition 11. *The dimension of the image of P is the same as the dimension of the image of $I - E$.*

Proof. Let $y \in Y$. We write $(Wy)(t) = \Psi(t)(a_1, a_2, \cdots, a_m)^T$ where $a_i \in \mathbb{R}$ is the i-th element of the vector $\left[\int_a^b \Psi^T(\tau)\Psi(\tau)\Delta\tau\right]^{-1}\int_a^b \Psi^T(\tau)y(\tau)\Delta\tau$. This implies that all the elements of the image of W are elements of the span of the columns of Ψ.

Now let y be a linear combination of the columns of Ψ; that is, $y(t) = \Psi(t)(b_1, b_2, \cdots, b_m)^T$, where $b_i \in \mathbb{R}$ for $i = 1, \cdots, m$.

$$\begin{aligned}(Wy)(t) &= \Psi(t)\left[\int_a^b \Psi(\tau)^T\Psi(\tau)\Delta\tau\right]^{-1}\int_a^b \Psi^T(\tau)y(\tau)\Delta\tau \\ &= \Psi(t)(b_1, b_2, \cdots, b_m)^T = y(t), \quad t \in [a,b]_{\mathbb{T}}.\end{aligned}$$

Therefore $y \in \mathrm{Im}(W)$, and hence the dimension of the image of P equals the dimension of the image of $I - E$. □

The following is clear.

Proposition 12. *$L : \mathrm{Im}(I-P) \cap D(L) \to \mathrm{Im}(L)$ is a bijection.*

Recall for $x \in X$ there exists a $u \in \ker(L)$ and a $v \in \mathrm{Im}(I-P)$ such that $x = u + v$. Since $L : \mathrm{Im}(I-P) \cap D(L) \to \mathrm{Im}(L)$ is a bijection, there exists a bounded linear map M :Im$(L) \to$ Im$(I-P) \cap D(L)$ such that

$$LMy = y \text{ for all } y \in \text{ Im}(L) \quad \text{ and } \quad MLx = v \text{ for all } x \in X\,.$$

Definition 13. Define $H : \mathbb{R}\times \text{Im}(P)\times \text{Im}(I-P) \to \text{Im}(I-E)\times \text{Im}(I-P)$ by

$$H(\epsilon, u, v) = \begin{pmatrix} WF(u+v) \\ v - Mh - \epsilon MEF(u+v) \end{pmatrix}.$$

Proposition 14. *Suppose there exists a solution to the linear boundary value problem given by* (1)–(2) *when* $\epsilon = 0$*. For* $\epsilon \neq 0$*,* $Lx = h + \epsilon Fx$ *if and only if* $H(\epsilon, u, v) = 0$.

Proof.

$$\begin{aligned} Lx = h + \epsilon Fx \;&\Leftrightarrow\; E(Lx - h - \epsilon Fx) = 0 \text{ and } (I-E)(Lx - h - \epsilon Fx) = 0 \\ &\Leftrightarrow\; Lx - h - \epsilon EFx = 0 \text{ and } WFx = 0 \\ &\Leftrightarrow\; v - Mh - \epsilon MEF(u+v) = 0 \text{ and } WF(u+v) = 0 \\ &\Leftrightarrow\; H(\epsilon, u, v) = 0. \end{aligned}$$

□

Proposition 15. *H is a continuously Fréchet differentiable map from* $\mathbb{R}\times$ Im$(P)\times$ Im$(I-P)$ *into* Im$(I-E)\times$ Im$(I-P)$*. If* $(\epsilon, u, v) \in \mathbb{R}\times$ Im$(P)\times$ Im$(I-P)$ *then for each* $(\alpha, p, q) \in \mathbb{R}\times$ Im$(P)\times$ Im$(I-P)$*,*

$$DH(\epsilon, u, v)(\alpha, p, q) = \begin{pmatrix} WDF(u+v)(p+q) \\ q - \alpha MEF(u+v) - \epsilon MEDF(u+v)(p+q) \end{pmatrix}.$$

The proof of this proposition is standard [5, 14] and will be omitted.

3. Main Results

The projection scheme which enabled us to reformulate the original problem as $H(\epsilon, u, v) = 0$ is commonly called the Lyapunov–Schmidt procedure. This approach has been successfully used in the study of periodic behavior in discrete and continuous dynamical systems [7, 5, 4, 6] as well as in boundary value problems for both differential and difference equations [12, 10, 14, 15]. In this section we will see how our analysis of boundary value problems on time scales allows us to establish the existence of solutions to problems which are neither completely continuous nor completely discrete.

Theorem 16. *Assume f is continuously differentiable,* $[B_1|B_2]$ *has full rank, and there exists a solution to* (1)–(2) *when* $\epsilon = 0$*. If there exists an*

$\hat{\alpha} \in \mathbb{R}^m$ such that

$$\int_a^b [f(\tau, S(\tau)\hat{\alpha} + Mh(\tau))]^T \Psi(\tau)\Delta\tau = 0$$

and

$$\int_a^b \Psi(\tau)^T \frac{\partial f}{\partial x}(\tau, S(\tau)\hat{\alpha} + Mh(\tau))S(\tau)\Delta\tau$$

is invertible, then for each ϵ small enough, there exists a solution, x_ϵ, to the boundary value problem (1)–(2)*. Furthermore,* $\lim_{\epsilon \to 0} ||x_\epsilon - S(\cdot)\hat{\alpha}|| = 0$.

Proof. F and H are continuously Fréchet differentiable as a result of f being continuously differentiable. Let $\hat{u} \in \mathrm{Im}(P)$ be given by $\hat{u}(t) = S(t)\hat{\alpha}$. $\int_a^b [f(\tau, \hat{u}(\tau) + Mh(\tau))]^T \Psi(\tau)\Delta\tau = 0$ implies that $F(\hat{u} + Mh)$ is in the image of L and therefore $H(0, \hat{u}, Mh) = 0$.

$\frac{\partial H}{\partial(u,v)}(0, \hat{u}, Mh) : \mathrm{Im}(P)\times \mathrm{Im}(I-P) \to \mathrm{Im}(I-E)\times \mathrm{Im}(I-P)$ is given by

$$\frac{\partial H}{\partial(u,v)}(0, \hat{u}, Mh)(z_1, z_2) = \begin{pmatrix} WDF(\hat{u} + Mh)(z_1 + z_2) \\ z_2 \end{pmatrix},$$

since the map $DH(\epsilon, u, v) : \mathbb{R}\times \mathrm{Im}(P)\times \mathrm{Im}(I-P) \to \mathrm{Im}(I-E)\times \mathrm{Im}(I-P)$ is given by

$$DH(\epsilon, u, v)(\alpha, p, q) = \begin{pmatrix} WDF(u+v)(p+q) \\ q - \alpha MEF(u+v) - \epsilon MEDF(u+v)(p+q) \end{pmatrix}.$$

Now we will prove that $WDF(\hat{u}+Mh) : \mathrm{Im}(P) \to \mathrm{Im}(I-E)$ is a bijection. Let z be an element of the kernel of $WDF(\hat{u} + Mh)$. Therefore,

$$\Psi(t) \left[\int_a^b \Psi(\tau)^T \Psi(\tau)\Delta\tau \right]^{-1} \int_a^b \Psi^T(\tau) \frac{\partial f}{\partial x}(\tau, \hat{u}(\tau) + Mh(\tau))z(\tau)\Delta\tau = 0$$

for all $t \in [a, b]_{\mathbb{T}}$. Since the columns of Ψ are linearly independent and since $z(t) = S(t)\alpha$ for some $\alpha \in \mathbb{R}^m$, by virtue of z being an element of $\mathrm{Im}(P)$,

$$\int_a^b \Psi^T(\tau) \frac{\partial f}{\partial x}(\tau, \hat{u}(\tau) + Mh(\tau))S(\tau)\alpha\Delta\tau = 0.$$

Since $\int_a^b \Psi(\tau)^T \frac{\partial f}{\partial x}(\tau, S(\tau)\hat{\alpha} + Mh(\tau))S(\tau)\Delta\tau$ is invertible α must be zero and hence $z = 0$.

Let (z_1, z_2) be an element of the kernel of $\frac{\partial H}{\partial(u,v)}(0, \hat{u}, Mh)$. This implies that $\begin{pmatrix} WDF(\hat{u} + Mh)(z_1 + z_2) \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $z_2 = 0$. $WDF(\hat{u}+Mh)(z_1+z_2) = 0$ implies that $z_1 + z_2 = 0$, and since $z_2 = 0$ we have that $z_1 = 0$ also. It is clear from this that $\frac{\partial H}{\partial(u,v)}(0, \hat{u}, Mh)$ is one-to-one. $\frac{\partial H}{\partial(u,v)}(0, \hat{u}, Mh)$ is also onto since $\mathrm{Im}(P)$ and $\mathrm{Im}(I - E)$ have the same dimension.

Since $H(0, \hat{u}, Mh) = 0$, $\frac{\partial H}{\partial(u,v)}(0, \hat{u}, Mh)$ is a bijection, and H is continuously differentiable, by the Implicit Function Theorem for ϵ small enough there exists a (u_ϵ, v_ϵ) such that $H(\epsilon, u_\epsilon, v_\epsilon) = 0$. Therefore $L(u_\epsilon + v_\epsilon) = h + \epsilon F(u_\epsilon + v_\epsilon)$ and thus $u_\epsilon + v_\epsilon$ is a solution of (1)–(2). Furthermore, $\lim_{\epsilon \to 0} ||u_\epsilon + v_\epsilon - \hat{u} - Mh|| = 0$. □

Our discussion leading to and including Theorem 1 has the same structure to analogous problems for both discrete and continuous dynamic systems [5, 6, 15]. Our presentation provides a framework for the analysis of a more general class of problems. In our next theorem we apply this analysis to the study of a boundary value problem on a time scale which is neither discrete nor continuous.

We present the following discussion as preparation for the next theorem. We consider the following time scale

$$\mathbb{T} = \left\{ \left[1 - \frac{1}{2^{2n}}, 1 - \frac{1}{2^{2n+1}} \right] : n = 0, 1, 2, \cdots \right\} \cup \{1\}.$$

It is clear that

$$\sigma(t) = \begin{cases} \frac{1}{2}(1 + t) = 1 - \frac{1}{2^{2n+2}} & \text{when } t = 1 - \frac{1}{2^{2n+1}}, n = 0, 1, 2, \cdots, \\ t & \text{otherwise}, \end{cases}$$

and

$$\mu(t) = \begin{cases} \frac{1}{2}(1 - t) = \frac{1}{2^{2n+2}} & \text{when } t = 1 - \frac{1}{2^{2n+1}}, n = 0, 1, 2, \cdots, \\ 0 & \text{otherwise}. \end{cases}$$

Now we consider the nonlinear second order scalar equation

$$au^{\Delta\Delta} + bu^{\Delta} + cu = \epsilon \bar{f}(u(t)) \tag{5}$$

subject to

$$\begin{gathered} b_1 x(0) + b_2 x'(0) + d_1 x(1) + d_2 x'(1) = 0 \\ \text{and} \\ b_3 x(0) + b_4 x'(0) + d_3 x(1) + d_4 x'(1) = 0, \end{gathered} \tag{6}$$

where $b_i, d_i \in \mathbb{R}$ for $i = 1, 2$. In order to carry out our analysis we write the boundary value problem in system form. Let $A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}$,

$B_1 = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, $B_2 = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$, and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \\ u^\Delta \end{bmatrix}$, where A is regressive and $[B_1|B_2]$ has full rank. We also let $f(x) = \begin{bmatrix} 0 \\ \bar{f}(x_1) \end{bmatrix}$ for all $t \in \mathbb{T}$. Now we consider the following system:

$$x^\Delta = Ax + \epsilon f(x) \tag{7}$$

subject to

$$B_1 x(0) + B_2 x(1) = 0. \tag{8}$$

Let Φ be the fundamental solution of $x^\Delta = Ax$ that satisfies $\Phi(0) = I$. Suppose

$$\ker((B_1 + B_2\Phi(b))^T) = \text{span}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \text{ and } \ker(B_1 + B_2\Phi(b)) = \text{span}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

So now we have,

$$\Psi(t) = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = [B_2\Phi(1)\Phi^{-1}(\sigma(t))]^T \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

and

$$S(t) = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \Phi(t)\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

There will be a solution to (7)-(8) (and thus (5)-(6)) if there exists an $\hat{\alpha} \in \mathbb{R}$ such that the following are true.

$$\int_0^1 f^T(t, S(t)\hat{\alpha})\Psi(t)\Delta t = \sum_{i=0}^{\infty} \int_{1-\frac{1}{2^{2i}}}^{1-\frac{1}{2^{2i+1}}} \bar{f}(S_1(t)\hat{\alpha})\Psi_2(t)\Delta t$$
$$+ \sum_{i=0}^{\infty} \frac{1}{2^{2i+2}} \bar{f}\left(S_1\left(1 - \frac{1}{2^{2i+1}}\right)\hat{\alpha}\right)\Psi_2\left(1 - \frac{1}{2^{2i+1}}\right) = 0,$$

and

$$\int_0^1 \Psi^T(t)\frac{\partial f}{\partial x}(t, S(t)\hat{\alpha})S(t)\Delta t = \sum_{i=0}^{\infty} \int_{1-\frac{1}{2^{2i}}}^{1-\frac{1}{2^{2i+1}}} \Psi_2(t)\frac{d\bar{f}}{dx_1}(S_1(t)\hat{\alpha})S_1(t)\Delta t$$
$$+ \sum_{i=0}^{\infty} \frac{1}{2^{2i+2}}\Psi_2\left(1 - \frac{1}{2^{2i+1}}\right)\frac{d\bar{f}}{dx_1}\left(S_1\left(1 - \frac{1}{2^{2i+1}}\right)\hat{\alpha}\right)S_1\left(1 - \frac{1}{2^{2i+1}}\right)$$

is invertible.

Theorem 17. *Suppose neither Ψ_2 nor S_1 change sign on the time scale, $\bar{f}$ is C^1 and strictly monotonic, and there exists an $M > 0$ such that $f(s)f(-s) <$*

0 whenever $s > M$. Then, there exists a solution u_0 of (5)-(6) when $\epsilon = 0$ and $\epsilon_0 > 0$ such that for each ϵ such that $|\epsilon| < \epsilon_0$ (5)-(6) has a solution u_ϵ and furthermore $u_\epsilon \to u$ uniformly as $\epsilon \to 0$.

Proof. It is clear that for there exists α_0 such that for $\alpha > \alpha_0$,

$$\int_0^1 f^T(t, S(t)\alpha)\Psi(t)\Delta t \text{ and } \int_0^1 f^T(t, S(t)(-\alpha))\Psi(t)\Delta t$$

have different signs. So by continuity there must exist an $\hat{\alpha}$ such that

$$\int_0^1 f^T(t, S(t)\hat{\alpha})\Psi(t)\Delta t = 0.$$

Since $\bar{f}$ is strictly monotonic,

$$\sum_{i=0}^{\infty} \int_{1-\frac{1}{2^{2i}}}^{1-\frac{1}{2^{2i+1}}} \Psi_2(t)\frac{d\bar{f}}{dx_1}(S_1(t)\hat{\alpha})S_1(t)\Delta t$$

$$+\sum_{i=0}^{\infty} \frac{1}{2^{2i+2}}\Psi_2\left(1-\frac{1}{2^{2i+1}}\right)\frac{d\bar{f}}{dx_1}\left(S_1\left(1-\frac{1}{2^{2i+1}}\right)\hat{\alpha}\right)S_1\left(1-\frac{1}{2^{2i+1}}\right) \neq 0.$$

The result follows from Theorem 1. □

References

[1] R.P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: A survey, *J. Comput. Appl. Math.*, **141** (2002), 1-26.

[2] S. Bancroft, J.K. Hale, D. Sweet, Alternative problems for nonlinear functional equations, *J. Differential Equations*, **4** (1968), 40-56.

[3] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser Boston, Inc., Boston, MA (2001).

[4] L. Cesari, Functional analysis and Galerkin's method, *Michigan Math. J.*, **11** (1964), 385-414.

[5] S.-N. Chow, J.K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, New York-Berlin (1982).

[6] D.L. Etheridge, J. Rodríguez, Periodic solutions of nonlinear discrete-time systems, *Appl. Anal.*, **62** (1996), 119-137.

[7] J.K. Hale, *Ordinary Differential Equations*, Second Edition, Robert E. Krieger Publishing Co. Inc., Huntington, NY (1980).

[8] J. Henderson, A. Peterson, C.C. Tisdell, On the existence and uniqueness of solutions to boundary value problems on time scales, *Adv. Difference Equ.*, **2004** (2004), 93-109.

[9] J. Rodríguez, An alternative method for boundary value problems with large nonlinearities, *J. Differential Equations*, **43** (1982), 157-167.

[10] J. Rodríguez, On resonant discrete boundary value problems, *Appl. Anal.*, **19** (1985), 265-274.

[11] J. Rodríguez, Galerkin's method for ordinary differential equations subject to generalized nonlinear boundary conditions, *J. Differential Equations*, **97** (1992), 112-126.

[12] J. Rodríguez, Nonlinear discrete Sturm-Liouville problems, *J. Math. Anal. Appl.*, **308** (2005), 380-391.

[13] J. Rodríguez, D. Sweet, Projection methods for nonlinear boundary value problems, *J. Differential Equations*, **58** (1985), 282-293.

[14] J. Rodríguez, P. Taylor, Weakly nonlinear discrete multipoint boundary value problems, *J. Math. Anal. Appl.*, **329** (2007), 77-91.

[15] J. Rodríguez, Multipoint boundary value problems for nonlinear ordinary differential equations, *Nonlinear Anal.*, **68** (2008), 3465-3474.

[16] P. Stehlík, Periodic boundary value problems on time scales, *Adv. Difference Eq-s.*, **2005** (2005), 81-92.