ON THE MULTIPLICATION MAP FOR
RANK 1 SHEAVES ON NODAL CURVES

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Abstract: Here we consider the multiplication map for depth 1 sheaves with
pure rank 1 on curves with only ordinary nodes or ordinary cusps as singulari-

ties.

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1. Introduction

Let $X$ be a projective curve and $F$ a depth 1 sheaf on $X$ with pure rank 1. Here
we look at the multiplication map $H^0(X, F) \otimes H^0(X, F) \to H^0(X, F^{\otimes 2})$. Of
course, $\text{Im}(\mu_F)$ is the image of the symmetric multiplication map $S^2(H^0(X, F))$
$\to H^0(X, F \otimes F)$. When $X$ is a smooth curve, then $F$ is a line bundle. In this
case the surjectivity of $\mu_F$ is a classical problem related to projectively normal
curves ([1], [3] and references quoted there or quoting these papers). There are
extensions of the classical case to the case in which $X$ is singular (and in some
extensions $X$ is allowed to be reducible). Here we show that the general case
may be reduced to this case for sheaves which are locally free at each point
of $X$ which is not an ordinary node or an ordinary cusp. Here we prove the
following result.

Theorem 1. Let $X$ be a reduced projective curve and $F$ a depth 1
sheaf on $X$ with pure rank 1. Set $S := \text{Sing}(F)$ and assume that $X$ has
an ordinary node or an ordinary cusp at each point of $S$. Let $v : C \to X$ be the partial normalization of $X$ in which we only normalize the points of $S$. Let $L := v^*(F)/\text{Tors}(v^*(L))$. Then $L \in \text{Pic}(C)$. Let $\mu_F : H^0(X, F) \otimes H^0(X, F) \to H^0(X, F \otimes F)$ and $\mu_L : H^0(C, L) \otimes H^0(C, L) \to H^0(C, L^\otimes 2)$ denote the multiplication maps. Let $a_F : H^0(X, F \otimes F) \to H^0(X, F \otimes F/\text{Tors}(F \otimes F))$ be the natural map. Set $\sigma_F : a_F \circ \mu_F$. Then:

(a) $a_F$ is surjective, $\text{rank}(\mu_L) = \text{rank}(a_F)$ and $\text{corank}(\mu_L) = \text{corank}(a_F)$.
(b) If $\mu_F$ is surjective, then $\mu_L$ is surjective.
(c) If $\mu_L$ is injective, then $\mu_F$ is injective.
(d) If $\mu_L$ is surjective and $h^1(X, I_S \otimes F) = h^1(X, F)$, then $\mu_F$ is surjective.
(e) If $\sharp(S) = 1$, $F$ is spanned and $\mu_L$ is surjective, then $\mu_F$ is surjective.

2. The Proof

**Lemma 1.** Let $X$ be a reduced projective curve. Fix $S \subseteq \text{Sing}(X)$ and assume that each point of $S$ is either an ordinary node or an ordinary cusp of $X$. Let $v : C \to X$ be the partial normalization of $X$ in which we only normalize the points of $S$. We have $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) + \sharp(S)$.

(a) For any coherent sheaf $G$ on $C$ with depth 1 and pure rank 1 the sheaf $v_*(G)$ has depth 1, pure rank 1, $\text{deg}(v_*(G)) = \text{deg}(G) + \sharp(S)$, and $h^i(X, v_*(G)) = h^i(X, G)$, $i = 1, 2$.

(b) For any depth 1 sheaf on $X$ with pure rank 1 set $F_S := v^*(F)/\text{Tors}(v^*(F))$. The sheaf $F_S$ has depth 1, pure rank 1, $\text{Sing}(F_S) = v^{-1}(\text{Sing}(F)\setminus S)$, $\text{deg}(F_S) = \text{deg}(F) - \sharp(S)$, $F_S \cong v_*(F_S)$ and $h^i(X, F_S) = h^i(C, F_S)$, $i = 0, 1$.

(c) If $F$ is spanned, then $F_S$ is spanned.

**Proof.** Since each point of $S$ is an ordinary node or an ordinary cusp of $X$, $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) + \sharp(S)$ ([4], pp. 164–166, for an ordinary node, [2] for both cases). Let $G$ be a coherent sheaf $G$ on $C$ with depth 1 and pure rank 1. Since $G$ has depth 1 and $v$ is finite, $v_*(G)$ has no non-zero subsheaf supported by a finite subset of $X$. Hence $v_*(G)$ has depth 1. It has pure rank 1, because $G$ has pure rank 1 and $v$ is an isomorphism outside a finite subset. Obviously, $h^0(X, v_*(G)) = h^0(C, G)$. Since $v$ is finite, $R^1v_*(G) = 0$. Hence the Leray spectral sequence of $v$ gives $h^1(X, v_*(G)) = h^1(C, G)$. Applying Riemann-Roch to $X$ and to $G$ we get $\text{deg}(G) - \text{deg}(v_*(G)) = -\chi(\mathcal{O}_C) + \chi(\mathcal{O}_X) = -\sharp(S)$, concluding the proof of part (a). Take $F$ as in (b). By construction $F_S$ has
depth 1. Obviously, it has pure rank 1. Since \( C \) is locally free at each point of \( v^{-1}(S) \) and \( v|C \backslash v^{-1}(S) \) is an isomorphism, \( \text{Sing}(F_S) = v^{-1}(\text{Sing}(F) \backslash S) \). Fix any \( P \in S \). Since \( C \) is smooth at each point of the classification of depth 1 modules with pure rank 1 singularities shows that the germ of \( F \) at \( P \) is an \( \mathcal{O}_{X,P} \)-module isomorphic to the maximal ideal of the local ring \( \mathcal{O}_X \). Hence a local computation gives \( \deg(F_S) = \deg(F) - \sharp(F_S) \) and \( F \cong v_*(F) \). Hence the remaining assertions of part (b) follow from part (a). Now assume that \( F \) is spanned. Since the tensor product is a right exact functor, \( v^*(F) \) is spanned. Hence any quotient of \( v^*(F) \) is spanned. Since \( F_S \) is a quotient of \( v^*(F) \), we get part (c).

Lemma 2. Let \( R \) be the completion of the local ring of an ordinary node or an ordinary cusp and \( \mathfrak{m} \) its maximal ideal. Then:

(a) \( \text{Tor}_1^R(\mathfrak{m}, \mathfrak{m}) \cong \mathbb{K} \).

(b) The multiplication map \( \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}^2 \) is surjective and its kernel is a 1-dimensional \( \mathbb{K} \)-vector space.

Proof. Part (a) is a local computation. Part (b) follows from part (a) by tensoring the exact sequence of \( R \)-modules

\[
0 \to \mathfrak{m} \to R \to \mathbb{K} \to 0 \tag{1}
\]

with \( \mathfrak{m} \).

Remark 1. Let \( Y \) be a reduced projective curve and \( G \) a coherent sheaf on \( Y \). Look at the exact sequence

\[
0 \to \text{Tors}(G) \to G \to G/\text{Tors}(G) \to 0. \tag{2}
\]

The sheaf \( G/\text{Tors}(G) \) has depth 1. Since \( \text{Tors}(G) \) is supported by a finite subset of \( Y \), \( h^1(Y, \text{Tors}(G)) = 0 \). Hence (2) gives \( h^0(Y, G) = h^0(Y, \text{Tors}(G)) + h^0(Y, G/\text{Tors}(G)) \) and the surjectivity of the natural map

\[
H^0(Y, G) \to H^0(Y, G/\text{Tors}(G)).
\]

Proof of Theorem 1. The surjectivity of \( a_F \) is the last line of Remark 1. Lemma 1 applied to \( F \) and to \( F \otimes F/Tors(F \otimes F) \) gives the second assertion of part (a). Part (a) implies parts (b) and (c). Assume the surjectivity of \( \mu_L \). Lemma 1 and Remark 2 show that \( \mu_F \) is surjective if and only if \( \text{Im}(\mu_F) \) contains the image \( \Gamma \) of \( H^0(X, \text{Tors}(F \otimes F)) \). Since \( \dim(X) = 1 \), \( h^1(X, \mathcal{I}_S \otimes F) = h^1(X, F) \) if and only if the restriction map \( \rho_{F,S} : H^0(X, F) \to H^0(S, F|S) \cong F|S \) is surjective. Hence (e) is a particular case of (d). Assume the surjectivity of \( \rho_{F,S} \). Obviously, the map \( \alpha : H^0(S, F|S) \otimes H^0(S, F|S) \to H^0(X, F \otimes F) \) is bijective. Since \( S \) is affine, the natural map \( \Gamma \to H^0(X, F \otimes F) \) is injective. The
surjectivity of $\rho_{F,S}$ implies that $\text{Im}(\mu_F)$ surjects onto $H^0(S,F|S) \to H^0(X,F \otimes F)$. Hence the bijectivity of $\alpha$ gives $\Gamma \subset \text{Im}(\mu_F)$, concluding the proof. \hfill \Box

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References


