International Journal of Pure and Applied Mathematics
Volume 54 No. 2 2009, 215-226

NONLINEAR MULTIPOINT BOUNDARY VALUE PROBLEMS AT RESONANCE

Jesús Rodríguez
Department of Mathematics
North Carolina State University
P.O. Box 8205, Raleigh, NC 27695-8205, USA
e-mail: rodrigu@math.ncsu.edu

Abstract: In this paper we establish sufficient conditions for the solvability of nonlinear ordinary differential equations subject to multipoint boundary conditions. Our analysis is devoted to the resonant case, that is, when the associated linear homogeneous boundary value problem has nontrivial solutions.

AMS Subject Classification: 34B10, 34B15
Key Words: boundary value problems, Schauder's Fixed Point Theorem

1. Introduction

Here we consider the solvability of nonlinear, multipoint boundary value problems of the form

$$u^{(n)}(t) + a_{n-1}(t)u^{(n-1)}(t) + \cdots + a_0(t)u(t) = g(u(t), u'(t)); \quad 0 \le t \le 1\,, \tag{1}$$

subject to

$$B_0 \begin{bmatrix} u(t_0) \\ \vdots \\ u^{n-1}(t_0) \end{bmatrix} + B_1 \begin{bmatrix} u(t_1) \\ \vdots \\ u^{n-1}(t_1) \end{bmatrix} + \cdots + B_N \begin{bmatrix} u(t_N) \\ \vdots \\ u^{n-1}(t_N) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2}$$

Throughout our discussion we assume the points $t_0, t_1, \dots, t_N$ are fixed and $0 = t_0 < t_1 < \cdots < t_N = 1$. Each matrix B_k is n by n and the n by $n(N+1)$ matrix

Received: May 2, 2009

$$\big[B_0|B_1|\cdots|B_N\big]$$

has rank n. This rank condition is equivalent to the fact that the boundary conditions are not redundant.

The function $g:\mathbb{R}^2\to\mathbb{R}$ is continuous and we assume that the limits $g(\infty,\infty), g(\infty,-\infty), g(-\infty,\infty)$, and $g(-\infty,-\infty)$ all exist and are finite.

In our discussion we will always assume that the linear homogeneous boundary value problem

$$u^{(n)}(t)+a_{n-1}(t)u^{(n-1)}(t)+\cdots+a_0(t)u(t)=0;\ 0\le t\le 1\,, \tag{3}$$

$$B_0\begin{bmatrix}u(t_0)\\ \vdots\\ u^{n-1}(t_0)\end{bmatrix}+B_1\begin{bmatrix}u(t_1)\\ \vdots\\ u^{n-1}(t_1)\end{bmatrix}+\cdots+B_N\begin{bmatrix}u(t_N)\\ \vdots\\ u^{n-1}(t_N)\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}. \tag{4}$$

has a one dimensional solution space.

Here we provide easily verifiable sufficient conditions for the existence of solutions to the boundary value problem (1)-(2). The conditions we present depend on the nature of the solution space of the linear problem (3)-(4) and on the limiting behavior of the nonlinearity g. These results are a significant generalization of those in [12]. The added generality is due to the fact that here we allow much more general nonlinearities.

2. Main Results

We will formulate the boundary value problem in system form. In order to do so we introduce some notation.

The n by n matrix $A(t)$ is given by

$$A(t)=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t)\end{bmatrix}.$$

If x is an element of $\mathbb{R}^n$, we will use $|x|$ to denote its Euclidean norm. If C is a matrix or more generally, any linear map, $||C||$ will be its operator norm. The transpose of a matrix Q will be written as Q^T. The map $f:\mathbb{R}^n\to\mathbb{R}^n$ will be defined as follows:

$$\text{For } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad f(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x_1, x_2) \end{bmatrix}.$$

If we write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} u \\ u' \\ \vdots \\ u^{n-1} \end{bmatrix}$$

it is clear that the boundary value problem (1)-(2) is equivalent to

$$x'(t) = A(t)x(t) + f(x(t));\ 0 \le t \le 1\,, \tag{5}$$

subject to

$$\sum_{k=0}^{N} B_k x(t_k) = 0. \tag{6}$$

The linear homogeneous boundary value problem (3)-(4) is equivalent to

$$x'(t) = A(t)x(t);\ 0 \le t \le 1\,, \tag{7}$$

subject to

$$\sum_{k=0}^{N} B_k x(t_k) = 0. \tag{8}$$

We write Φ to denote the unique fundamental matrix solution of $x'(t) = A(t)x(t)$ that satisfies $\Phi(0) = I$. The matrix D is given by

$$D = \sum_{k=0}^{N} B_k \Phi(x(t_k)).$$

Throughout the paper V will be an arbitrary, but fixed, subspace of $\mathbb{R}^n$ such that

$$\ker(D) \bigoplus V = \mathbb{R}^n.$$

Since the solutions of the linear, homogeneous boundary value problem (7)-(8) are those of the form

$$x(t) = \Phi(t)x_0\,,$$

where

$$0=\sum_{k=0}^{N}B_kx(t_k),$$

it is clear that the solutions of (7)-(8) are of the form

$$x(t)=\Phi(t)p\,,$$

where p is in the kernel of D.

In this paper we are assuming the solution space of (3)-(4) is one dimensional. This, of course, is equivalent to $\ker(D)$ being one dimensional. From now on we assume that p spans $\ker(D)$ and that it has been chosen so that

$$\int_0^1|\Phi(t)p|^2dt=1.$$

The space $C[0,1]$ is given by

$$C[0,1]=\{\phi:[0,1]\to\mathbb{R}^n\mid \phi \text{ is continuous }\}$$

and if ϕ is an element of $C[0,1]$, its norm is $||\phi||=\sup_{0\le t\le 1}|\phi(t)|$.

We define $\psi:[0,1]\to\mathbb{R}^n$ as follows:

$$\psi(t)=\begin{cases}[(B_1\Phi(t_1)+B_2\Phi(t_2)+\cdots+B_N\Phi(t_N))\Phi^{-1}(t)]^Tc; & 0\le t\le t_1,\\ \\ [(B_2\Phi(t_2)+\cdots+B_N\Phi(t_N))\Phi^{-1}(t)]^Tc; & t_1<t\le t_2,\\ \\ \vdots \quad \vdots & \\ \\ [B_N\Phi(t_N))\Phi^{-1}(t)]^Tc; & t_{N-1}<t\le t_N,\end{cases}$$

where c is an element of $\ker(D^T)$ which has been chosen so that

$$\int_0^1|\psi(t)|^2dt=1.$$

Remark 1. The function ψ appears in [12] and [14] where it is used to characterize the range of a differential operator. In [12] this characterization is used in the formulation of an existence analysis based on the Lyapunov-Schmidt procedure. In this paper we circumvent the Lyapunov-Schmidt procedure, and by doing so we are able to consider more general nonlinearities.

Proposition 2. *For each continuous function, $x:[0,1]\to\mathbb{R}^n$ there exists* *a unique v_x in V such that*

$$Dv_x = -\sum_{k=1}^{N} B_k\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)[f(x(s)) - (\int_0^1 \psi^T(u)f(x(u))du)\psi(s)]ds.$$

Furthermore, there is a constant Δ such that $|v_x| \leq \Delta$ for all x in $C[0,1]$.

Proof.

$$\begin{aligned}
& B_1\Phi(t_1)\int_0^{t_1}\Phi^{-1}(s)\Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u)f\big(x(u)\big)du\Big)\psi(s)\Big]ds \\
+ \quad & B_2\Phi(t_2)\int_0^{t_2}\Phi^{-1}(s)\Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u)f\big(x(u)\big)du\Big)\psi(s)\Big]ds \\
& \vdots \qquad \vdots \\
+ \quad & B_N\Phi(t_N)\int_0^{t_N}\Phi^{-1}(s)\Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u)f\big(x(u)\big)du\Big)\psi(s)\Big]ds \\
= \quad & \int_0^{t_1}\sum_{j=1}^{N} B_j\Phi(t_j)\Phi^{-1}(s)\Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u)f\big(x(u)\big)du\Big)\psi(s)\Big]ds \\
+ \quad & \int_{t_1}^{t_2}\sum_{j=2}^{N} B_j\Phi(t_j)\Phi^{-1}(s)\Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u)f\big(x(u)\big)du\Big)\psi(s)\Big]ds \\
+ \quad & \int_{t_2}^{t_3}\sum_{j=2}^{N} B_j\Phi(t_j)\Phi^{-1}(s)\Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u)f\big(x(u)\big)du\Big)\psi(s)\Big]ds \\
& \vdots \qquad \vdots \\
+ \quad & \int_{t_{N-1}}^{t_N} B_N\Phi(t_N)\Phi^{-1}(s)\Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u)f\big(x(u)\big)du\Big)\psi(s)\Big]ds.
\end{aligned}$$

Let c be the element in the nullspace of D^T that appears in the definition of the function ψ. Then

$$c^T\sum_{k=1}^{N} B_k\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)\Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u)f\big(x(u)\big)du\Big)\psi(s)\Big]ds$$

$$\begin{aligned}
&= \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \sum_{j=k}^{N} c^T B_j \Phi(t_j) \Phi^{-1}(s) \Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u) f\big(x(u)\big) du \Big) \Big] \psi(s) ds \\
&= \int_0^1 \psi^T(t) \Big[f\big(x(t)\big) - \Big(\int_0^1 \psi^T(u) f\big(x(u)\big) du \Big) \psi(t) \Big] dt \\
&= \int_0^1 \psi^T(t) f\big(x(t)\big) dt - \Big(\int_0^1 \psi^T(u) f\big(x(u)\big) du \Big) \int_0^1 |\psi(t)|^2 dt \\
&= 0
\end{aligned}$$

because $\int_0^1 |\psi(t)|^2 dt = 1$.

Since $\ker(D^T)$ =span$\{c\}$ it follows that

$$\sum_{k=1}^{N} B_k \Phi(t_k) \int_0^{t_k} \Phi^{-1}(s) \Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u) f\big(x(u)\big) du \Big) \psi(s) \Big] ds$$

is orthogonal to the nullspace of D^T. Therefore there is an element y in $\mathbb{R}^n$ such that

$$Dy = -\sum_{k=1}^{N} B_k \Phi(t_k) \int_0^{t_k} \Phi^{-1}(s) \Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u) f\big(x(u)\big) du \Big) \psi(s) \Big] ds.$$

Since

$$\mathbb{R}^n = \ker(D) \bigoplus V$$

it is clear that there is a unique v_x in V such that

$$Dv_x = -\sum_{k=1}^{N} B_k \Phi(t_k) \int_0^{t_k} \Phi^{-1}(s) \Big[f\big(x(s)\big) - \Big(\int_0^1 \psi^T(u) f\big(x(u)\big) du \Big) \psi(s) \Big] ds.$$

The existence of a number Δ such that $|v_x| \leq \Delta$ for all x in $C[0,1]$ is an immediate consequence of the previous argument and the fact that f is a bounded map from $\mathbb{R}^n$ into $\mathbb{R}^n$. □

In order to simplify our presentation we introduce the following notation.

For $i = 1, 2$, we define $s_i(t)$ to be the i-th component of $\Phi(t)p$, and we let $w_i\big(x(t)\big)$ denote the i-th component of

$$\Phi(t) v_x + \Phi(t) \int_0^t \Phi^{-1}(s) \left[f\big(x(s)\big) - \left(\int_0^1 \psi^T(u) f\big(x(u)\big) du \right) \psi(s) \right] ds.$$

The n-th component of $\psi(t)$ will be denoted by $h(t)$.

Since

$$f(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x_1, x_2) \end{bmatrix},$$

it is clear that $\int_0^1 \psi^T(s) f\big(x(s)\big) ds = \int_0^1 h(t) g\big(x_1(s), x_2(s)\big) ds$.

Definition 3. $H : \mathbb{R} \times C[0,1] \to \mathbb{R} \times C[0,1]$ is defined by

$$H(\alpha, x) = \begin{bmatrix} H_1(\alpha, x) \\ H_2(\alpha, x) \end{bmatrix},$$

where $H_1 : \mathbb{R} \times C[0,1] \to \mathbb{R}$ is given by

$$H_1(\alpha, x) = \alpha - \int_0^1 h(t) g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big) dt,$$

and $H_2 : \mathbb{R} \times C[0,1] \to C[0,1]$ is defined by

$$H_2(\alpha, x) = \Phi(t)\big[\alpha p + v_x\big] + \Phi(t) \int_0^t \Phi^{-1}(s) \left[f\big(x(s)\big) - \left(\int_0^1 \psi^T(u) f\big(x(u)\big) du \right) \psi(s) \right] ds.$$

Proposition 4. *If H has a fixed point $(\bar{\alpha}, \bar{x})$ then $\bar{x}$ is a solution of the boundary value problem (5)-(6).*

Proof. If $(\bar{\alpha}, \bar{x})$ is a fixed point of H,

$$\bar{x}(t) = \Phi(t)[\bar{\alpha} p + v_{\bar{x}}] + \Phi(t) \int_0^t \Phi^{-1}(s) \left[f\big(x(s)\big) - \left(\int_0^1 \psi^T(u) f\big(x(u)\big) du \right) \psi(s) \right] ds.$$

Therefore, $\bar{x}_i(t) = \bar{\alpha} s_i(t) + w_i\big(\bar{x}(t)\big)$ for i=1, 2.

Also,

$$\int_0^1 h(t) g\Big(\bar{\alpha} s_1(t) + w_1\big(\bar{x}(t)\big), \bar{\alpha} s_2(t) + w_2\big(\bar{x}(t)\big)\Big) dt = 0.$$

Therefore,

$$\int_0^1 \psi^T(u) f\big(x(u)\big) du = 0,$$

and consequently

$$\bar{x}(t) = \Phi(t)[\bar{\alpha} p + v_{\bar{x}}] + \Phi(t) \int_0^t \Phi^{-1}(s) f\big(\bar{x}(s)\big) ds.$$

This implies, of course, that $\bar{x}$ solves the differential equation

$$x'(t) = A(t)x(t) + f\big(x(t)\big).$$

To verify that $\bar{x}$ satisfies the boundary conditions, we compute

$$\begin{aligned}\sum_{k=0}^{N} B_k\bar{x}(t_k) &= \sum_{k=0}^{N} B_k\Phi(t_k)[\bar{\alpha}p + v_{\bar{x}}] + \sum_{k=0}^{N} B_k\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)f\big(\bar{x}(s)\big)ds \\ &= \bar{\alpha}\sum_{k=0}^{N} B_k\Phi(t_k)p + \sum_{k=0}^{N} B_k\Phi(t_k)v_{\bar{x}} + \sum_{k=1}^{N} B_k\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)f\big(\bar{x}(s)\big)ds \\ &= Dv_{\bar{x}} + \sum_{k=1}^{N} B_k\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)f\big(\bar{x}(s)\big)ds.\end{aligned}$$

Since

$$Dv_{\bar{x}} = -\sum_{k=1}^{N} B_k\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)\left[f\big(\bar{x}(s)\big) - \left(\int_0^1 \psi^T(u)f\big(\bar{x}(u)\big)du\right)\psi(s)\right]ds,$$

and we have seen that

$$\int_0^1 \psi^T(u)f\big(\bar{x}(u)\big)du = 0,$$

it follows that

$$\sum_{k=0}^{N} B_k\bar{x}(t_k) = 0.$$

That is, the boundary conditions are also satisfied. □

The following notation will be used in the next theorem.

$$A_1 = \{t : s_1(t) > 0\} \text{ and } A_2 = \{t : s_2(t) > 0\},$$

$$\begin{aligned}C_1 \;=\; &\Big(\int_{A_1\cap A_2} h(t)dt\Big)g(\infty,\infty) + \Big(\int_{A_1\cap(\mathbb{R}\setminus A_2)} h(t)dt\Big)g(\infty,-\infty) \\ &+\Big(\int_{(\mathbb{R}\setminus A_1)\cap A_2} h(t)dt\Big)g(-\infty,\infty) + \Big(\int_{(\mathbb{R}\setminus A_1)\cap(\mathbb{R}\setminus A_2)} h(t)dt\Big)g(-\infty,-\infty),\\ C_2 \;=\; &\Big(\int_{A_1\cap A_2} h(t)dt\Big)g(-\infty,-\infty) + \Big(\int_{A_1\cap(\mathbb{R}\setminus A_2)} h(t)dt\Big)g(-\infty,\infty) \\ &+\Big(\int_{(\mathbb{R}\setminus A_1)\cap A_2} h(t)dt\Big)g(\infty,-\infty) + \Big(\int_{(\mathbb{R}\setminus A_1)\cap(\mathbb{R}\setminus A_2)} h(t)dt\Big)g(\infty,\infty).\end{aligned}$$

Theorem 5. *Suppose the solution space of (3)-(4) is one dimensional and that $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous. If C_1 and C_2 have different signs, then the* *boundary value problem*

$$u^{(n)}(t) + a_{n-1}(t)u^{(n-1)}(t) + \cdots + a_0(t)u(t) = g(u(t), u'(t));\ \ 0 \le t \le 1$$

subject to

$$B_0 \begin{bmatrix} u(t_0) \\ \vdots \\ u^{n-1}(t_0) \end{bmatrix} + B_1 \begin{bmatrix} u(t_1) \\ \vdots \\ u^{n-1}(t_1) \end{bmatrix} + \cdots + B_N \begin{bmatrix} u(t_N) \\ \vdots \\ u^{n-1}(t_N) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

has a solution.

Proof. Without loss of generality, we assume $C_1 > 0$ and $C_2 < 0$.

$$\begin{aligned}
&\int_0^1 h(t) g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big) dt \\
&= \int_{A_1 \cap A_2} h(t) g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big) dt \\
&+ \int_{A_1 \cap (\mathbb{R} \setminus A_2)} h(t) g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big) dt \\
&+ \int_{(\mathbb{R} \setminus A_1) \cap A_2} h(t) g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big) dt \\
&+ \int_{(\mathbb{R} \setminus A_1) \cap (\mathbb{R} \setminus A_2)} h(t) g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big) dt.
\end{aligned}$$

Since w_1 and w_2 are bounded and the limits $g(\infty, \infty), g(\infty, -\infty), g(-\infty, \infty)$ and $g(-\infty, -\infty)$ exist, it follows from Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned}
&\lim_{\alpha \to \infty} \int_0^1 h(t) g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big) dt \\
&\quad = \left(\int_{A_1 \cap A_2} h(t) dt \right) g(\infty, \infty) + \left(\int_{A_1 \cap (\mathbb{R} \setminus A_2)} h(t) dt \right) g(\infty, -\infty) \\
&+ \left(\int_{(\mathbb{R} \setminus A_1) \cap A_2} h(t) dt \right) g(-\infty, \infty) + \left(\int_{(\mathbb{R} \setminus A_1) \cap (\mathbb{R} \setminus A_2)} h(t) dt \right) g(-\infty, -\infty).
\end{aligned}$$

In the same manner we see that

$$\begin{aligned}
&\lim_{\alpha \to -\infty} \int_0^1 h(t) g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big) dt \\
&\quad = \left(\int_{A_1 \cap A_2} h(t) dt \right) g(-\infty, -\infty) + \left(\int_{A_1 \cap (\mathbb{R} \setminus A_2)} h(t) dt \right) g(-\infty, \infty)
\end{aligned}$$

$$+\left(\int_{(\mathbb{R}\setminus A_1)\cap A_2} h(t)dt\right)g(\infty,-\infty)+\left(\int_{(\mathbb{R}\setminus A_1)\cap(\mathbb{R}\setminus A_2)} h(t)dt\right)g(\infty,\infty).$$

It is clear that there are constants M_1, M_2, and M_3 such that for any (α, x) in $\mathbb{R}\times C[0,1]$,

$$\| H_2(\alpha,x) \| \leq M_1|\alpha| + M_2,$$

and

$$\sup_{0\leq t\leq 1} |h(t)g\big(x_1(t),x_2(t)\big)| \leq M_3.$$

For $(\alpha, x) \in \mathbb{R}\times C[0,1]$ we will use the norm $||(\alpha,x)|| = \max\{|\alpha|, ||x||\}$ where $||x|| = \sup_{0\leq t\leq 1}|x(t)|$.

We choose $r > 2M_3$ and such that

$$\int_0^1 h(t)g\Big(\alpha s_1(t)+w_1\big(x(t)\big),\alpha s_2(t)+w_2\big(x(t)\big)\Big)dt > 0$$

for $\alpha \geq \frac{r}{2}$ and

$$\int_0^1 h(t)g\Big(\alpha s_1(t)+w_1\big(x(t)\big),\alpha s_2(t)+w_2\big(x(t)\big)\Big)dt < 0$$

for $\alpha \leq -\frac{r}{2}$.

We define

$$\mathcal{B} = \{(\alpha,x) : |\alpha| \leq r \text{ and } ||x|| \leq M_1 r + M_2\}.$$

We will now show that H maps $\mathcal{B}$ into itself.

If $(\alpha, x) \in \mathcal{B}$ and $\alpha \geq \frac{r}{2}$ then, clearly

$$H_1(\alpha,x) \leq \frac{\alpha}{2}$$

because

$$\int_0^1 h(t)g\Big(\alpha s_1(t)+w_1\big(x(t)\big),\alpha s_2(t)+w_2\big(x(t)\big)\Big)dt > 0.$$

Also,

$$\Big|\int_0^1 h(t)g\Big(\alpha s_1(t)+w_1\big(x(t)\big),\alpha s_2(t)+w_2\big(x(t)\big)\Big)dt\Big|$$
$$\leq \int_0^1 \Big|h(t)g\Big(\alpha s_1(t)+w_1\big(x(t)\big),\alpha s_2(t)+w_2\big(x(t)\big)\Big)\Big|dt \leq M_3 < \frac{r}{2}.$$

Therefore,

$$H_1(\alpha,x) \geq 0.$$

If $0 \leq \alpha \leq \frac{r}{2}$, we see that

$$|H_1(\alpha, x)| \leq |\alpha| + \int_0^1 \left|h(t)g\Big(\alpha s_1(t) + w_1\big(x(t)\big), \alpha s_2(t) + w_2\big(x(t)\big)\Big)\right| dt$$
$$\leq |\alpha| + M_3 \leq \frac{r}{2} + \frac{r}{2} = r.$$

In a similar way we prove that if $(\alpha, x) \in \mathcal{B}$ and $-r \leq \alpha \leq 0$

$$|H_1(\alpha, x)| \leq r.$$

For any (α, x) in $\mathcal{B}$,

$$||H_2(\alpha, x)|| \leq M_1|\alpha| + M_2 \leq M_1 r + M_2.$$

Therefore, H maps $\mathcal{B}$ into itself. The existence of a fixed point of H, and hence of the solvability of the boundary value problem (5)-(6) is a consequence of Schauder's Theorem. □

Remark 6. The techniques and ideas used in the proof of Theorem 5 have been successfully used in a variety of different cases. In [2] and [10] they are used to establish the existence of periodic behavior in discrete dynamical systems, and in [5], [9], and [12] to provide sufficient conditions for the solvability of continuous and discrete boundary value problems. For related approaches the reader is referred to [1], [3], [4], [5], [6], [7], [8], [11], [13], and [14].

References

[1] S.N. Chow, J.K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, New York (1982).

[2] D.L. Etheridge, J. Rodriguez, Periodic solutions of nonlinear discrete-time systems, *Applicable Analysis*, **62** (1996), 119-137.

[3] J.K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York (1969).

[4] J.K. Hale, Applications of alternative problems, *Lecture Notes* (1971), 71-1.

[5] E.M. Landesman, A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *Journal of Mathematics and Mechanics*, **19** (1970), 609-623.

[6] J. Rodríguez, Nonlinear differential equations under stieltjes boundary conditions, *Nonlinear Analysis*, **7** (1983), 107-116.

[7] J. Rodríguez, On resonant discrete boundary value problems, *Applicable Analysis*, **19** (1985), 265-274.

[8] J. Rodríguez, Galerkin's method for ordinary differential equations subject to generalized nonlinear boundary conditions, *J. Differential Equations*, **97** (1992), 112-126.

[9] J. Rodríguez, Nonlinear discrete Sturm-Liouville problems, *J. Math. Anal. and Appl.*, **308** (2005), 380-391.

[10] J. Rodríguez, D.L. Etheridge, Periodic solutions of nonlinear, second-order difference equations, *Advances in Difference Equations*, **2005:2** (2005), 173-192.

[11] J. Rodríguez, D. Sweet, Projection methods for nonlinear boundary value problems, *J. Differential Equations*, **58** (1985), 282-293.

[12] J. Rodríguez, P. Taylor, Multipoint boundary value problems for nonlinear ordinary differential equations, *Nonlinear Analysis: Theory Methods and Applications* (2008), To Appear.

[13] N. Rouche, J. Mawhin, *Ordinary Differential Equations: Stability and Periodic Solutions*, Pitman, London (1980).

[14] W. Spealman, D. Sweet, The alternative method for solutions in the kernel of a bounded linear functional, *Journal of Differential Equations*, **37** (1980), 297-302.