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STATIONARY MAGNETIC FLOW OF A SECOND GRADE FLUID PAST A ROTATING PLANE: EXACT SOLUTIONS

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Abstract: Exact solutions are given for the steady flow of a second grade fluid occupying the halfspace $\mathcal{S}$ past the plane $z = 0$ uniformly rotating about a fixed normal axis ($\equiv z-$axis). No conditions on the sign of the material parameters characterizing the second grade fluid are imposed. The solutions are obtained in a velocity field of the form considered by Berker in [2]. Then a uniform magnetic field $\mathbf{H}_0$ orthogonal to the (electrically non conducting) rotating plane is impressed. The induced magnetic field is supposed depending only on z. The results are compared with those corresponding to the Newtonian case [3] and some numerical simulations are given.

AMS Subject Classification: 76A05, 76U05, 76W05
Key Words: second grade fluids, exact solutions, rotating fluids, magnetohydrodynamics

1. Introduction

During the past years there has been considerable interest in rotating magnetohydrodynamic flows of non Newtonian fluids. Actually magnetohydrodynamics finds practical use in many areas of engineering, biology and medicine: viscous fluids with polymer additives and blood flowing in the cardiovascular system can be influenced by the application of an external magnetic field (see [13],

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[12]). Moreover generalizations of the Navier-Stokes model in order to explain a lot of non-standard features of real fluids have been proposed starting from Rivlin-Ericksen fluids of differential type (see [19]).

In this paper we consider second-grade fluids which have been object of many papers in the last few years (see [20], [9], [22], [8], etc.). The most important applications of these fluids concern biological fluids, liquid foams, slurries, food products and many others materials. The second grade fluids can be seen either as a self-consistent model or as a second order approximating model of the simple fluids with fading memory. In the first case all the flows have to meet the Clausius-Duhem inequality (see [5]) so that the material moduli characterizing the fluid must verify the conditions

$$\mu \geq 0, \qquad \alpha_1 + \alpha_2 = 0.$$

Moreover if $\mu > 0, \alpha_1 \geq 0$, then the fluid exhibits asymptotic stability (see [4] and the references quoted herein) and the specific Helmoltz free energy has a minimum at equilibrium if and only if $\alpha_1 \geq 0$ (see [5]). However some experimental studies have not confirmed the previous restrictions. A comprehensive discussion of this argument as well as a critical review on the fluids of differential type can be found in [6]. In any case we develop our study without any assumptions on the sign and on the size of the normal stress moduli α_1, α_2 and we show that the conclusions are not influenced, in a significant manner, by the sign of α_1, α_2.

The flow induced by an infinite disk (the plane $z = 0$) rotating in its own plane in a fluid occupies a central position in fluid dynamics beginning from the work of T. von Karman (see [11]) (swirling flow) because it has immediate technical applications (i.e. rotating machines) and, from a mathematical point of view, the geometry of the flow is so simple to make possible the determination of an exact solution. The solutions relative to this problem, apart from the rigid rotation, can be divided into two classes: solutions such that the velocity field is symmetric and solutions such that the velocity field is asymmetric about the $z-$axis (i.e. the rotation axis): von Karman flow belongs to the first family while the flow we are going to study belongs to the second one. So we are interested to velocity fields which are not symmetric about the rotation axis and we suppose the induced magnetic field depending only on z.

In order to clarify the characteristics of the motion taken into consideration, we recall that asymmetric solutions were introduced in [2] in the study of the steady motion for a Newtonian incompressible fluid confined between two parallel planes rotating about a fixed normal axis with the same angular velocity. These flow problems have relevance to the determination of the material moduli in viscometric experiments (rheometers).

We follow the procedure outlined in [14], [3] in which Newtonian non electrically conducting and electrically conducting fluids respectively are considered. These papers treat the problem in a significant physical manner different from that followed in [2]. More precisely a modified pressure field (i.e. fluid pressure plus a suitable term) is assigned in order to generalize its expression relative to the rigid motion. Therefore we assume the modified pressure independent of z and the velocity field such that the streamlines are concentric circles in planes parallel to rigid boundary and the locus of the circles centers is no longer the $z-$semi-axis (as in the rigid rotation) but a curve Λ. The gradient of this pressure field differs from that corresponding to the rigid motion through a constant vector field which is parallel to the rotating plane and is arbitrarily fixed. As in Poiseuille flow between two fixed planes there is a pressure drop to allow a non-trivial flow (i.e. with non-zero velocity field), here we have the constant term of pressure gradient to determine a non-trivial flow (i.e. non-rigid rotation) and the deformation of the locus of the streamlines centers into the curve Λ.

As far as second grade fluids are concerned, Rajagopal-Gupta investigated the flow of a homogeneous incompressible second grade fluid between two parallel rotating plates in [16]. The results obtained are similar to those of [2], the normal stress modulus α_1 influences the velocity field and the pressure field depends on z. For motions confined between two parallel infinite plates rotating about two noncoincident axes perpendicular to them we refer to [17]. There are some papers in the literature in which the second grade fluid (in steady motion) is permeated by a uniform magnetic field (see [12], [18], [21], [1], [10]). In these studies the geometry of the problem is different from that here considered and the induced magnetic field is neglected completely.

In the present paper first we determine the exact solution for the steady flow of a homogeneous incompressible second grade fluid in a halfspace, bounded by a rigid infinite plane rotating with constant angular velocity $\mathbf{\Omega}$ about a fixed axis ($z-$axis) normal to it. Then the exact solution is found when the fluid is supposed electrically conducting and an external magnetic field of constant strength H_0 is applied in the $z-$direction. The plane is assumed to be non electrically conducting.

The exact solutions are found imposing no-slip condition for the velocity field on the rotating plane. As far as the magnetic field is concerned it is continuous across the boundary $z = 0$ because we suppose the halfspace $\mathcal{S}^- = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}$ to be vacuum (free space) and the magnetic permeability of the fluid is taken equal to that of free space. Moreover we assume the fields bounded with respect to z.

We find that in both cases there is a boundary layer for the velocity (BLV) in which the flow is not a rigid rotation. As one can see by means of numerical simulations, the thickness of (BLV) increases if the viscoelastic parameter Γ increases and decreases if the angular velocity Ω increases. Outside this layer, as $z \to +\infty$, the flow tends to a rigid rotation about the straight line Λ_∞, parallel to the $z-$axis. It is interesting to remark that the constant pressure drop in the $(x, y)-$direction determines the translation of rotation axis from the $z-$axis, as $z \to +\infty$.

In the second case there is also a boundary layer (BLH) for the total magnetic field $\mathbf{H}$ in which the angle $\varphi \in (0, \frac{\pi}{2})$, between $\mathbf{H}$ and the the external magnetic field $\mathbf{H}_0$ depends on z.

We notice that (BLH) is larger than (BLV) and if the angular velocity increases, then its thickness grows thinner. Moreover outside (BLH) the total magnetic field tends to the external one.

We find that (BLV) becomes thinner when an external magnetic field is impressed to the fluid. The influence of $\mathbf{H}_0$ is less evident if the angular velocity increases.

The paper is organized in the following manner:

In Section 2 we formulate the problem and prescribe the form of the modified pressure field in which we include also a term depending on the symmetric part of the velocity gradient.

In Section 3 we obtain the exact solution of the problem. It depends on the material constants, on the angular velocity Ω and on the constant part of the modified pressure gradient.

In Section 4 we illustrate some interesting consequences of the solution and we furnish some numerical examples of the results.

Sections 5, 6 are devoted to study the analogous problem when an uniform magnetic field $\mathbf{H}_0$ orthogonal to the (electrically non conducting) plane is impressed. The induced magnetic field is supposed depending only on z. The results obtained are compared with those corresponding to the Newtonian case and some numerical simulations are given.

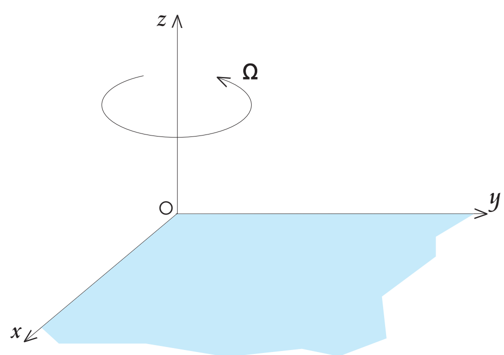


Figure 1: Description flow

2. Statement of the Problem

Consider the stationary flow of a second-grade fluid confined in the halfspace $\mathcal{S}$ bounded by a rigid infinite plane rotating with constant angular velocity $\mathbf{\Omega}$ about a fixed axis normal to it. A Cartesian rectangular coordinate system $Oxyz$, with the $z-$axis coincident with the axis of rotation, is introduced so that $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$, $z = 0$ is the equation of the rigid wall and $\mathbf{\Omega} = \Omega(0, 0, 1)$, $\Omega > 0$, without loss of generality. The equations governing the steady flow of the homogeneous incompressible fluid (supposing the body forces to be conservative) are (see [19], [5])

$$\begin{aligned} &\nu\triangle\mathbf{v} + \alpha_1\mathbf{v}\cdot\nabla\triangle\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} - \alpha_1\nabla\cdot[\nabla\mathbf{v}\,\mathbf{A}] \\ &+(2\alpha_1+\alpha_2)\{\nabla\cdot\mathbf{A}^2 - \frac{1}{2}\nabla|\mathbf{A}|^2\} = \nabla p, \\ &\nabla\cdot\mathbf{v} = 0, \qquad \text{on}\,\mathcal{S}. \end{aligned} \tag{1}$$

In (1) $\mathbf{v}$ is the velocity field, $\mathbf{A} = \nabla\mathbf{v} + \nabla\mathbf{v}^T$,

$$p = \frac{1}{\rho}\Big(p^* + P\Big) - (2\alpha_1 + \alpha_2)\frac{1}{2}|\mathbf{A}|^2,$$

p^* is the pressure field and $-\nabla P$ is the external body force, ρ is the constant mass density; $\nu > 0$ is the kinematic viscosity coefficient and α_1, α_2 are material constants (normal stress moduli).

For sake of simplicity, in the sequel, p will be called modified pressure field.

At the moment we do not impose any restriction to the values of the material coefficients α_1, α_2.

To (1) we adjoin the no-slip boundary condition

$$\mathbf{v} = \Omega(-y, x, 0), \quad \text{at} \;\; z = 0. \tag{2}$$

We notice that the equations of motion are higher order than the Navier-Stokes equations: so the usual no-slip boundary conditions could not be sufficient for determinacy (see [15]). Nevertheless, in many cases the no-slip boundary conditions can be sufficient to avoid ill-posed problems. An example for that is the flow studied in the present paper.

As it is easy to verify, under condition (2), system (1) admits the simple solution (rigid body motion)

$$\mathbf{v}_R = \Omega(-y, x, 0), \quad p_R = \frac{1}{2}\Omega^2(x^2 + y^2) + p_0\,, \tag{3}$$

where p_0 is an arbitrary constant.

Moreover, as is well known, the streamlines in any $z =$ constant plane are concentric circles with center on the $z-$axis.

In our analysis we fix $f_0, g_0 \in \mathbb{R}$ arbitrarily and we assume that the fluid is subjected to a modified pressure field p given by

$$p = \frac{1}{2}\Omega^2[(x - f_0)^2 + (y - g_0)^2] + p_0\,, \tag{4}$$

where p_0 is an arbitrary inessential constant.

We shall search classical solutions $(\mathbf{v}, p)$ of (1) with p given by (4) such that:

i) the streamlines in any $z =$ constant plane are concentric circles;

ii) $\mathbf{v}$ satisfies the boundary condition (2);

iii) $\mathbf{v} \in \mathcal{M}$, $\mathcal{M}$ being the class of functions which are bounded with respect to z, $z \in [0, +\infty)$.

We notice that $\nabla p = \nabla p_R + \nabla p_\Lambda$ with $\nabla p_\Lambda \equiv (-\Omega^2 f_0, -\Omega^2 g_0, 0)$.

Because of i), we shall seek sufficiently smooth solutions of (1), (2) satisfying the previous conditions with the components of $\mathbf{v}$ given by

$$v_1 = -\Omega(y - g(z)), \quad v_2 = \Omega(x - f(z)), \quad v_3 = 0, \;\forall z \geq 0. \tag{5}$$

Of course $\mathbf{v}$ is divergence free.

By virtue of assumptions (5) the space curve Λ, given by the locus of the points at which the velocity is zero in each of the plane parallel to the plane $z = 0$, has Cartesian equations

$$x = f(z), \quad y = g(z), \quad z \in [0, +\infty). \tag{6}$$

3. Exact Solution

On substituting (4), (5) into (1) and taking into account (2), we obtain

$$\begin{aligned} \nu g'' - \Omega f - \alpha_1 \Omega f'' &= -\Omega\ f_0, \\ \nu f'' + \Omega g + \alpha_1 \Omega g'' &= \Omega\ g_0, \end{aligned} \tag{7}$$

together with

$$f(0) = g(0) = 0.$$

Moreover we notice that

$$p^* + P = \rho\, p + \rho(2\alpha_1 + \alpha_2)\ \Omega^2 (f'^2 + g'^2) \equiv \rho\, p + p_1. \tag{8}$$

Therefore f, g determine the pressure term p_1.

Now we have to find f, g. To this end put

$$\mathcal{F} = f + i\ g, \qquad \mathcal{F}_0 = f_0 + i\ g_0.$$

The equations $(7.1), (7.2)$ can be written as:

$$(\nu - i\alpha_1\Omega)\ \mathcal{F}'' - i\ \Omega\ \mathcal{F} = -i\Omega\ \mathcal{F}_0. \tag{9}$$

Define now the following parameters:

$$k_1 = \frac{\Omega}{\nu}, \qquad \Gamma = \alpha_1 k_1 = \frac{\alpha_1 \Omega}{\nu}.$$

We notice that the non-dimensional parameter Γ is a viscoelastic parameter characterizing the ratio of the elastic forces to the viscous forces and k_1^{-1} has the physical dimensions of the square of a length.

After some calculations, we obtain that the general solution to equation (9) is given by

$$\mathcal{F} = C_1\ e^{-(\beta + i\gamma)z} + C_2\ e^{(\beta + i\gamma)z} + \mathcal{F}_0\,, \tag{10}$$

where C_1, C_2 are arbitrary constants and

$$\beta = \sqrt{\frac{k_1}{2(1+\Gamma^2)}\Big(\sqrt{1+\Gamma^2} - \Gamma\Big)}, \qquad \gamma = \sqrt{\frac{k_1}{2(1+\Gamma^2)}\Big(\sqrt{1+\Gamma^2} + \Gamma\Big)}. \tag{11}$$

Since we do not make any hypothesis on the sign of α_1, then the parameter Γ can be either ≥ 0 or < 0. In any case β, γ are positive.

On the other hand

$$\mathcal{F}(0) = 0, \qquad \mathcal{F} \text{ bounded as } \ z \to +\infty, \tag{12}$$

so that the solution $\mathcal{F}(z)$ of our problem becomes

$$\mathcal{F}(z) = \mathcal{F}_0[1 - e^{-(\beta + i\gamma)z}], \quad \forall z \geq 0. \tag{13}$$

Finally we can separate the real and imaginary parts of (13) to obtain

$$\begin{aligned} f(z) &= f_0(1 - e^{-\beta z}\cos\gamma z) - g_0 e^{-\beta z}\sin\gamma z\,, \\ g(z) &= g_0(1 - e^{-\beta z}\cos\gamma z) + f_0 e^{-\beta z}\sin\gamma z\,, \end{aligned} \tag{14}$$

and calculate $p_1(z)$ from (8) to get

$$p_1(z) = \frac{(2\alpha_1 + \alpha_2)\rho\Omega^2 k_1}{\sqrt{1+\Gamma^2}}\,(f_0^2 + g_0^2)\,e^{-2\beta z}. \tag{15}$$

We summarize the previous results in the following:

Theorem 1. *Let a second grade homogeneous incompressible fluid occupy the halfspace* $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ *bounded by the rigid plane* $z = 0$ *rotating about the fixed* $z-$*axis with constant angular velocity* $\mathbf{\Omega} = (0, 0, \Omega), (\Omega > 0)$. *If the modified pressure field has the form (4), then the velocity field solution to the problem (1), (2) satisfying i), ii), iii) is given by (5), (14), (11).*

We notice that $\forall(f_0, g_0)$ (i.e. for any choice of the pressure field of the form (4)) there exists a unique flow satisfying the previous conditions.

4. Remarks and Numerical Simulations

In this section we discuss some properties of the solution and analyze some numerical examples.

1. First of all we note that f, g do not depend on α_2. Moreover we recall that in Sections 2, 3 no sign hypotheses on α_1, α_2 have been made. If we suppose that the thermodynamical restrictions are satisfied (i.e. $\mu \geq 0,\ \alpha_1 + \alpha_2 = 0$), then f, g have the same expression (14) while the modified pressure field is slightly different because $\alpha_1 + \alpha_2 = 0$ in $p_1(z)$.

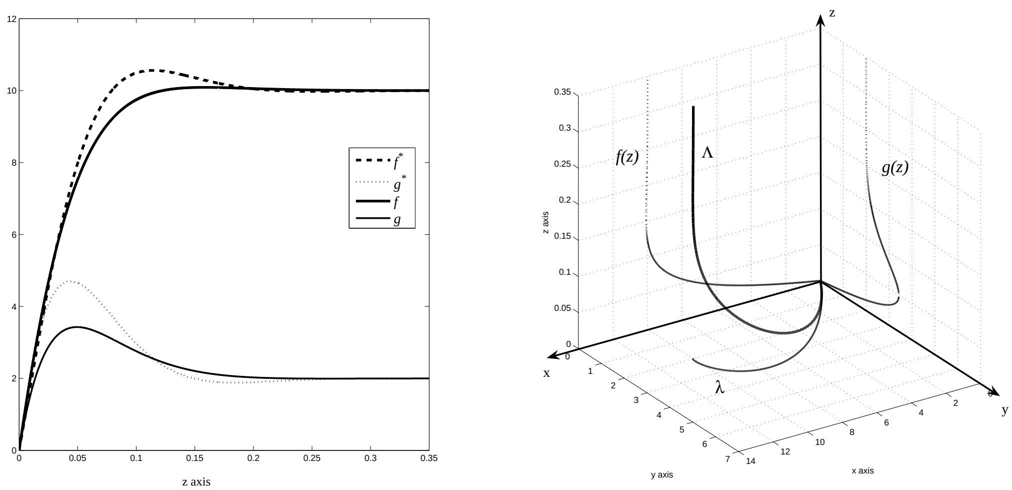
2. We observe that if $\Gamma = 0$, i.e. $\alpha_1 = 0$, but $\alpha_2 \neq 0$, then we find the results concerning the Newtonian case (see [14]) as far as $\mathbf{v}$ is concerned. Actually we deduce

$$\beta = \gamma = \sqrt{\frac{\Omega}{2\nu}} = m\,,$$

so that f, g of (14) reduce to f^*, g^* respectively given by

$$\begin{aligned} f^*(z) &= f_0(1 - e^{-mz}\cos mz) - g_0 e^{-mz}\sin mz\,, \\ g^*(z) &= f_0\, e^{-mz}\sin mz + g_0(1 - e^{-mz}\cos mz), \qquad \forall z \geq 0\,, \end{aligned}$$

which are the functions found in (see [14]).



a) b)

Figure 2: Figure 2a shows the comparison between f, f^* and g, g^* when $\Gamma = 0.6$, $\Omega = 1\, rad\, sec^{-1}$. Figure 2b shows Λ and its projections λ, f, g.

3. As far as the pressure field $p^* + P$ is concerned we find that it depends on z by means of $p_1(z)$ unlike the Newtonian case. Of course in order to obtain the results of [14] also α_2 has to be assumed equal to zero.

The fact that $\frac{\partial(p^*+P)}{\partial z} \neq 0$ states that the contribution due to the pressure to the normal forces exerted on the planes $z =$constant is different at any plane and decreases with z if $2\alpha_1 + \alpha_2 > 0$.

In any case this contribution tends to zero as $z \to +\infty$.

4. If we take the limit as $z \to +\infty$ in both members of (14), (15) we get

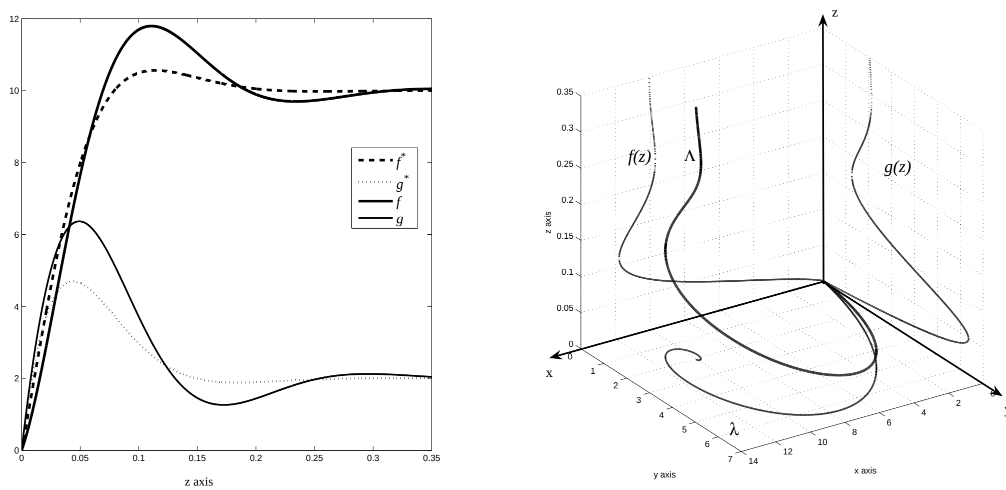
$$\lim_{z\to+\infty} f(z) = f_0, \quad \lim_{z\to+\infty} g(z) = g_0, \quad \lim_{z\to+\infty} p_1(z) = 0.$$

These results state that, as $z \to +\infty$, $\mathbf{v}$ differs from the rigid body velocity $\mathbf{v}_R$ through the constant vector $\mathbf{v}_0 = \Omega(g_0, -f_0, 0)$ orthogonal to the pressure drop ∇p_Λ and the pressure $p^* + P$ tends to

$$p_\infty = \frac{1}{2}\rho\Omega^2[(x - f_0)^2 + (y - g_0)^2] + \rho p_0.$$

Moreover the curve Λ tends, as $z \to +\infty$, to the straight line Λ_∞, parallel to the $z-$axis, passing through the point of coordinates $(f_0, g_0, 0)$.

5. Here we give some numerical examples.



a) b)

Figure 3: Figure 3a shows the comparison between f, f^* and g, g^* when $\Gamma = -0.6$, $\Omega = 1\, rad\, sec^{-1}$. Figure 3b shows Λ and its projections λ, f, g.

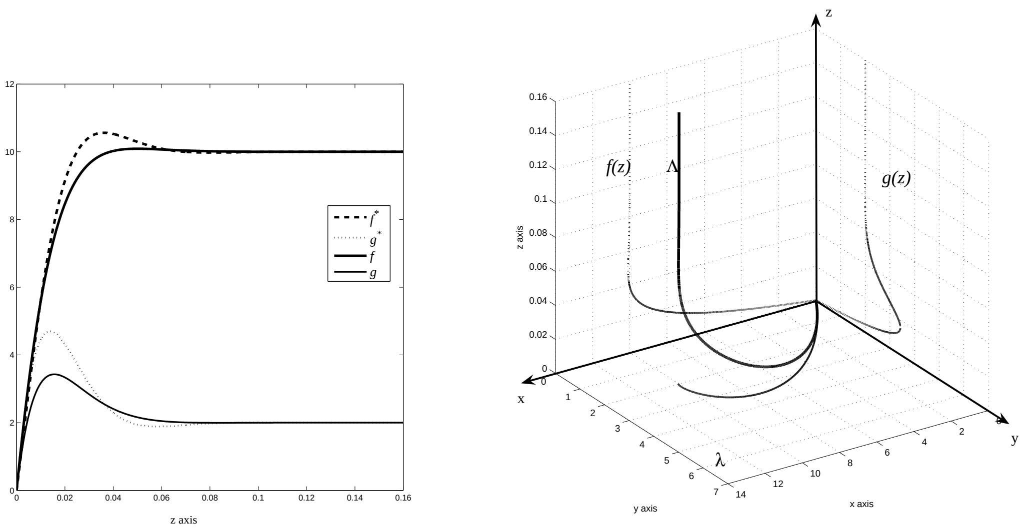
The graphs are given for $f_0 = 10.0\, m$, $g_0 = 2.0\, m$, and supposing the parameter k_1 equal to $10^3\, m^{-2}(\Omega = 1\, rad\, sec^{-1})$ and $10^4\, m^{-2}(\Omega = 10\, rad\, sec^{-1})$ while the viscoelastic parameter Γ assumes the values $-0.6,\ 0.6$ respectively.

We recall that the values given for $f_0,\ g_0$ are purely indicative because they are completely arbitrary.

Figures $2a, 3a, 4a, 5a$ show the comparison between f^*, f and g^*, g when the values of Γ, Ω increase. We see that the maximum value of f and g is less than the maximum value of f^*, g^* when $\Gamma < 0$, while it is greater when $\Gamma > 0$. Figures $2b, 3b, 4b, 5b$ show the curve Λ and its projections.

We notice that there is a boundary layer whose thickness increases if Γ increases and decreases if Ω increases. Outside this layer Λ tends, as $z \to +\infty$, to the straight line Λ_∞.

By observing also the curve λ (projection of Λ in the (x, y)-plane), we note that the distortion of Λ is more accentuated as Γ increases.



a) b)

Figure 4: Figure 4a shows the comparison between f, f^* and g, g^* when $\Gamma = -0.6$, $\Omega = 10\, rad\, sec^{-1}$. Figure 4b shows Λ and its projections λ, f, g.

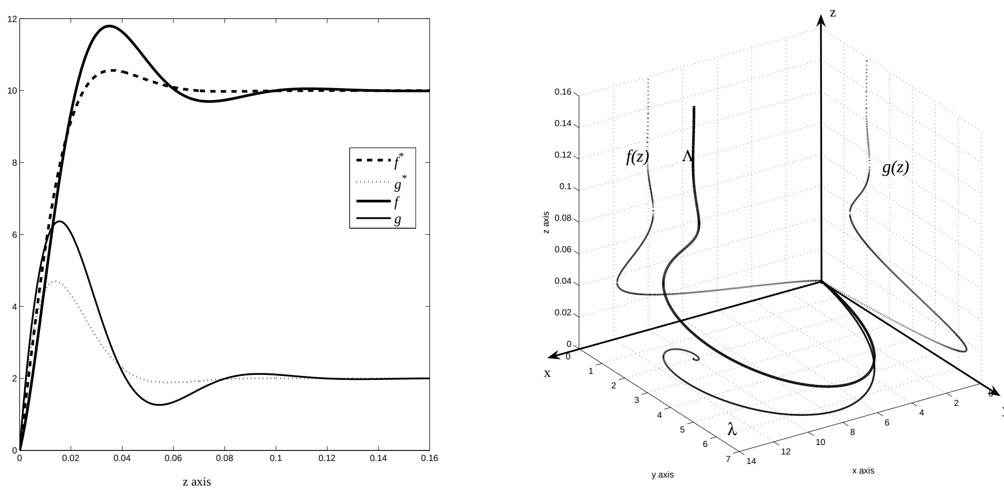
5. Preliminaries in the Presence of the Magnetic Field

In this section we assume that a uniform magnetic field $\mathbf{H}_0 = (0, 0, H_0)$ is impressed upon the fluid. The magnetic field is orthogonal to the rotating plane which is supposed non-electrically conducting. In this case the governing equations are (see [7]):

$$\begin{aligned}
&\nu \triangle \mathbf{v} + \alpha_1 \mathbf{v} \cdot \nabla \triangle \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{v} - \alpha_1 \nabla \cdot [\nabla \mathbf{v}\, \mathbf{A}] \\
&+(2\alpha_1 + \alpha_2)\{\nabla \cdot \mathbf{A}^2 - \frac{1}{2}\nabla |\mathbf{A}|^2\} + \mu^* \,\mathbf{H} \cdot \nabla \mathbf{H} = \nabla p, \\
&\eta \triangle \mathbf{H} + \nabla \times (\mathbf{v} \times \mathbf{H}) = \mathbf{0}, \\
&\nabla \cdot \mathbf{v} = 0, \\
&\nabla \cdot \mathbf{H} = 0 \qquad \text{on } \mathcal{S}.
\end{aligned} \tag{16}$$

In (16) $\mathbf{H}$ is the magnetic field,

$$p = \frac{1}{\rho}\Big(p^* + P + \frac{\mu_e H^2}{2}\Big) - (2\alpha_1 + \alpha_2)\frac{1}{2}|\mathbf{A}|^2,$$



a) b)

Figure 5: Figure 5a shows the comparison between f, f^* and g, g^* when $\Gamma = 0.6$, $\Omega = 10\, rad\, sec^{-1}$. Figure 5b shows Λ and its projections λ, f, g.

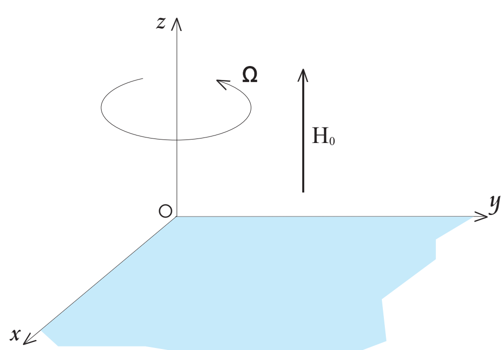


Figure 6: Description MH-flow

$\mu^* = \frac{\mu_e}{\rho}$, μ_e magnetic permeability (it is not restrictive to take μ_e equal to the magnetic permeability of free space), $\eta = \frac{1}{\mu_e \sigma_e}$ is the magnetic resistivity and σ_e is the electrical conductivity.

To (16) we adjoin the boundary conditions

$$\mathbf{v} = \Omega(-y, x, 0), \quad \mathbf{H}_\tau = \mathbf{0}, \quad \text{at} \quad z = 0. \tag{17}$$

The second condition of (17) means that the tangential component $\mathbf{H}_\tau$ of the magnetic field is continuous across the boundary $z = 0$.

As it is easy to verify, under conditions (17), system (16) admits the simple solution (rigid body motion)

$$\mathbf{v}_R = \Omega(-y, x, 0), \quad \mathbf{H}_R = (0, 0, H_0), \quad p_R = \frac{1}{2}\Omega^2(x^2 + y^2) + p_0. \tag{18}$$

We shall search classical solutions $(\mathbf{v}, \mathbf{H}, p)$ of (16) with p given by (4), such that:

i) the streamlines in any $z =$ constant plane are concentric circles;

ii) $(\mathbf{v}, \mathbf{H})$ satisfies the boundary conditions (17);

iii) $(\mathbf{v}, \mathbf{H}) \in \mathcal{M}$, $\mathcal{M}$ being the class of functions $(\mathbf{v}_{\mathcal{M}}, \mathbf{H}_{\mathcal{M}})$ which are bounded with respect to z, $z \in [0, +\infty)$, and $\mathbf{H}_{\mathcal{M}} \to (0, 0, H_0)$ as $z \to +\infty$, uniformly in x, y.

Therefore, we shall seek sufficiently smooth solutions of (16), (17) belonging to $\mathcal{M}$ of the form

$$v_1 = -\Omega(y - g(z)), \qquad v_2 = \Omega(x - f(z)), \qquad v_3 = 0, \tag{19}$$

$$H_1 = h_1(z), \quad H_2 = h_2(z), \quad H_3 = H_0 + h_3(z) \qquad \forall z \geq 0, \tag{20}$$

where $(h_1(z), h_2(z), h_3(z))$ is the unknown induced magnetic field.

6. Exact Solution in the Presence of H_0

First of all we note that (16.4) implies that h_3 is constant so that

$$h_3 = 0, \qquad \forall z \geq 0 \tag{21}$$

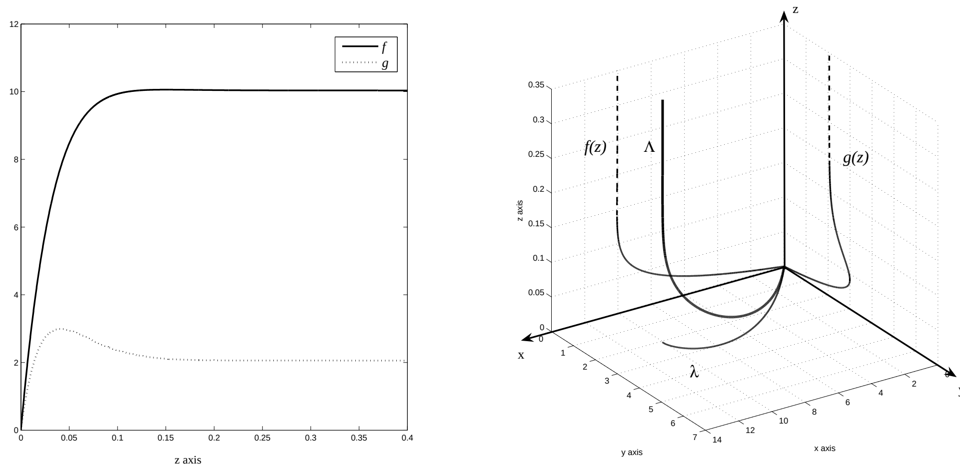
by virtue of condition iii) and (20.3).

On substituting (19), (20), (4) into (16) and taking into account (17), after putting

$$k_2 = \frac{\Omega}{\eta}$$

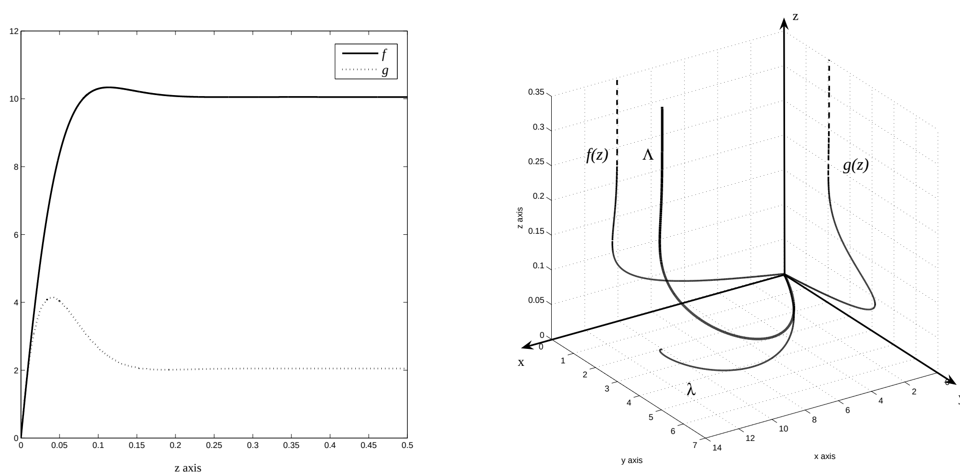
(k_2^{-1} has the physical dimensions of the square of a length) we obtain

$$\begin{aligned} g'' - k_1 f - \Gamma f'' + \frac{\mu^* H_0}{\nu\Omega} h_1' &= -k_1 \ f_0, \\ f'' + k_1 g + \Gamma g'' - \frac{\mu^* H_0}{\nu\Omega} h_2' &= \ k_1 \ g_0, \\ h_1'' + k_2 H_0 \ g' - k_2 \ h_2 &= 0, \end{aligned} \tag{22}$$



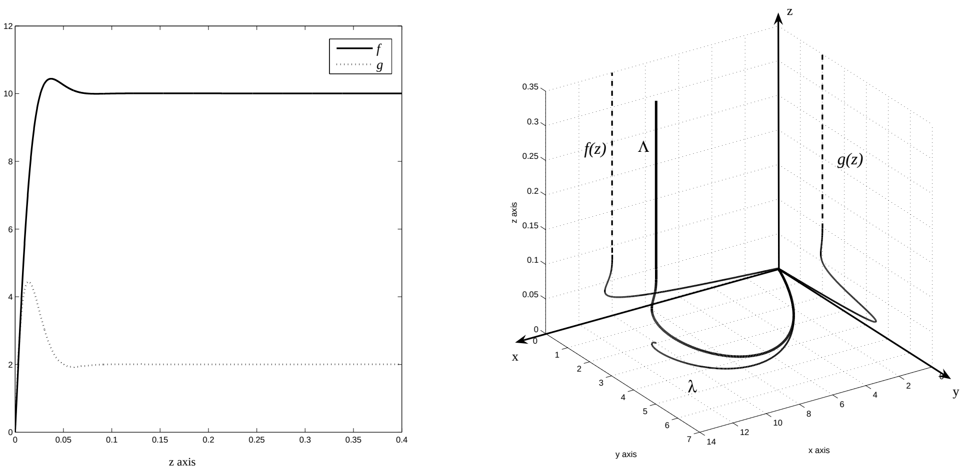
a) b)

Figure 7: Figure 7a shows f, g when $H_0 = 10^5 A\, m^{-1}, \Gamma = 0, \Omega = 1\, rad\, sec^{-1}$. Figure 7b shows Λ and its projections.



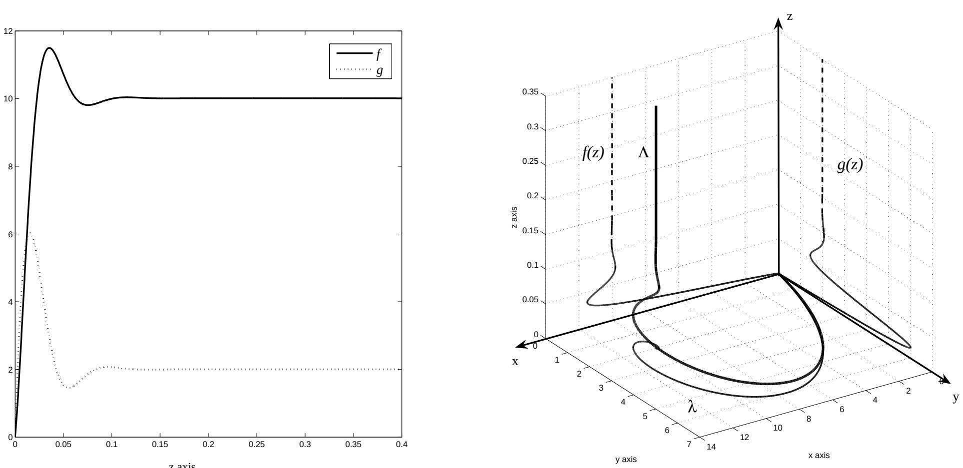
a) b)

Figure 8: Figure 8a shows f, g when $H_0 = 10^5 A\, m^{-1}, \Gamma = 0.6, \Omega = 1\, rad\, sec^{-1}$. Figure 8b shows Λ and its projections.



a) b)

Figure 9: Figure 9a shows f, g when $H_0 = 10^5 A\, m^{-1}, \Gamma = 0$, $\Omega = 10\, rad\, sec^{-1}$. Figure 9b shows Λ and its projections.



a) b)

Figure 10: Figure 10a shows f, g when $H_0 = 10^5 A\, m^{-1}, \Gamma = 0.6$, $\Omega = 10\, rad\, sec^{-1}$. Figure 10b shows Λ and its projections.

$$h_2'' - k_2 H_0 \; f' + k_2 \; h_1 \;\; = \;\; 0\,, \qquad\qquad z \in (0, +\infty)\,,$$

together with

$$f(0) = g(0) = 0, \qquad h_1(0) = h_2(0) = 0.$$

Moreover we have that $p^* + P = \rho p - \dfrac{\mu_e H^2}{2} + p_1$.

Now we have to find f, g.

On setting

$$\mathcal{F} = f + i\; g, \quad \mathcal{H} = h_1 + i\; h_2, \quad \mathcal{F}_0 = f_0 + i\; g_0,$$

the system (22) can be written as:

$$\begin{aligned} (1 - i\,\Gamma)\mathcal{F}'' + i\,\frac{\mu^*\,H_0}{\Omega\nu}\mathcal{H}' - i\;k_1\mathcal{F} &= -i\;k_1\mathcal{F}_0, \\ \mathcal{H}'' - iH_0k_2\mathcal{F}' + i\;k_2\mathcal{H} &= 0. \end{aligned} \tag{23}$$

After some calculations, from (23) we deduce that $\mathcal{F}$ satisfies the following 4-th order ODE:

$$(1 - i\Gamma)\mathcal{F}^{IV} - [N - \Gamma k_2 + i(k_1 - k_2)]\mathcal{F}'' + k_1k_2\mathcal{F} = k_1k_2\mathcal{F}_0\,, \tag{24}$$

where $N = \frac{\mu^*\,H_0^2}{\nu\eta}$ (N^{-1} has the physical dimensions of the square of a length).

We notice that, from a physical point of view, $k_1 \gg k_2$. Therefore, in the sequel, we shall suppose

$$k_1 - k_2 > 0.$$

Moreover at this point we assume

$$\Gamma \geq 0, \qquad \text{i.e.} \qquad \alpha_1 \geq 0.$$

By means of long and cumbersome algebraic calculations we obtain that the four complex roots of the characteristic equation associated to (24) have the following form

$$m_1 = \beta_1 + i\;\gamma_1, \quad -m_1, \quad m_2 = \beta_2 - i\;\gamma_2, \quad -m_2$$

with $\beta_1, \beta_2, \gamma_1, \gamma_2 > 0$ given by

$$\begin{aligned} 2\beta_1 &= \tfrac{1}{\sqrt[4]{2}\sqrt{1+\Gamma^2}}\Big\{\Big[\rho_+\sqrt{1+\Gamma^2} + \sqrt{2}\Big(N - \Gamma k_1\Big) + \sqrt{\rho + a} - \Gamma\sqrt{\rho - a}\Big]\Big\}^{1/2}, \\ 2\gamma_1 &= \tfrac{1}{\sqrt[4]{2}\sqrt{1+\Gamma^2}}\Big\{\Big[\rho_+\sqrt{1+\Gamma^2} - \sqrt{2}\Big(N - \Gamma k_1\Big) - \sqrt{\rho + a} + \Gamma\sqrt{\rho - a}\Big]\Big\}^{1/2}, \\ 2\beta_2 &= \tfrac{1}{\sqrt[4]{2}\sqrt{1+\Gamma^2}}\Big\{\Big[\rho_-\sqrt{1+\Gamma^2} + \sqrt{2}\Big(N - \Gamma k_1\Big) - \sqrt{\rho + a} + \Gamma\sqrt{\rho - a}\Big]\Big\}^{1/2}, \\ 2\gamma_2 &= \tfrac{1}{\sqrt[4]{2}\sqrt{1+\Gamma^2}}\Big\{\Big[\rho_-\sqrt{1+\Gamma^2} - \sqrt{2}\Big(N - \Gamma k_1\Big) + \sqrt{\rho + a} - \Gamma\sqrt{\rho - a}\Big]\Big\}^{1/2}, \end{aligned} \tag{25}$$

where

$$\rho = \sqrt{a^2+b^2},$$
$$a = (N-\Gamma k_2)^2 - (k_1+k_2)^2, \qquad b = 2[N(k_1-k_2)+\Gamma k_2(k_1+k_2)],$$
$$\rho_+ = \left\{\left[\sqrt{2}(N-\Gamma k_2)+\sqrt{\rho+a}\right]^2 + \left[\sqrt{2}(k_1-k_2)+\sqrt{\rho-a}\right]^2\right\}^{1/2},$$
$$\rho_- = \left\{\left[\sqrt{2}(N-\Gamma k_2)-\sqrt{\rho+a}\right]^2 + \left[\sqrt{2}(k_1-k_2)-\sqrt{\rho-a}\right]^2\right\}^{1/2}. \quad (26)$$

Therefore the solution of (24) satisfying the conditions

$$\mathcal{F}(0)=0, \qquad \mathcal{F} \text{ bounded as } z\to+\infty,$$

is

$$\mathcal{F}(z) = C(e^{-m_1 z}-e^{-m_2 z}) + \mathcal{F}_0(1-e^{-m_2 z}), \qquad \forall z \geq 0,\ C\in\mathbb{C}. \quad (27)$$

At this point we integrate (23.1) taking into account (27) and the hypothesis $\mathcal{H}(z)\to 0$ as $z\to+\infty$.

We get

$$\mathcal{H}(z) = \frac{H_0 k_2}{N}\Big[C\Big(-\frac{k_1+(\Gamma+i)m_1^2}{m_1}\ e^{-m_1 z} + \frac{k_1+(\Gamma+i)m_2^2}{m_2}\ e^{-m_2 z}\Big) + \mathcal{F}_0\frac{k_1+(\Gamma+i)m_2^2}{m_2}\ e^{-m_2 z}\Big], \qquad \forall z\geq 0. \quad (28)$$

The boundary condition $\mathcal{H}(0)=0$ allows to determine the complex constant C:

$$C = \frac{m_1[k_1+(\Gamma+i)m_2^2]}{(m_2-m_1)[k_1-(\Gamma+i)m_1 m_2]}\mathcal{F}_0. \quad (29)$$

Finally we can separate the real and imaginary parts of (27), (28) with C given by (29) to obtain

$$\begin{aligned} f(z) &= f_0\{1+e^{-\beta_1 z}(u\cos\gamma_1 z - v\sin\gamma_1 z) - e^{-\beta_2 z}[(1+u)\cos\gamma_2 z + v\sin\gamma_2 z]\} \\ &\quad + g_0\{e^{-\beta_1 z}(v\cos\gamma_1 z + u\sin\gamma_1 z) + e^{-\beta_2 z}[(1+u)\sin\gamma_2 z - v\cos\gamma_2 z]\}, \\ g(z) &= f_0\{-e^{-\beta_1 z}(v\cos\gamma_1 z + u\sin\gamma_1 z) - e^{-\beta_2 z}[(1+u)\sin\gamma_2 z - v\cos\gamma_2 z]\} \\ &\quad + g_0\{1+e^{-\beta_1 z}(u\cos\gamma_1 z - v\sin\gamma_1 z) - e^{-\beta_2 z}[(1+u)\cos\gamma_2 z + v\sin\gamma_2 z]\}, \end{aligned} \quad (30)$$

$$\begin{aligned} h_1(z) &= \tfrac{H_0 k_2}{N}\Big\{f_0\Big[-e^{-\beta_1 z}(c\cos\gamma_1 z + d\sin\gamma_1 z) + e^{-\beta_2 z}(c\cos\gamma_2 z - d\sin\gamma_2 z)\Big] + \\ &\quad + g_0\Big[e^{-\beta_1 z}(d\cos\gamma_1 z - c\sin\gamma_1 z) - e^{-\beta_2 z}(d\cos\gamma_2 z + c\sin\gamma_2 z)\Big]\Big\}, \\ h_2(z) &= \tfrac{H_0 k_2}{N}\Big\{f_0\Big[e^{-\beta_1 z}(-d\cos\gamma_1 z + c\sin\gamma_1 z) + e^{-\beta_2 z}(d\cos\gamma_2 z + c\sin\gamma_2 z)\Big] + \\ &\quad + g_0\Big[-e^{-\beta_1 z}(c\cos\gamma_1 z + d\sin\gamma_1 z) + e^{-\beta_2 z}(c\cos\gamma_2 z - d\sin\gamma_2 z)\Big]\Big\}, \end{aligned} \quad (31)$$

with

$$u = \frac{A_1(A_2-A_1)+B_1(B_2-B_1)}{(A_2-A_1)^2+(B_2-B_1)^2}, \quad v = \frac{B_1A_2-A_1B_2}{(A_2-A_1)^2+(B_2-B_1)^2},$$

$$\begin{aligned} A_1 &= (\beta_2^2-\gamma_2^2)(\beta_1+\Gamma\gamma_1)+\gamma_1(k_1+2\beta_2\gamma_2)-2\Gamma\beta_1\beta_2\gamma_2, \\ A_2 &= (\beta_1^2-\gamma_1^2)(\beta_2-\Gamma\gamma_2)-\gamma_2(k_1-2\beta_1\gamma_1)+2\Gamma\beta_1\beta_2\gamma_1, \\ B_1 &= (\beta_2^2-\gamma_2^2)(\Gamma\beta_1-\gamma_1)+\beta_1(2\beta_2\gamma_2+k_1)+2\Gamma\beta_2\gamma_1\gamma_2, \\ B_2 &= (\beta_1^2-\gamma_1^2)(\Gamma\beta_2+\gamma_2)-\beta_2(2\beta_1\gamma_1-k_1)+2\Gamma\beta_1\gamma_1\gamma_2, \end{aligned}$$

$$c = uD_1+vE_1 = (1+u)D_2+vE_2, \quad d = uE_1-vD_1 = (1+u)E_2-vD_2,$$

$$D_1 = \frac{k_1\beta_1+(\Gamma\beta_1-\gamma_1)(\beta_1^2+\gamma_1^2)}{\beta_1^2+\gamma_1^2}, \; E_1 = \frac{-k_1\gamma_1+(\beta_1+\Gamma\gamma_1)(\beta_1^2+\gamma_1^2)}{\beta_1^2+\gamma_1^2},$$

$$D_2 = \frac{k_1\beta_2+(\gamma_2+\Gamma\beta_2)(\beta_2^2+\gamma_2^2)}{\beta_2^2+\gamma_2^2}, \; E_2 = \frac{k_1\gamma_2+(\beta_2-\Gamma\gamma_2)(\beta_2^2+\gamma_2^2)}{\beta_2^2+\gamma_2^2}. \qquad (32)$$

We conclude this section summarizing the previous results in the following:

Theorem 2. *Let an electrically conducting homogeneous incompressible second grade fluid occupy the halfspace $\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0\}$ bounded by the rigid plane $z=0$ rotating about the fixed $z-$axis with constant angular velocity $\mathbf{\Omega} = (0,0,\Omega), (\Omega > 0)$. We suppose that a uniform magnetic field $\mathbf{H}_0 = (0,0,H_0)$ orthogonal to the electrically non conducting plane is impressed upon the fluid and the body forces are conservative. If the modified pressure field has the form (4), then $(\mathbf{v}, \mathbf{H})$ solution to the problem (16), (17) satisfying i), ii), iii) is given by (19), (20), (21), (30), (31), (32).*

We notice that $\forall(f_0, g_0)$ (i.e. for any choice of the pressure field of the kind (4)) there exists a unique magnetohydrodynamic flow satisfying the previous conditions.

7. Remarks and Numerical Examples in the Presence of H_0

We conclude with some remark and some numerical examples:

1. The term $p_1(z)$ in the pressure field assumes now the expression

$$\begin{aligned} p_1(z) &= \rho(2\alpha_1+\alpha_2)\,\Omega^2(f'^2+g'^2) \\ &= \rho(2\alpha_1+\alpha_2)\,\Omega^2(f_0^2+g_0^2)\Big[(u^2+v^2)(\beta_1^2+\gamma_1^2)e^{-2\beta_1 z} \\ &\quad +[(1+u)^2+v^2](\beta_2^2+\gamma_2^2)e^{-2\beta_2 z}\Big]. \end{aligned} \qquad (33)$$

Also in this case $\frac{\partial(p^*+P)}{\partial z} \neq 0$ so that the contribution due to the pressure to the normal forces exerted on the planes $z =$constant is different at any plane and decreases with z if $2\alpha_1 + \alpha_2 > 0$. This contribution tends to zero as $z \to +\infty$.

2. Also in the presence of the magnetic field we have that, if we take the limit as $z \to +\infty$ in both members of (30), we get

$$\lim_{z\to+\infty} f(z) = f_0, \quad \lim_{z\to+\infty} g(z) = g_0,$$

$$\lim_{z\to+\infty} h_1(z) = \lim_{z\to+\infty} h_2(z) = 0.$$

Moreover

$$\lim_{z\to+\infty} p_1(z) = 0.$$

These results state that, as $z \to +\infty$, $\mathbf{v}$ differs from the rigid body velocity $\mathbf{v}_R$ through the constant vector $\mathbf{v}_0 = \Omega(g_0, -f_0, 0)$ orthogonal to the pressure drop ∇p_Λ and the pressure $p^* + P$ tends to

$$p_\infty = \frac{1}{2}\Omega^2[(x - f_0)^2 + (y - g_0)^2] + \rho p_0 - \frac{\mu_e H_0^2}{2}$$

and the curve Λ tends, as $z \to +\infty$, to the straight line Λ_∞, parallel to the $z-$axis, which passes through the point of coordinates $(f_0, g_0, 0)$.

Finally, being

$$h_3 = H_0, \quad \forall z \geq 0\,,$$

we have that the pressure drop does not influence the induced magnetic field in $z-$direction.

3. If $H_0 = 0$, i.e. $N = 0$, then we find the results of Section 3 again. Actually we have

$$\beta_1 \equiv \beta, \quad \gamma_1 \equiv \gamma, \quad \beta_2 = \sqrt{\frac{k_2}{2}} = \gamma_2, \quad u = -1, \quad v = 0\,,$$

so that the expressions of f, g of (30.1), (30.2) reduce to f, g given by (14.1), (14.2) respectively.

Further a simple calculation shows that $h_i \to 0,\ i = 1, 2$, as $H_0 \to 0$.

4. If $\Gamma = 0$, i.e. $\alpha_1 = 0$, then the roots m_1, m_2 reduce to the roots m_1, m_2 found in [3] for the Newtonian fluids and f, g, h_1, h_2 are exactly the same as in [3]. Therefore in the limit $\alpha_1 \to 0$, only the pressure field has a different form because of the presence of the term $p_1(z)$ given by (33).

5. Finally we briefly examine the case $\Gamma < 0$, i.e. $\alpha_1 < 0$. In this case the characteristic equation associated to (24) has again four complex roots $m_1, -m_1, m_2, -m_2$ whose expressions are similar to those given in (25). Therefore the form of f, g, h_1, h_2 is analogous to that given in Theorem 2 and there

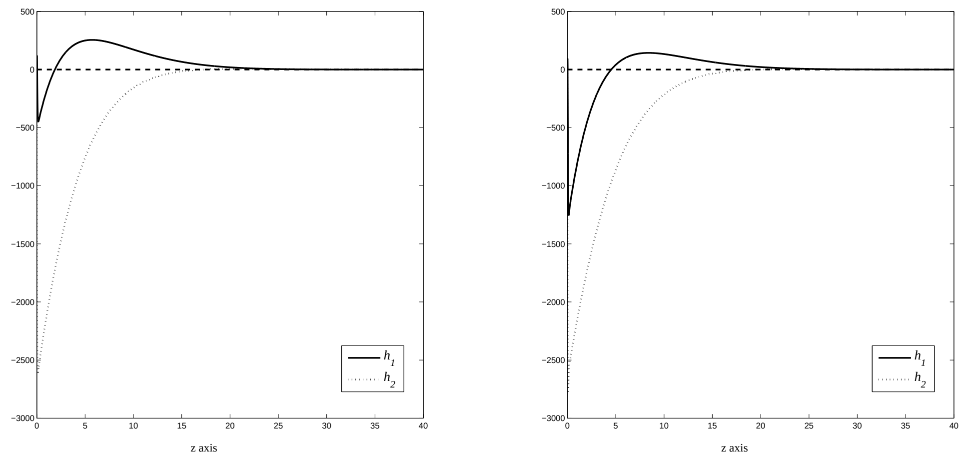


Figure 11: Plots showing the behavior of h_1, h_2 when $H_0 = 10^5 A\, m^{-1}, k_1 = 10^3, k_2 = 10^{-1}$, $\Omega = 1$ rad/sec and $\Gamma = 0$, 0.6 respectively.

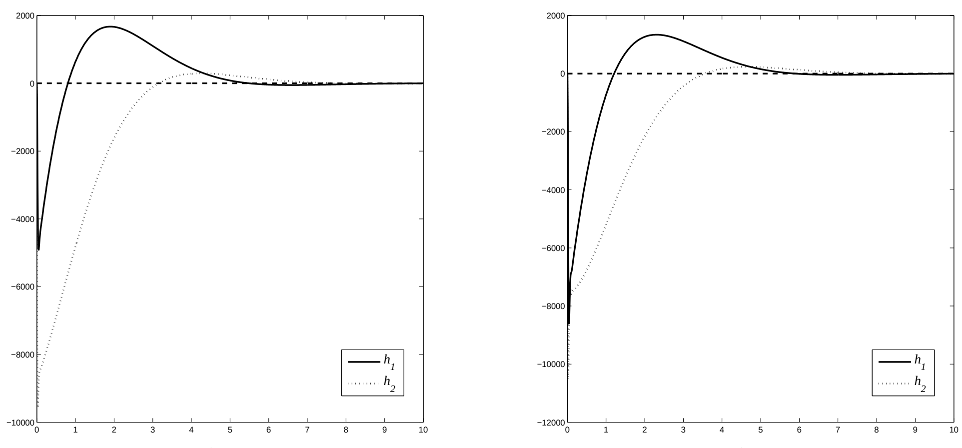


Figure 12: Plots showing the behavior of h_1, h_2 when $H_0 = 10^5 A\, m^{-1}, k_1 = 10^3, k_2 = 10^{-1}$, $\Omega = 10$ rad/sec and $\Gamma = 0$, 0.6 respectively.

are no significative changes in the behavior of the solution to the problem.

6. Here we give some numerical examples assuming $f_0 = 10.0\,m$, $g_0 = 2.0\,m$ and and supposing the parameter k_1 equal to $10^3\,m^{-2}(\Omega = 1\,rad\,sec^{-1})$ and $10^4\,m^{-2}(\Omega = 10\,rad\,sec^{-1})$ while the viscoelastic parameter Γ assumes the values 0, 0.6.

As far as the external magnetic field is concerned we assume $H_0 = 10^5 A\,m^{-1}$ (induction magnetic field ~ 1 Tesla) and suppose $k_2 = 10^{-1}\,m^{-2},\ k_2 = 1\,m^{-2}$, i.e. $\Omega = 1,\ \Omega = 10\ rad\,sec^{-1}$ (i.e. $\sim 0.15,\ \sim 1.5\,revolution/sec$) respectively and $N = 10^3\,m^{-2}$.

Figures 7a, 8a, 9a, 10a show that the graphs of the functions f and g when the values of Γ, Ω increase; Figures 7b, 8b, 9b, 10b show the curve Λ (and its projections) for the values above considered.

We recall that the values given for $f_0,\ g_0$ are purely indicative because they are completely arbitrary.

We can see that the boundary layer relative to **v** (BLV) in which the curve Λ is distorted is thinner in the presence of H_0; moreover if the angular velocity increases then the influence of the external magnetic field is less manifest. Further the thickness increases if the viscoelastic parameter Γ increases and decreases if the angular velocity Ω increases. Outside this layer Λ tends, as $z \to +\infty$, to the straight line Λ_∞.

Figures 11, 12 show the behavior of h_1, h_2. We can see that the strength of the induced magnetic field is much smaller than H_0; the angle $\varphi = \varphi(z)$ between the total magnetic field $\mathbf{H}$ and the external magnetic field $\mathbf{H}_0$ changes with z in a boundary layer (BLH) whose thickness depends on Ω and Γ.

This thickness decreases when Ω increases as for the boundary layer relative to the velocity field. The width of (BLH) is much larger than the width of (BLV). We can note that φ (and hence the direction of $\mathbf{H}$) changes fast near the boundary while outside the boundary layer, the total magnetic field reduces to $\mathbf{H}_0$ which is parallel to the $z-$axis.

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