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SCROLLAR INVARIANTS OF PENCILS ON BINARY CURVES

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Abstract: Here for all positive integers k_1, k_2, g such that $g \geq 2(k_1 + k_2 - 1)$ we prove the existence of a binary curve X and a line bundle L on X with multidegree (k_1, k_2) and expected scrollar invariants, i.e. with $h^0(X, L^{\otimes c}) = c + 1$ for all c such that $1 \leq c \leq \lfloor g/(k_1 + k_2 - 1)\rfloor$.

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1. Introduction

Let X be a reduced projective curve and $L \in \mathrm{Pic}(X)$. The knowledge of the *scrollar invariants* of L in the classical sense is essentially equivalent to the knowledge of the sequence $\{h^0(X, L^{\otimes c})\}_{c\geq 1}$. In the classical case $h^0(Y, L) = 2$. In this case it is very important to know for which integers $c \geq 2$ we have $h^0(Y, L^{\otimes c}) = c + 1$ (see [1], [3], [4]). Here we consider the case of a binary curve X of genus g, i.e. X is nodal, $X = X_1 \cup X_2$, $X_1 \cong X_2 \cong \mathbb{P}^1$ and $\sharp(X_1 \cap X_2) = g+1$ (see [2]). Each line bundle on X has a multidegree $(\deg(L|X_1), \deg(L|X_2))$. As in [2] we only consider line bundles with sections and multidegress $k_1 > 0$, $k_2 > 0$. We first prove the following result.

Theorem 1. *Fix positive integers k_1, k_2, c, g. There are a binary curve $X = X_1 \cup X_2$ and $L \in Pic(X)$ such that $\deg(L|X_i) = k_i$, $i = 1, 2$, and $h^0(X, L^{\otimes c}) = c + 1$ if and only if $c \leq \lfloor g/(k - 1)\rfloor$.*

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The geometrical interpretation of the scrollar invariants of a line bundle requires that the line bundles induces a finite morphism $X \to \mathbb{P}^1$. In the spanned case we are only able to prove the following result.

Theorem 2. *Fix integers $k \geq 3$ and $g \geq 2k$. There are a binary curve $X = X_1 \cup X_2$ of genus g and $L \in Pic(X)$ of multidegree $(1, k-1)$ such that L is spanned and $h^0(X, L^{\otimes c}) = c+1$ for all positive integers $c \leq \lfloor g/(k-1) \rfloor$.*

See [2], Lemma 21 and Proposition 22, for much more in the case $k_1 = k_2 = 1$, i.e. the case of hyperelliptic binary curves.

2. The Proofs

Lemma 1. *Let Y be a reduced and projective curve. Let L be a spanned line bundle on Y such that $L \neq \emptyset$. Then $h^0(X, L^{\otimes c}) \geq c+1$ for all integers $c \geq 1$.*

Proof. Since $\dim(X) = 1$ and L is a spanned line bundle and its not trivial $h^0(X, L) \geq 2$, Since $\dim(X) = 1$ and L is a spanned line bundle, there is a linear subspace $V \subseteq H^0(X, L)$ such that V spans L and $\dim(V) = 2$. Thus V induces a surjective morphim $f : X \to \mathbb{P}^1$. Fix any irreducible component T of X such that $f(T) = \mathbb{P}^1$. Notice that $\dim(S^c(V)) = c+1$ (symmetric product). It is sufficient to prove that the evaluation map $\rho : S^c(V) \to H^0(X, L^{\otimes c})$ has image of dimension $c+1$. Let $\beta : H^0(X, L^{\otimes c}) \to H^0(T, L^{\otimes c}|T)$ be the restriction map. Since $f(T) = \mathbb{P}^1$, the restriction map $\eta : V \to H^0(T, L|T)$ is injective. Since the lemma is obvious for the integral curve T, the map $\alpha : S^c(\eta(V)) \to H^0(T, L^{\otimes c}|T)$ has image of dimension $c+1$. Thus $\beta \circ \rho$ has image of dimension $c+1$. □

Lemma 2. *Let X be a stable curve of genus g such that there is an ample and spanned degree 2 line bundle R on X such that $R^{\otimes(g-1)} \cong \omega_X$ and $h^0(X, R) = 2$. Assume that the natural map $\eta : S^{(g-1)}(H^0(X, R)) \to H^0(X, \omega_X)$ (symmetric product) is surjective, i.e. assume that the canonical map factors through the morphism $f : X \to \mathbb{P}^1$ induced by $|D|$. Fix integers $a \geq 0$, $b \geq 0$ such that $2a + b \leq 2g - 2$ and an effective degree b divisor D of X supported by X_{reg} and such that no point appearing with multiplicity of ≥ 2 in D is a ramification point of D and no two points in the same fiber of f appears in D. Then $h^0(X, R^{\otimes a}(D)) = a+1$.*

Proof. Since $h^0(X, \omega_X) = g$, the surjectivity of η is equivalent its injectivity.

By Riemann-Roch we need to prove $h^0(X, R^{\otimes(g-1-a)}(-D)) = \max\{0, g-a-b\}$. We know that R is spanned and that the natural map $S^{(g-1-a)}(H^0(X,R)) \to H^0(R^{\otimes(g-1-a)})$ is bijective. Hence it is sufficient to use that $f(D)$ is an effective degree b divisor of $\mathbb{P}^1$ seen as embedded as a rational normal curve of $\mathbb{P}^{(g-1-a)}$ and use that any effective divisor of degree $x \le r+1$ of a rational normal curve of $\mathbb{P}^r$ spans an $(x-1)$-dimensional linear space. □

Since $h^0(X, \omega_X) = g$, the surjectivity of η is equivalent its injectivity. It would be easy to modify the statement of Lemma 2 assuming the injectivity of η, but dropping the condition $h^0(X,R) = 2$. However, this generalization would be illusory.

As an immediate consequence of Lemma 2 we get the following results (in the third one we may even drop that X is nodal).

Corollary 1. *Let $X = X_1 \cup X_2$ be a hyperelliptic binary curve of genus g. Let R be the hyperelliptic line bundle and $f : X \to \mathbb{P}^1$ be the degree 2 associated covering. Fix non-negative integers a, b such that $2a + b \le 2g - 2$. Take an effective degree b divisor $D \subset X_{reg}$ such that no two points in the support of D are in the same fiber of f. Then $h^0(X, R^{\otimes b}(D)) = a + 1$.*

Corollary 2. *Let $X = X_1 \cup X_2$ be a hyperelliptic binary curve of genus g and R its hyperelliptic line bundle. Fix integers k_1, k_2, c such that $2 \le k_1 \le g-1$, $2 \le k_2 \le g-1$ and $1 \le c \le \lfloor g/(k_1+k_2-1)\rfloor$. Fix a general $S_i \subset X_i$ such that $\sharp(S_i) = k_i - 2$ and set $L := R(S_1 \cup S_2)$. Then $h^0(X, L^{\otimes c}) = c + 1$.*

Corollary 3. *Let X be an integral genus $g \ge 2$ Gorenstein curve such that there is a degree 2 morphism $f : X \to \mathbb{P}^1$. Fix integers c, k such that $k \ge 2$ and $1 \le c \le \lfloor g/(k-1)\rfloor$. Let $S \subset X$ be a general subset such that $\sharp(S) = k-2$. Set $L := R(S)$. Then $h^0(X, L^{\otimes c}) = c + 1$.*

Proof of Theorem 1. Set $k := k_1 + k_2$. Assume the existence of (X, L) with $p_a(X) = g$, L spanned and $\deg(L) = k$. Lemma 1 gives $h^0(X, L^{\otimes c}) \ge c + 1$. Riemann-Roch gives $\chi(L^{\otimes c}) = ck + 1 - g$. Notice that $ck + 1 - g \le c + 1$ if and only $c \le \lfloor g/(k-1)\rfloor$. Hence the "only if" part is obvious. Now we prove the "if" part. We take a binary hyperelliptic curve and apply Corollary 2. □

Lemma 3. *Fix positive integers k, a, c such that $c \le a - 2$, a reduced curve $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ of type (k, a) and $S \subseteq Sing(Y)$ such that Y has an ordinary node at each point of S. Let $u : X \to Y$ be the partial normalization of Y in which we normalize only the points of S. Set $L := u^*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1))$.*

(a) L is a degree k spanned line bundle.

(b) $h^0(X, L^{\otimes c}) = c + 1 + h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{I}_S(k-2, a-2-c))$.

Proof. Part (a) is obvious. We have $\omega_{\mathbb{P}^1\times\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2)$. Hence the adjunction formula gives $\omega_Y \cong \mathcal{O}(k-2, a-2)$. Since $h^1(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2)) = 0$ (use Knünneth). The restriction map $H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k-2, a-2) \to H^0(Y, \omega_Y)$ is surjective. The adjunction theory of nodal singularities gives part (b). □

Proof of Theorem 2. Let a be the minimal integer such that $(k-1)(a-1) \geq g$. Hence

$$g \leq (k-1)(a-1) \leq g + k - 2 \tag{1}$$

Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a general union of a general curve $Y_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ of type $(1, a-1)$ and a general curve $Y_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$of type $(k-1, a)$. Hence $Y_1 \cong Y_2 \cong \mathbb{P}^1$, Y is nodal and $\sharp(Y_1 \cap Y_2) = (k-1)(a-1)+1$. Fix $S \subset Y_1 \cap Y_2$ such that $\sharp(S) = (k-1)(a-1) - g$. The inequality (1) gives $0 \leq \sharp(S) \leq k-2$. Let $u : X \to Y$ be the partial normalization of Y in which we normalize only the points of S. Call X_i, $i = 1, 2$, the irreducible component of X such that $u(X_i) = Y_i$. Set $L := u^*(\mathcal{O}_Y(0,1))$. L has multidegree $(1, k-1)$ and it is spanned. Since $h^0(Y, \mathcal{O}_Y(0, c)) = c+1$ for all $c \leq a-2$, it is sufficient to find S such that $h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{I}_S(k-2, 0)) = 0$. This is possible, because the restriction to S of each projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is injective. □

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References

[1] E. Ballico, A remark on linear series on general k-gonal curves, *Boll. U.M.I.*, **7**, No. 3-A (1989), 195-197.

[2] L. Caporaso, Brill-Noether theory of binary curves, *ArXiv:* math/0807.1484.

[3] M. Coppens, G. Martens, Linear series on 4-gonal curves, *Math. Nachr.*, **213**, No. 1 (2000), 35-55.

[4] M. Coppens, G. Martens, Linear series on a general k-gonal curve, *Abhandl. Math. Sem. Univ. Hamburg*, **69** (1999), 347-371.