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ON THE MULTIPLICATION MAP FOR RANK 1 SHEAVES ON NODAL CURVES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we consider the multiplication map for depth 1 sheaves with pure rank 1 on curves with only ordinary nodes or ordinary cusps as singularities.

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1. Introduction

Let X be a projective curve and F a depth 1 sheaf on X with pure rank 1. Here we look at the multiplication map $H^0(X, F) \otimes H^0(X, F) \to H^0(X, F^{\otimes 2})$. Of course, $\mathrm{Im}(\mu_F)$ is the image of the symmetric multiplication map $S^2(H^0(X, F)) \to H^0(X, F \otimes F)$. When X is a smooth curve, then F is a line bundle. In this case the surjectivity of μ_F is a classical problem related to projectively normal curves ([1], [3] and references quoted there or quoting these papers). There are extensions of the classical case to the case in which X is singular (and in some extensions X is allowed to be reducible). Here we show that the general case may be reduced to this case for sheaves which are locally free at each point of X which is not an ordinary node or an ordinary cusp. Here we prove the following result.

Theorem 1. *Let X be a reduced projective curve and F a depth 1 sheaf on X with pure rank 1. Set $S := Sing(F)$ and assume that X has*

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an ordinary node or an ordinary cusp at each point of S. *Let* $v : C \to X$ *be the partial normalization of* X *in which we only normalize the points of* S. *Set* $L := v^*(F)/Tors(v^*(L))$. *Then* $L \in Pic(C)$. *Let* $\mu_F : H^0(X,F) \otimes H^0(X,F) \to H^0(X,F \otimes F)$ *and* $\mu_L : H^0(C,L) \otimes H^0(C,L) \to H^0(C,L^{\otimes 2})$ *denote the multiplication maps. Let* $a_F : H^0(X,F \otimes F) \to H^0(X, F \otimes F/\mathrm{Tors}(F \otimes F))$ *be the natural map. Set* $\sigma_F : a_F \circ \mu_F$. *Then:*

(a) a_F *is surjective,* $\mathrm{rank}(\mu_L) = \mathrm{rank}(a_F)$ *and* $corank(\mu_L) = corank(a_F)$.

(b) *If* μ_F *is surjective, then* μ_L *is surjective.*

(c) *If* μ_L *is injective, then* μ_F *is injective.*

(d) *If* μ_L *is surjective and* $h^1(X, \mathcal{I}_S \otimes F) = h^1(X,F)$, *then* μ_F *is surjective.*

(e) *If* $\sharp(S) = 1$, F *is spanned and* μ_L *is surjective, then* μ_F *is surjective.*

2. The Proof

Lemma 1. *Let* X *be a reduced projective curve. Fix* $S \subseteq Sing(X)$ *and assume that each point of* S *is either an ordinary node or an ordinary cusp of* X. *Let* $v : C \to X$ *be the partial normalization of* X *in which we only normalize the points of* S. *We have* $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) + \sharp(S)$.

(a) *For any coherent sheaf* G *on* C *with depth* 1 *and pure rank* 1 *the sheaf* $v_*(G)$ *has depth* 1, *pure rank* 1, $\deg(v_*(G)) = \deg(G) + \sharp(S)$, *and* $h^i(X, v_*(G)) = h^i(X,G)$, $i = 1,2$.

(b) *For any depth* 1 *sheaf on* X *with pure rank* 1 *set* $F_S := v^*(F)/\mathrm{Tors}(v^*(F))$. *The sheaf* F_S *has depth* 1, *pure rank* 1, $Sing(F_S) = v^{-1}(Sing(F) \backslash S)$, $\deg(F_S) = \deg(F) - \sharp(S)$, $F \cong v_*(F_S)$ *and* $h^i(X,F) = h^i(C,F_S)$, $i = 0,1$.

(c) *If* F *is spanned, then* F_S *is spanned.*

Proof. Since each point of S is an ordinary node or an ordinary cusp of X, $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) + \sharp(S)$ ([4], pp. 164–166, for an ordinary node, [2] for both cases). Let G be a coherent sheaf G on C with depth 1 and pure rank 1. Since G has depth 1 and v is finite, $v_*(G)$ has no non-zero subsheaf supported by a finite subset of X. Hence $v_*(G)$ has depth 1. It has pure rank 1, because G has pure rank 1 and v is an isomorphism outside a finite subset. Obviously, $h^0(X, v_*(G)) = h^0(C,G)$. Since v is finite, $R^1 v_*(G) = 0$. Hence the Leray spectral sequence of v gives $h^1(X, v_*(G)) = h^1(C,G)$. Applying Riemann-Roch to X and to G we get $\deg(G) - \deg(v_*(G)) = -\chi(\mathcal{O}_C) + \chi(\mathcal{O}_X) = -\sharp(S)$, concluding the proof of part (a). Take F as in (b). By construction F_S has depth 1. Obviously, it has pure rank 1. Since C is locally free at each poin of $v^{-1}(S)$ and $v|C\backslash v^{-1}(S)$ is an isomorphism, $\mathrm{Sing}(F_S) = v^{-1}(\mathrm{Sing}(F)\backslash S)$. Fix any $P \in S$. Since C is smooth at each point of the classification of depth 1 modules with pure rank 1 singularities shows that the germ of F at P is an $\mathcal{O}_{X,P}$-module isomorphic to the maximal ideal of the local ring $\mathcal{O}_X$. Hence a local computation gives $\deg(F_S) = \deg(F) - \sharp(F_S)$ and $F \cong v_*(F_S)$. Hence the remaining assertions of part (b) follow from part (a). Now assume that F is spanned. Since the tensor product is a right exact functor, $v^*(F)$ is spanned. Hence any quotient of $v^*(F)$ is spanned. Since F_S is a quotient of $v^*(F)$, we get part (c). □

Lemma 2. *Let R be the completion of the local ring of an ordinary node or an ordinary cusp and* $\mathbf{m}$ *its maximal ideal. Then:*

(a) $Tor_1^R(\mathbf{m}, \mathbf{m}) \cong \mathbb{K}$.

(b) The multiplication map $\mathbf{m} \otimes \mathbf{m} \to \mathbf{m}^2$ *is surjective and its kernel is a* 1*-dimensional* $\mathbb{K}$*-vector space.*

Proof. Part (a) is a local computation. Part (b) follows from part (a) by tensoring the exact sequence of R-modules

$$0 \to \mathbf{m} \to R \to \mathbb{K} \to 0 \tag{1}$$

with $\mathbf{m}$. □

Remark 1. Let Y be a reduced projective curve and G a coherent sheaf on Y. Look at the exact sequence

$$0 \to \mathrm{Tors}(G) \to G \to G/\mathrm{Tors}(G) \to 0\,. \tag{2}$$

The sheaf $G/\mathrm{Tors}(G)$ has depth 1. Since $\mathrm{Tors}(G)$ is supported by a finite subset of Y, $h^1(Y, \mathrm{Tors}(G)) = 0$. Hence (2) gives $h^0(Y, G) = h^0(Y, \mathrm{Tors}(G)) + h^0(Y, G/\mathrm{Tors}(G))$ and the surjectivity of the natural map

$$H^0(Y, G) \to H^0(Y, G/\mathrm{Tors}(G)).$$

Proof of Theorem 1. The surjectivity of a_F is the last line of Remark 1. Lemma 1 applied to F and to $F \otimes F/\mathrm{Tors}(F \otimes F)$ gives the second assertion of part (a). Part (a) implies parts (b) and (c). Assume the surjectivity of μ_L. Lemma 1 and Remark 2 show that μ_F is surjective if and only if $\mathrm{Im}(\mu_F)$ contains the image Γ of $H^0(X, \mathrm{Tors}(F \otimes F))$. Since $\dim(X) = 1$, $h^1(X, \mathcal{I}_S \otimes F) = h^1(X, F)$ if and only if the restriction map $\rho_{F,S} : H^0(X, F) \to H^0(S, F|S) \cong F|S$ is surjective. Hence (e) is a particular case of (d). Assume the surjectivity of $\rho_{F,S}$. Obviously, the map $\alpha : H^0(S, F|S) \otimes H^0(S, F|S) \to H^0(X, F \otimes F)$ is bijective. Since S is affine, the natural map $\Gamma \to H^0(X, F \otimes F)$ is injective. The surjectivity of $\rho_{F,S}$ implies that $\mathrm{Im}(\mu_F)$ surjects onto $H^0(S, F|S) \to H^0(X, F \otimes F)$. Hence the bijectivity of α gives $\Gamma \subset \mathrm{Im}(\mu_F)$, concluding the proof. $\square$

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