SCROLLAR INVARIANTS OF PENCILS ON BINARY CURVES

E. Ballico

Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here for all positive integers \( k_1, k_2, g \) such that \( g \geq 2(k_1 + k_2 - 1) \) we prove the existence of a binary curve \( X \) and a line bundle \( L \) on \( X \) with multidegree \((k_1, k_2)\) and expected scrollar invariants, i.e. with \( h^0(X, L^c) = c + 1 \) for all \( c \) such that \( 1 \leq c \leq \lfloor g/(k_1 + k_2 - 1) \rfloor \).

AMS Subject Classification: 14H51, 14H10, 14H20
Key Words: binary curve, scrollar invariant

1. Introduction

Let \( X \) be a reduced projective curve and \( L \in \text{Pic}(X) \). The knowledge of the scrollar invariants of \( L \) in the classical sense is essentially equivalent to the knowledge of the sequence \( \{h^0(X, L^c)\}_{c>1} \). In the classical case \( h^0(Y, L) = 2 \). In this case it is very important to know for which integers \( c \geq 2 \) we have \( h^0(Y, L^c) = c + 1 \) (see [1], [3], [4]). Here we consider the case of a binary curve \( X \) of genus \( g \), i.e. \( X \) is nodal, \( X = X_1 \cup X_2, X_1 \cong X_2 \cong \mathbb{P}^1 \) and \#(X_1 \cap X_2) = g+1 (see [2]). Each line bundle on \( X \) has a multidegree \((\deg(L|X_1), \deg(L|X_2))\). As in [2] we only consider line bundles with sections and multidegree \( k_1 > 0, k_2 > 0 \). We first prove the following result.

Theorem 1. Fix positive integers \( k_1, k_2, c, g \). There are a binary curve \( X = X_1 \cup X_2 \) and \( L \in \text{Pic}(X) \) such that \( \deg(L|X_i) = k_i, i = 1, 2 \), and \( h^0(X, L^c) = c + 1 \) if and only if \( c \leq \lfloor g/(k_1 + k_2 - 1) \rfloor \).
The geometrical interpretation of the scrollar invariants of a line bundle requires that the line bundles induces a finite morphism \( X \to \mathbb{P}^1 \). In the spanned case we are only able to prove the following result.

**Theorem 2.** Fix integers \( k \geq 3 \) and \( g \geq 2k \). There are a binary curve \( X = X_1 \cup X_2 \) of genus \( g \) and \( L \in \text{Pic}(X) \) of multidegree \( (1, k - 1) \) such that \( L \) is spanned and \( h^0(X, L^{\otimes c}) = c + 1 \) for all positive integers \( c \leq \lfloor g/(k - 1) \rfloor \).

See [2], Lemma 21 and Proposition 22, for much more in the case \( k_1 = k_2 = 1 \), i.e. the case of hyperelliptic binary curves.

### 2. The Proofs

**Lemma 1.** Let \( Y \) be a reduced and projective curve. Let \( L \) be a spanned line bundle on \( Y \) such that \( L \neq \emptyset \). Then \( h^0(X, L^{\otimes c}) \geq c + 1 \) for all integers \( c \geq 1 \).

**Proof.** Since \( \dim(X) = 1 \) and \( L \) is a spanned line bundle and its not trivial \( h^0(X, L) \geq 2 \). Since \( \dim(X) = 1 \) and \( L \) is a spanned line bundle, there is a linear subspace \( V \subseteq H^0(X, L) \) such that \( V \) spans \( L \) and \( \dim(V) = 2 \). Thus \( V \) induces a surjective morphism \( f : X \to \mathbb{P}^1 \). Fix any irreducible component \( T \) of \( X \) such that \( f(T) = \mathbb{P}^1 \). Notice that \( \dim(S^c(V)) = c + 1 \) (symmetric product). It is sufficient to prove that the evaluation map \( \rho : S^c(V) \to H^0(X, L^{\otimes c}) \) has image of dimension \( c + 1 \). Let \( \beta : H^0(X, L^{\otimes c}) \to H^0(T, L^{\otimes c}|T) \) be the restriction map. Since \( f(T) = \mathbb{P}^1 \), the restriction map \( \eta : V \to H^0(T, L|T) \) is injective. Since the lemma is obvious for the integral curve \( T \), the map \( \alpha : S^c(\eta(V)) \to H^0(T, L^{\otimes c}|T) \) has image of dimension \( c + 1 \). Thus \( \beta \circ \rho \) has image of dimension \( c + 1 \). \( \square \)

**Lemma 2.** Let \( X \) be a stable curve of genus \( g \) such that there is an ample and spanned degree 2 line bundle \( R \) on \( X \) such that \( R^{\otimes (g-1)} \cong \omega_X \) and \( h^0(X, R) = 2 \). Assume that the natural map \( \eta : S^{(g-1)}(H^0(X, R)) \to H^0(X, \omega_X) \) (symmetric product) is surjective, i.e. assume that the canonical map factors through the morphism \( f : X \to \mathbb{P}^1 \) induced by \(|D|\). Fix integers \( a \geq 0, b \geq 0 \) such that \( 2a + b \leq 2g - 2 \) and an effective degree \( b \) divisor \( D \) of \( X \) supported by \( X_{\text{reg}} \) and such that no point appearing with multiplicity of \( \geq 2 \) in \( D \) is a ramification point of \( D \) and no two points in the same fiber of \( f \) appears in \( D \). Then \( h^0(X, R^{\otimes a}(D)) = a + 1 \).

**Proof.** Since \( h^0(X, \omega_X) = g \), the surjectivity of \( \eta \) is equivalent its injectivity.
By Riemann-Roch we need to prove \( h^0(X, R^{g-1-a})(-D)) = \max\{0, g-a-b\}. \) We know that \( R \) is spanned and that the natural map \( S^{(g-1-a)}(H^0(X, R)) \to H^0(R^{g-1-a}) \) is bijective. Hence it is sufficient to use that \( f(D) \) is an effective degree \( b \) divisor of \( \mathbb{P}^1 \) seen as embedded as a rational normal curve of \( \mathbb{P}^{g-1-a} \) and use that any effective divisor of degree \( x \leq r+1 \) of a rational normal curve of \( \mathbb{P}^r \) spans an \((x-1)\)-dimensional linear space.

Since \( h^0(X, \omega_X) = g \), the surjectivity of \( \eta \) is equivalent its injectivity. It would be easy to modify the statement of Lemma 2 assuming the injectivity of \( \eta \), but dropping the condition \( h^0(X, R) = 2 \). However, this generalization would be illusory.

As an immediate consequence of Lemma 2 we get the following results (in the third one we may even drop that \( X \) is nodal).

**Corollary 1.** Let \( X = X_1 \cup X_2 \) be a hyperelliptic binary curve of genus \( g \). Let \( R \) be the hyperelliptic line bundle and \( f : X \to \mathbb{P}^1 \) be the degree 2 associated covering. Fix non-negative integers \( a, b \) such that \( 2a + b \leq 2g - 2 \). Take an effective degree \( b \) divisor \( D \subset X_{reg} \) such that no two points in the support of \( D \) are in the same fiber of \( f \). Then \( h^0(X, R^{\otimes b}(D)) = a + 1 \).

**Corollary 2.** Let \( X = X_1 \cup X_2 \) be a hyperelliptic binary curve of genus \( g \) and \( R \) its hyperelliptic line bundle. Fix integers \( k_1, k_2, c \) such that \( 2 \leq k_1 \leq g-1 \), \( 2 \leq k_2 \leq g-1 \) and \( 1 \leq c \leq |g/(k_1 + k_2 - 1)| \). Fix a general \( S_i \subset X_i \) such that \( \sharp(S_i) = k_i - 2 \) and set \( L := R(S_1 \cup S_2) \). Then \( h^0(X, L^{\otimes c}) = c + 1 \).

**Corollary 3.** Let \( X \) be an integral genus \( g \geq 2 \) Gorenstein curve such that there is a degree 2 morphism \( f : X \to \mathbb{P}^1 \). Fix integers \( c, k \) such that \( k \geq 2 \) and \( 1 \leq c \leq |g/(k-1)| \). Let \( S \subset X \) be a general subset such that \( \sharp(S) = k - 2 \). Set \( L := R(S) \). Then \( h^0(X, L^{\otimes c}) = c + 1 \).

**Proof of Theorem 1.** Set \( k := k_1 + k_2 \). Assume the existence of \((X, L)\) with \( p_a(X) = g \), \( L \) spanned and \( \deg(L) = k \). Lemma 1 gives \( h^0(X, L^{\otimes c}) \geq c + 1 \). Riemann-Roch gives \( \chi(L^{\otimes c}) = ck + 1 - g \). Notice that \( ck + 1 - g \leq c + 1 \) if and only \( c \leq |g/(k-1)| \). Hence the “only if” part is obvious. Now we prove the “if” part. We take a binary hyperelliptic curve and apply Corollary 2. \( \square \)

**Lemma 3.** Fix positive integers \( k, a, c \) such that \( c \leq a - 2 \), a reduced curve \( Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \) of type \((k, a)\) and \( S \subset \text{Sing}(Y) \) such that \( Y \) has an ordinary node at each point of \( S \). Let \( u : X \to Y \) be the partial normalization of \( Y \) in which we normalize only the points of \( S \). Set \( L := u^*(O_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)) \).

(a) \( L \) is a degree \( k \) spanned line bundle.

(b) \( h^0(X, L^{\otimes c}) = c + 1 + h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{I}_S(k-2, a-2-c)) \).
Proof. Part (a) is obvious. We have $\omega_{\mathbb{P}^1 \times \mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2)$. Hence the adjunction formula gives $\omega_Y \cong \mathcal{O}(k-2, a-2)$. Since $h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2)) = 0$ (use Kn"unneth). The restriction map $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k-2, a-2)) \to H^0(Y, \omega_Y)$ is surjective. The adjunction theory of nodal singularities gives part (b).

Proof of Theorem 2. Let $a$ be the minimal integer such that $(k-1)(a-1) \geq g$. Hence
\[ g \leq (k-1)(a-1) \leq g + k - 2 \quad (1) \]
Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a general union of a general curve $Y_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ of type $(1, a-1)$ and a general curve $Y_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ of type $(k-1, a)$. Hence $Y_1 \cong Y_2 \cong \mathbb{P}^1$, $Y$ is nodal and $\sharp(Y_1 \cap Y_2) = (k-1)(a-1) + 1$. Fix $S \subset Y_1 \cap Y_2$ such that $\sharp(S) = (k-1)(a-1) - g$. The inequality (1) gives $0 \leq \sharp(S) \leq k - 2$. Let $u : X \to Y$ be the partial normalization of $Y$ in which we normalize only the points of $S$. Call $X_i$, $i = 1, 2$, the irreducible component of $X$ such that $u(X_i) = Y_i$. Set $L := u^*(\mathcal{O}_Y(0, 1))$. $L$ has multidegree $(1, k-1)$ and it is spanned. Since $h^0(Y, \mathcal{O}_Y(0, c)) = c + 1$ for all $c \leq a - 2$, it is sufficient to find $S$ such that $h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{I}_S(k-2, 0)) = 0$. This is possible, because the restriction to $S$ of each projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is injective.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References