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GENERALIZED MINIMAL LIOUVILLE SURFACES

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Abstract: The variational principle leading to generalized minimal surfaces in a class of Liouville surfaces was formulated. It was proved that extremal surfaces satisfy generalized Laplace condition almost everywhere.

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1. Generalized Laplace Condition

The study of soap films resulted in the theory of minimal surfaces. The regular ones are identified with those whose mean curvature is equal to zero. They satisfy Laplace condition stating that the mean curvature of a stationary film is proportional to the difference of pressures applied to the different sides of it.

It is now known, Boruvka et al [1], that this condition is not adequate for the regions of large values of the equilibrium surface‘s curvatures. In the paper Boruvka et al [1] the classical Laplace condition was substituted by the linear combination of mean and Gauss curvature leading to the generalized Laplace condition. Later Korovkin et al [8] the similar condition was deduced from the second thermodynamical law and it acquired the following form

$$p_L - p_V = 2 \cdot \sigma \cdot H + l \cdot \sigma \cdot K\,. \tag{1}$$

This is equillibrium condition of the system consisting of liquid (L) and vapour

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phases (V). It is written on the interface dividing different phases. It takes into account the width (l_p) of intermediate layer.

The letter H corresponds to the mean curvature of interface and the letter K — to its Gaussian curvature.

The letter σ denotes capillary tension on the surface.

We attribute the name of the generalized minimal surfaces to those ones that satisfy the condition (1) with the difference of the pressures equal to zero.

It is quite clear that we get classical Laplace condition for the values of l_p equal to zero. It is also clear that the generalized Laplace condition comes into play when the layer‘s thickness l_p is comparable with one of the curvatures radii of the interface.

2. Variational Problem

The above mentioned theory of the minimal surfaces shows us how we could possibly solve the problem of equilibrium of two phased systems using variational principles Finn [5] and Dierks et al [4].

We formulate in this section the variational principle for the generalized minimal surfaces. But unlikely to the minimal surfaces case it is useless to try to involve the isometric parametrization for them as the main tool of the study. We cannot use in our case the Dirichlet integral instead of the integral of the area of the surface. Thus we cannot use the properties of harmonic functions in order to study the minimal sequences and smoothness of the new surfaces. In order to overcome these difficulties we introduce the special class of the surfaces thus restricting the generality of the study. But it is necessary to say that the method we propose here in our opinion could be used for the investigation of the general case.

The first problem we encounter is the construction of the functional which under a proper variation yields us the term proportional to the Gauss curvature and is conjugated in a suitable way to the functional of the area of the surface.

In the case of axisymmetrical surfaces we have constructed the functional Chtchterbakov et al [3]

$$M(\Sigma) = \int\limits_{\Sigma} f(\dot{y}) \cdot \mathrm{d}s \,,$$

$$f(t) := \frac{1}{2} \cdot \Bigg\{ -\sqrt{1-t^2} \cdot \int\limits_0^t \Bigg(\arcsin\sigma + \sigma \cdot \sqrt{1-\sigma^2} - \frac{\pi}{2} \Bigg) \cdot (1-\sigma^2)^{-\frac{3}{2}} \cdot \mathrm{d}\sigma + E_0 \cdot \sqrt{1-t^2} \Bigg\}. \quad (2)$$

Here Σ denotes a line generating an axisymmetrical surface and $z = x(s) + i \cdot y(s)$ — its natural parametrization.

Let S be area of surface. We can prove that the variational problem for the functional $S + \theta \cdot M$ in a suitable class of admissible surfaces leads us to the generalized Laplace condition. Instead of exposing this idea in all the details we consider here a slightly more general case, that is the case of Liouville surfaces.

Let us describe now the class $\aleph$ of admissible surfaces of the kind. We suppose that the functions $\varphi = \varphi(u)$, $\psi = \psi(v)$ of the first quadratic form $\mathrm{d}s^2 = [\phi(u) + \psi(v)] \cdot (du^2 + dv^2)$ of the surfaces from $\aleph$ satisfy the following conditions:

L.1. The functions ψ defined over $[-\pi, \pi]$ are decreasing on $[0, \pi]$.

L.2. $\psi(v) = \psi(-v)$, $v \leq 0$.

L.3. The functions φ are increasing on $[0, 1]$, $\varphi(0) > 0$.

L.4. The functions $\varphi = \varphi(u), \psi = \psi(v)$ posses generalized derivatives satisfying the following conditions

$$\frac{\sqrt{\varphi(u) + \psi(\pi)} \cdot \sqrt{\psi(v) - \psi(\pi)}}{\varphi(u) + \psi(v)} \cdot \left[-\left(\sqrt{\psi(v) - \psi(\pi)}\right)' + \left(\sqrt{\varphi(u) + \psi(\pi)}\right)' \right] \leq 1\,.$$

We suppose also that admissible surfaces pass through two closed regular curves C_1, C_2 lying in two different cubes whose intersection is void.

Now we pass to the construction of the functional to be defined over $\aleph$. Let (t, τ) be semi-geodesic coordinates on admissible surface $\bar{X} \in \aleph$. Later we shall describe semi-geodesic parametrization of Liouville surfaces in details. The first quadratic form $\mathrm{d}s^2$ in semi-geodesic coordinates has the following representation

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + G(t, \tau) \cdot \mathrm{d}\tau^2\,. \quad (3)$$

Let us introduce the following functional Ξ on $\aleph$

$$\Xi(\bar{X}) := A(\bar{X}) + \theta \cdot K_1(X)\,. \quad (4)$$

The letter A denotes the area of the surface $\bar{X} \in \aleph$ and K_1 is the functional defined by the following expression

$$K_1(\bar{X}) := \int\limits_{-T}^{T} \mathrm{d}t \cdot \int\limits_{\Gamma_t} f_1\big(g(t,\tau)\big) \cdot \mathrm{d}\tau\,. \tag{5}$$

Here Γ_t denotes geodesic line corresponding to the parameter t, $[-T, T]$ be the projection of the image of the domain Π in semi-geodesic coordinate's onto axis t and

$$g(t,\tau) := \left(\sqrt{G}\right)_\tau\,. \tag{6}$$

The function f_1 is defined by the expression (2) and satisfies the following second order ordinary differential equation

$$\frac{\mathrm{d}^2 f_1}{\mathrm{d}t^2} \cdot \sqrt{1-t^2} - \frac{\mathrm{d}f_1}{\mathrm{d}t} \cdot \frac{t}{\sqrt{1-t^2}} + f_1 \cdot \frac{t}{\sqrt{1-t^2}} = -1\,. \tag{7}$$

We are now capable to formulate the variational problem.

Variational Problem V. To find $\bar{X}_e \in \aleph$ such that the following equality takes place

$$\inf\left\{\Xi(\bar{X})\,\middle|\,\bar{X} \in \aleph\right\} = \Xi(\bar{X}_e)\,. \tag{8}$$

We are going to prove the following theorem.

Theorem 1. *Let $\aleph$ be the set of admissible surfaces introduced earlier for the variational problem* **V** *and Ξ the functional (4) defined on it. Then there exists a surface $\bar{X}_e \in \aleph$ such that the following condition holds almost everywhere in $(0,1) \times [-\pi, \pi]$*

$$H + \theta \cdot K = 0\,. \tag{9}$$

Here $H = H(u,v)$ denotes the mean curvature of $\bar{X}_e$ and K — its Gaussian one. At the interior points of the sets $\{g = 1\}$ and $\{g = 0\}$ this surface is minimal.

3. Semi-Geodesic Parametrization of Liouvile Surfaces

Let us consider the following expressions

$$\Phi(u) := \int\limits_0^u \frac{\mathrm{d}x}{\sqrt{\phi(x) + \psi(\pi)}}\,, \tag{10}$$

$$\Lambda(v) := \int\limits_0^v \frac{\mathrm{d}y}{\sqrt{\psi(y) - \psi(\pi)}} \,. \tag{11}$$

It is well known that the geodesic lines of Liouville surface can be written in the form

$$\Phi(u) = \Lambda(v) \pm t, \quad t \in R\,. \tag{12}$$

We select signal minus in the representation (12).

Let $(0, \hat{t})$ be a point of intersection of a geodesic line Γ_t with the axis $\{u = 0\}$. It is clear that the function $t(\hat{t})$ defined by the following expression

$$t(\hat{t}) := \int\limits_0^{\hat{t}} \frac{\mathrm{d}y}{\sqrt{\psi(y) - \psi(\pi)}}, \quad \hat{t} \in [0, \pi]\,,$$

is monotone one. This means that any of the two geodesic passing through the different points $(0, \hat{t}_1)$, $(0, \hat{t}_2)$ cannot intersect in the interior part of Π. Thus we can cover it by the family of disjoint geodesic lines Γ_t, $t \in (0, \pi)$. If the function ψ satisfies additional condition

$$\psi'(\pi) = 0$$

then the line $\{v = \pi\}$ is also geodesic one. Now we shall construct a family of lines orthogonal to the lines of the family $\{\Gamma_t\}$.

Let $B = B(u, v)$ be a function whose level lines are orthogonal to the lines of $\{\Gamma_t\}$. The condition for the lines to be orthogonal means that the following equation takes place

$$(\varphi(u) + \psi(v)) \cdot \Big(B_u \cdot \sqrt{\psi(v) - \psi(\pi)} + B_v \cdot \sqrt{\varphi(u) + \psi(\pi)} \Big) = 0\,. \tag{13}$$

It is clear that the function of the following type

$$B(u, v) = -\int\limits_0^u \sqrt{\varphi\,(x) + \psi(\pi)} \cdot \mathrm{d}x + \int\limits_0^v \sqrt{\psi(y) - \psi(\pi)} \cdot \mathrm{d}y \tag{14}$$

satisfies the equation (13). It is also clear that the lines of the family $\{\Gamma_t\}$ are orthogonal to the lines of the family $\{B(u, v) = \tau\}$ thus constituting orthogonal coordinate system. The coordinates (u, v) are isometric coordinates and (t, τ) — semi-geodesic ones. The following system maintain the correspondence between these coordinates

$$\Phi(u) + \Lambda(v) = t\,, \tag{15}$$

$$B(u, v) = \tau\,. \tag{16}$$

Now let us express the function G from the equation (3) and g from (6) in terms of the functions φ, ψ and their derivatives. The direct calculations lead us to the representations

$$G(t,\tau) = \big(\varphi(u) + \psi(\pi)\big) \cdot \big(\psi(v) - \psi(\pi)\big)\,, \tag{17}$$

$$g(t,\tau) = \frac{\sqrt{\varphi(u)+\psi(\pi)}\cdot\sqrt{\psi(v)-\psi(\pi)}}{\varphi(u)+\psi(v)} \cdot \Big[\Big(\sqrt{\psi(v)-\psi(\pi)}\Big)' - \Big(\sqrt{\varphi(u)+\psi(\pi)}\Big)' \Big]. \tag{18}$$

From the condition L.4 we now get that

$$|g(t,\tau)| \le 1\,. \tag{19}$$

The inequality (19) implies that the functional Ξ is well defined over the set $\aleph$.

4. Minimizing System and its Compactness

Let us prove here the folowing lemma.

Lemma 1. *Let $\{\bar{X}_n\}$ be a minimizing sequence for the variational problem V. Then it is compact in $\aleph$ in the sense of uniform convergence over Π.*

Proof. Let φ_n, ψ_n be the functions of the first quadratic form of the surface $\bar{X}_n$. For a given θ we can select E_0 in such a way that the sequence $\big\{A\{\bar{X}_n\}\big\}$ is to be bounded. As the functions φ_n are monotone ones than the sequence $\{\varphi_n\}$ is limited in the interior points of the interval $(0,1)$. The same is valid for the sequence $\{\psi_n\}$ in the interior points of the interval $(-\pi,\pi)$.

The monotone character of the functions φ_n, ψ_n also means that at the interior points of the line (15) the sums $\varphi_n(u)+\psi_n(v)$ are bounded from below. The same is valid for the functions $\sqrt{\varphi_n(u)+\psi(\pi)}+\sqrt{\psi(v)-\psi(\pi)}$. We can select the functions φ_n, ψ_n as absolutely continuous ones. Then from the condition L.4 we get that their sequences are compact over the segments $[0,1]$ and $[-\pi,\pi]$ respectively.

Let φ and ψ be the limits of the convergent subsequences of the considered sequences. The condition L.4 means that the functions

$$\frac{\sqrt{\varphi_n(u)+\psi_n(\pi)}\cdot\sqrt{\psi_n(v)-\psi_n(\pi)}}{\varphi_n(u)+\psi_n(v)} \cdot \Big[\Big(\sqrt{\psi_n(v)-\psi_n(\pi)}\Big)' - \Big(\sqrt{\varphi_n(u)+\psi_n(\pi)}\Big)' \Big]$$

are limited in every functional space $L^p(\pi)$, $p>1$, Gilbarg et al [6]. It means that the sequence of these functions is weakly compact in each of these spaces. The weak limit function of the convergent subsequence can be written in the form

$$\frac{\sqrt{\varphi(u)+\psi(\pi)}\cdot\sqrt{\psi(v)-\psi(\pi)}}{\varphi(u)+\psi(v)}\cdot\chi((u,v)),$$

$$\chi(u,v)=w-\lim_{n\to\infty}\left[\left(\sqrt{\psi_n(v)-\psi_n(\pi)}\right)'-\left(\sqrt{\varphi_n(u)+\psi_n(\pi)}\right)'\right].$$

We can consider the sequences $\left\{\sqrt{\varphi_n(u)+\psi_n(\pi)}\right\}$, $\left\{\sqrt{\psi_n(v)-\psi_n(\pi)}\right\}$ which are compact in each $W_p^1(\Pi)$, $p>1$, space as uniformly convergent. This means that the function χ can be written in the form

$$\chi(u,v)=\left[\left(\sqrt{\psi(v)-\psi(\pi)}\right)'-\left(\sqrt{\varphi(u)+\psi(\pi)}\right)'\right].$$

From the properties of weak convergent sequences we get, Hutson et al [7], that

$$\begin{aligned}&\left\{\iint\limits_{\Pi}\left|\frac{\sqrt{\varphi(u)+\psi(\pi)}\cdot\sqrt{\psi(v)-\psi(\pi)}}{\varphi(u)+\psi(v)}\cdot\left[\left(\sqrt{\psi(v)-\psi(\pi)}\right)'\right.\right.\right.\\&\qquad\left.\left.\left.-\left(\sqrt{\varphi(u)+\psi(\pi)}\right)'\right]\right|^p\mathrm{d}u\cdot\mathrm{d}v\right\}^{\frac{1}{p}}\\&\le\liminf_{n\to\infty}\left\{\iint\limits_{\Pi}\left|\frac{\sqrt{\varphi_n(u)+\psi_n(\pi)}\cdot\sqrt{\psi_n(v)-\psi_n(\pi)}}{\varphi_n(u)+\psi_n(v)}\cdot\left[\left(\sqrt{\psi_n(v)-\psi_n(\pi)}\right)'\right.\right.\right.\\&\qquad\left.\left.\left.-\left(\sqrt{\varphi_n(u)+\psi_n(\pi)}\right)'\right]\right|^p\mathrm{d}u\cdot\mathrm{d}v\right\}^{\frac{1}{p}}\le\pi^{\frac{1}{p}},\quad\forall p>1\,.\end{aligned}\tag{20}$$

Passing to the limit with p tending to infinity we now get from the inequality (20) that the functions φ, ψ satisfy the condition L.4. As the functions $\left\|\bar{X}_{nu}\right\|^2, \left\|\bar{X}_{nv}\right\|^2$ are uniformly limited than the sequence $\left\{\bar{X}_n\right\}$ is compact in the sense of uniform convergence over Π. The functions φ, ψ correspond to the limit surface $\bar{X}_e$ of the convergent subsequence of $\{\bar{X}_n\}$. □

5. Mean and Gaussian Curvatures of the Extremal Surface

We prove here the following lemma.

Lemma 2. *Mean curvature and Gaussian curvature of the extremal sur-*

face exist almost everywhere and belong to the space $L^p(\Pi)$, $p > 1$.

Proof. Let $\{\bar{X}_n\}$ be minimizing sequence of the surfaces for the variational problem V. Without loss of generality we can select them as twice differentiable ones. Than for any finite differentiable function $\bar{\Phi}$ with the supporter inside of Π we have

$$\iint\limits_{\Pi} \left(\bar{X}_{nuu} + \bar{X}_{nvv}\right) \cdot \bar{\Phi} \cdot \mathrm{d}u \cdot \mathrm{d}v = \iint\limits_{\Pi} \left(\bar{X}_{nu} \cdot \bar{\Phi}_u + \bar{X}_{nv} \cdot \bar{\Phi}_v\right) \cdot \mathrm{d}u \cdot \mathrm{d}v \,. \tag{21}$$

The functions $\left\|\bar{X}_{nu}\right\|^2$, $\left\|\bar{X}_{nv}\right\|^2$ are uniformly bounded inside of Π. This means that the sequences $\{\bar{X}_{nu}\}$ and $\{\bar{X}_{nv}\}$ are weakly convergent in the space $L^p(\Pi)$, $p > 1$. Now from the equation (21) we get that the sequences $\{\bar{X}_{nuu}t\}\{\bar{X}_{nvv}\}$ are also weakly convergent in these spaces.

The functions of the sequence $\{\bar{X}_n\}$ are uniformly bounded in $W_2^1(\Pi)$ and compact in the sense of uniform convergence. Taking this into account we arrive at the following equation

$$\iint\limits_{\Pi} \left(\bar{X}_{euu} + \bar{X}_{evv}\right) \cdot \bar{\Phi} \cdot \mathrm{d}u \cdot \mathrm{d}v = \iint\limits_{\Pi} \left(\bar{X}_{eu} \cdot \bar{\Phi}_u + \bar{X}_{ev} \cdot \bar{\Phi}_v\right) \cdot \mathrm{d}u \cdot \mathrm{d}v \,. \tag{22}$$

The coordinates (u, v) are isometric ones. It means that the following equation takes place almost everywhere in Π

$$\bar{X}_{euu} + \bar{X}_{evv} = \lambda^2 \cdot H \cdot \bar{N}, \lambda^2 = \varphi + \psi \,. \tag{23}$$

From the equation (23) we get that the mean curvature belongs to the space $L^p(\pi)$, $p > 1$.

The same result is valid for the Gaussian curvature.

The lemma is proved. □

6. Variations of the Surfaces of Liouville Class

Using class of Liouville surfaces we easily got compactness of the minimizing sequence and certain regularity of the limit surface. Of course we must to pay for this. We can easily verify that the normal displacement of the Liouville surfaces in the general case leads us outside of their class. The same is valid for the surfaces obtained using inner variations.

Here we introduce special variations which correspond to the variations of the surfaces obtained by the transformation of independent variables conjugated by the variation of the functions φ, ψ. They are performed in such a way that transformed geodesic line of the surface is a geodesic line for the new one.

Let $(u_0, v_0) \in \Pi$ be a fixed interior point. In the neighbourhood $\Delta \subset \bar{\Delta} \subset \Pi$ of this point, $\Delta = \{|u - u_0| < \delta\} \times \{|v - v_0| < \delta\}$, we consider the local topological transformation of the form

$$u^* = u + \varepsilon \cdot P(t, \tau) + o(\varepsilon), \quad \varepsilon \to 0\,, \tag{24}$$

$$v^* = v + \varepsilon \cdot Q(t, \tau) + o(\varepsilon), \quad \varepsilon \to 0\,. \tag{25}$$

Here P and Q are finite differentiable functions which will be defined later in a suitable way. The coordinates (u, v) and semi-geodesic ones are connected by the following equations

$$t = \int_0^u \frac{\mathrm{d}x}{\sqrt{\varphi(x) + b}} + \int_0^v \frac{\mathrm{d}y}{\sqrt{\psi(y) - b}}\,, \tag{26}$$

$$\tau = -\int_0^u \sqrt{\varphi(x) + b} \cdot \mathrm{d}x + \int_0^u \sqrt{\psi(y) - b} \cdot \mathrm{d}y\,. \tag{27}$$

Here $b = \psi(\pi)$. Now we introduce new functions φ^* , ψ^* in such a way that the following condition is satisfied

$$t = \int_0^{u^*} \frac{\mathrm{d}x}{\sqrt{\varphi^*(x) + b}} + \int_0^{v^*} \frac{\mathrm{d}y}{\sqrt{\psi^*(y) - b}}\,. \tag{28}$$

This condition means that the point (u^*, v^*) from (28) lie on geodesic line of the surface with the first quadratic forms defined by the function $\varphi^*(u^*) + \psi^*(v^*)$. Of course the transformed surface corresponding to this form is not uniquely determined by it. But using Bonnet Theorem do Carmo [2] and Pogorelov et al [9] we can construct a lot of them. The geodesic lines (26), (28) of the surfaces under consideration correspond to the equal values of t. Besides the inclinations of these lines are the same at the points $(0, v)$ of the axis u.

Now let us select function φ^* as follows

$$\sqrt{\varphi^*(u^*) + b} := \sqrt{b + \varphi(u)} \cdot \left\{1 + \varepsilon \cdot \frac{\mathrm{d}\hat{P}}{\mathrm{d}u} + \varepsilon \cdot \frac{\mathrm{d}\hat{R}}{\mathrm{d}u}\right\} + o(\varepsilon)\,. \tag{29}$$

Here $\hat{P}(u) := P\big(t(u, v(u)), \tau(u, v(u))\big)$, $\hat{R}(u) := R\big(t(u, v(u)), \tau(u, v(u))\big)$. A function R is also to be defined later in a suitable way.

It can be easily checked that for the points $(u, v(u))$ of geodesic lines corresponding to $t = \text{const}$ the following equalities take place

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}u} = \frac{\partial P}{\partial \tau} \cdot \left(\frac{\partial \tau}{\partial u} + \frac{\partial \tau}{\partial v} \cdot v'(u) \right) = -\frac{\varphi(u) + \psi(v)}{\sqrt{\varphi(u) + b}} \cdot \frac{\partial P}{\partial \tau}\,, \tag{30}$$

$$\frac{\mathrm{d}\hat{Q}}{du} = -\frac{\varphi(u) + \psi(v)}{\sqrt{\varphi(u) + b}} \cdot \frac{\partial Q}{\partial \tau}\,. \tag{31}$$

From the equations (26), (29), (30), (31) we arrive at the following representations

$$\sqrt{\varphi^*(u^*) + b} = \sqrt{\varphi(u) + b} - \varepsilon \cdot (\varphi + \psi) \frac{\partial P}{\partial \tau} - \varepsilon \cdot (\varphi + \psi) \cdot \frac{\partial R}{\partial \tau} + o(\varepsilon), \quad \varepsilon \to 0, \tag{32}$$

$$\begin{aligned} \sqrt{\psi^*(v^*) - b} = \sqrt{\psi(v) - b} + \varepsilon \cdot (\varphi + \psi) \cdot \frac{\partial P}{\partial \tau} \\ - \varepsilon \cdot (\varphi + \psi) \cdot \frac{\partial R}{\partial \tau} \cdot \frac{\sqrt{\psi(v) - b}}{\sqrt{\varphi(u) + b}} + o(\varepsilon), \quad \varepsilon \to 0\,. \end{aligned} \tag{33}$$

We prove now the following assertion.

Lemma 3. *Let G be coefficient of the first quadratic form of the surface $\bar{X}$ corresponding to the function $\varphi(u) + \psi(v)$ and G^* — the similar coefficient for the surface $\bar{X}^*$corresponding to the function $\varphi^*(u^*) + \psi^*(v^*)$ constructed before. Then we have the following representation*

$$\begin{aligned} \sqrt{G^*} - \sqrt{G} = \varepsilon \left[\varphi(u) + \psi(v) \right] \cdot \Bigg[- \sqrt{\psi - b} \cdot \frac{\partial P}{\partial \tau} + \sqrt{\varphi + b} \cdot \frac{\partial P}{\partial \tau} \\ - 2 \cdot \sqrt{\psi(v) - b} \cdot \frac{\partial R}{\partial \tau} \Bigg] + o(\varepsilon), \quad \varepsilon \to 0\,, \end{aligned} \tag{34}$$

and

$$\begin{aligned} g^*(t, \tau) - g(t, \tau) = \left(\sqrt{G^*} \right)_{\tau^*} - \left(\sqrt{G} \right)_{\tau} = 2 \cdot g \cdot \varepsilon \cdot \Bigg\{ - \sqrt{\varphi + b} \cdot \frac{\partial P}{\partial \tau} - \frac{\varphi + \psi}{\sqrt{\varphi + b}} \cdot \frac{\partial R}{\partial \tau} \\ + \sqrt{\psi - b} \cdot \frac{\partial Q}{\partial \tau} \Bigg\} + \varepsilon \cdot \Bigg\{ (\varphi + \psi) \cdot \Bigg[- \sqrt{\psi(v) - b} \cdot \frac{\partial P}{\partial \tau} - 2 \cdot \sqrt{\psi - b} \cdot \frac{\partial R}{\partial \tau} \\ + \sqrt{\varphi + b} \cdot \frac{\partial Q}{\partial \tau} \Bigg] \Bigg\}_{\tau} + o(\varepsilon), \quad \varepsilon \to 0\,. \end{aligned} \tag{35}$$

Here

$$\tau^* = -\int\limits_0^{u^*} \sqrt{\varphi^*(x) + b} \cdot \mathrm{d}x + \int\limits_0^{v^*} \sqrt{\psi^*(y) - b} \cdot \mathrm{d}y\,. \tag{36}$$

Proof. The formula (34) follows immediately from the representation

$$G^* = \sqrt{\varphi^*(u^*) + b} \cdot \sqrt{\psi^*(v^*) - b}$$

and from the equations (32), (33). The transformation (24), (25) is a topological one. The same can be said about correspondence $(t, \tau) \to (t^*, \tau^*)$. Then from equation (36) we get

$$\frac{\mathrm{d}\tau}{\mathrm{d}\tau^*} = 1 + 2 \cdot \varepsilon \cdot \Bigg[\sqrt{\varphi + b} \cdot \frac{\partial P}{\partial \tau} + \frac{\varphi(u) + \psi(v)}{2 \cdot \sqrt{\varphi(u) + b}} \cdot \frac{\partial R}{\partial \tau} - \sqrt{\psi(v) - b} \cdot \frac{\partial Q}{\partial \tau}\Bigg] + o(\varepsilon), \quad \varepsilon \to 0\,. \quad (37)$$

Combining the equation (37) with the representation (34) we get that (35) is valid.

The lemma is proved. □

7. Proof of Theorem 1

First of all let us prove the following lemma.

Lemma 4. *Let ς be an arbitrary finite and differentiable function given in the neighborhood Δ of the point $(u_0, v_0) \in \Pi$ such that $g(u, v) \neq 1$ on Δ. Then there exist functions P, Q, R differentiable in the domain Δ such that the following equalities take place almost everywhere in the domain Δ*

$$\Bigg\{(\varphi + \psi) \cdot \Bigg[- \sqrt{\psi(v) - b} \cdot \frac{\partial P}{\partial \tau} - 2 \cdot \sqrt{\psi - b} \cdot \frac{\partial R}{\partial \tau} + \sqrt{\varphi + b} \cdot \frac{\partial Q}{\partial \tau}\Bigg]\Bigg\}_\tau + 2 \cdot g \cdot \Bigg\{- \sqrt{\varphi + b} \cdot \frac{\partial P}{\partial \tau} - \frac{\varphi + \psi}{\sqrt{\varphi + b}} \cdot \frac{\partial R}{\partial \tau} + \sqrt{\psi - b} \cdot \frac{\partial Q}{\partial \tau}\Bigg\} = -g \cdot \sqrt{1 - g^2} \cdot \frac{\partial \varsigma}{\partial \tau}\,, \quad (38)$$

$$2 \cdot \Bigg\{- \sqrt{\varphi + b} \cdot \frac{\partial P}{\partial \tau} - \frac{\varphi + \psi}{\sqrt{\varphi + b}} \cdot \frac{\partial R}{\partial \tau} + \sqrt{\psi - b} \cdot \frac{\partial Q}{\partial \tau}\Bigg\} = \sqrt{1 - g^2} \cdot \frac{\partial \varsigma}{\partial \tau}\,, \quad (39)$$

$$(\varphi + \psi) \cdot \Bigg[- \sqrt{\psi(v) - b} \cdot \frac{\partial P}{\partial \tau} - 2 \cdot \sqrt{\psi - b} \cdot \frac{\partial R}{\partial \tau} + \sqrt{\varphi + b} \cdot \frac{\partial Q}{\partial \tau}\Bigg] + 2 \cdot \sqrt{G} \cdot \Bigg[-\sqrt{\varphi + b} \cdot \frac{\partial P}{\partial \tau} - \frac{\varphi + \psi}{\sqrt{\varphi + b}} \cdot \frac{\partial R}{\partial \tau} + \sqrt{\psi - b} \cdot \frac{\partial Q}{\partial \tau}\Bigg] = \varsigma \cdot g \cdot \sqrt{G} \cdot H\,. \quad (40)$$

Proof. Let A and B be the following expressions

$$A := (\varphi+\psi)\cdot\left[-\sqrt{\psi(v)-b}\cdot\frac{\partial P}{\partial\tau}-2\cdot\sqrt{\psi-b}\cdot\frac{\partial R}{\partial\tau}+\sqrt{\varphi+b}\cdot\frac{\partial Q}{\partial\tau}\right],$$

$$B := -\sqrt{\varphi+b}\cdot\frac{\partial P}{\partial\tau}-\frac{\varphi+\psi}{\sqrt{\varphi+b}}\cdot\frac{\partial R}{\partial\tau}+\sqrt{\psi-b}\cdot\frac{\partial Q}{\partial\tau}.$$

Then the system (38)-(40) can be written in the following form

$$A_\tau(t,\tau)-2g\cdot B=-g\cdot\sqrt{1-g^2}\cdot\frac{\partial\varsigma}{\partial\tau}, \tag{41}$$

$$2\cdot B=\sqrt{1-g^2}\cdot\frac{\partial\varsigma}{\partial\tau}, \tag{42}$$

$$A+2\cdot\sqrt{G}\cdot B=\varsigma\cdot g\cdot\sqrt{G}\cdot H. \tag{43}$$

From the equations (42), (43) we can express the functions $\frac{\partial P}{\partial\tau}$, $\frac{\partial Q}{\partial\tau}$ in terms of the function $\frac{\partial R}{\partial\tau}$. Substituting these relations into the equation (41) we get the second order ordinary differentiable equations for the function $R(.,\tau)$ on the geodesics corresponding to almost all values of t. Solving these equations we get the functions P, Q, R we need.

The lemma is proved. □

Proof of Theorem 1. From Lemma 2 it follows that the function g is continuous on Π. Let us consider for the first the set $\{g=1\}$. In the interior point of this set (if any exists) the extremal surface $\bar{X}_e$ has Gaussian curvature equal to zero almost everywhere. This means that the functional K_1 does not influence the values of the functional Ξ over this set. It follows that we can substitute a part of the surface $\bar{X}_e$ by the plane set conjugating it with the rest of the surface to guarantee the continuity of the functions φ, ψ. It implies that the surface $\bar{X}_e$ is the minimal surface over the interior part of the set $\{g=1\}$. It is evident that the condition (9) is satisfied almost everywhere over this set. Now let us consider the points of the open set $\{g<1\}$. Let $\bar{X}^*$ be variation of the surface $\bar{X}_e$ in the class of Liouville surfaces. From Lemma 4 it follows for the differentiable function ς with support contained in the set Δ that we can find the functions P, Q, R satisfying the system of the equations (38)-(40). The variation $\bar{X}^*$of the surface $\bar{X}_e$ in this case belongs to the class of Liouville surfaces for the values of ε sufficiently small.

Let $[-T,T]$ be the projection of the image of the domain Π in semi-geodesic coordinate's onto axis t for the surface under consideration.

Now we evidently have the following equalities

$$
\begin{aligned}
K_1(\bar{X}) - K_1(\bar{X}_e) &= \int\limits_{-T}^{T} dt \int\limits_{\Gamma_t^*} f_1\left(g^*(t,\tau^*)\right) \mathrm{d}\tau^* - \int\limits_{-T}^{T} dt \int\limits_{\Gamma_t} f_1\left(g(t,\tau)\right) \mathrm{d}\tau \\
&= \int\limits_{-T}^{T} dt \int\limits_{\Gamma_t^*} f_{1g}\left[g^*(t,\tau^*) - g(t,\tau)\right] \cdot \mathrm{d}\tau^* - \int\limits_{-T}^{T} \mathrm{d}t \int\limits_{\Gamma_t^*} f_1\left(g\left(t,\tau\right)\right) \mathrm{d}\tau^* \\
&\qquad - \int\limits_{-T}^{T} \mathrm{d}t \int\limits_{\Gamma_t} f_1(g(t,\tau))\mathrm{d}\tau + o(\varepsilon) \\
= \varepsilon \cdot \int\limits_{-T}^{T} \mathrm{d}t \int\limits_{\Gamma_t} &\left[-f_{1gg} \cdot g \cdot \sqrt{1-g^2} + \frac{g^2}{\sqrt{1-g^2}} \cdot f_{1g} - \frac{g \cdot f_1}{\sqrt{1-g^2}}\right] \cdot g_\tau \cdot \varsigma(t,\tau) \cdot d\tau \\
&+ o(\varepsilon) = \varepsilon \cdot \int\limits_{-T}^{T} \mathrm{d}t \int\limits_{\Gamma_t} g \cdot g_\tau \cdot \varsigma(t,\tau) \cdot \mathrm{d}\tau + o(\varepsilon), \quad \varepsilon \to 0\,. \qquad (44)
\end{aligned}
$$

Now we proceed with calculation of the variation $\delta A(\bar{X}_e)$ of the area of the surface $\bar{X}_e$. We see that

$$
\begin{aligned}
\delta A(\bar{X}_e) &= \int\limits_{-T}^{T} \mathrm{d}t \int\limits_{\Gamma_t^*} \sqrt{G^*} \cdot \mathrm{d}\tau^* - \int\limits_{-T}^{T} \mathrm{d}t \int\limits_{\Gamma_t} \sqrt{G} \cdot \mathrm{d}\tau \\
&= \varepsilon \cdot \int\limits_{-T}^{T} \int\limits_{\Gamma_t} g \cdot \sqrt{G} \cdot H \cdot \varsigma \cdot \mathrm{d}t \cdot \mathrm{d}\tau + o(\varepsilon), \quad \varepsilon \to 0. \qquad (45)
\end{aligned}
$$

Using the formulas (44), (45) we arrive at the necessary condition for $\bar{X}_e$ to be extremal

$$
\int\limits_{-T}^{T} \mathrm{d}t \int\limits_{\Gamma_t} (H + \theta \cdot K) \cdot g \cdot \sqrt{G} \cdot \varsigma \cdot d\tau = 0.
$$

It means that the condition (9) is satisfied almost everywhere on the set $\{g < 1\}$.

Now the function $f_1(g)$ can be written in the form

$$
f_1(g) = \frac{1}{2} \cdot \Bigg\{ - \sqrt{1-g^2} \cdot \int\limits_0^g \left(\arcsin\sigma + \sigma \cdot \sqrt{1-\sigma^2} - \frac{\pi}{2}\right) \cdot \left(1-\sigma^2\right)^{-\frac{3}{2}} \cdot d\sigma
$$

$$+ E_0 \cdot \sqrt{1-g^2}\Big\}.$$

It implies that on the set $\{g = 0\}$ the surface $\bar{X}_e$ is a minimal one and $H = K = 0$ on this set.

The theorem is proved. $\square$

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