International Journal of Pure and Applied Mathematics
Volume 54 No. 2 2009, 233-240

CURVES IN $\mathbb{P}^r$ THROUGH A FINITE SET

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Fix a finite set $S \subset \mathbb{P}^r$ and let $u_S : W_S \to \mathbb{P}^r$ be the blowing-up of S. Here we study the postulation of the strict transform in W_S of a sufficiently general curve $C \subset \mathbb{P}^r$ with fixed degree and genus such that $S \subset C_{reg}$.

AMS Subject Classification: 14H50
Key Words: postulation, curve through a set

1. Introduction

Fix a finite $S \subset \mathbb{P}^r$, $r \geq 2$. Quite often it is interesting to study the set $\Theta(d, g, r, S)$ of all smooth and irreducible curves $C \subset \mathbb{P}^r$ such that $S \subset C$, $\deg(C) = d$, $p_a(C) = g$ and C spans $\mathbb{P}^r$. Set $s := \sharp(S)$. Let $u_S : W_S \to \mathbb{P}^r$ be the blowing-up of $\mathbb{P}^r$ along S and $E_S := u_S^{-1}(S)$ the union of the s exceptional divisors. For all integers k, S set $\mathcal{L}_{k,S,t} := u_S^*(\mathcal{O}_{\mathbb{P}^r}(k))(-tE_S)$ and $\mathcal{L}_{k,S} := \mathcal{L}_{k,S,1}$. For any $C \in \Theta(d, r, S)$ let C_S be its strict transform in W_S. Since C is smooth at each point of S and $S \subset C$, $u_S|C_S : C_S \to C$ is an isomorphism, $p_a(C_S) = g$ and $\deg(\mathcal{L}_{k,S,t}|C_S) = kd - ts$ for all k, t. Let $\rho_{C,S,k,t} : H^0(W_S, \mathcal{L}_{k,S,t}) \to H^0(C_S, \mathcal{L}_{k,S,t})$ denote the restriction map. Set $\rho_{C,S,k} := \rho_{C,S,k,1}$. Here we consider the rank of $\rho_{C,S,k}$ for a "general" $C \in \Theta(d, g, r, S)$. Of course, we first restrict the data r, d, g, S, identify an interesting irreducible component Θ of $\Theta(d, g, r, S)$ and look at the general element of Θ. The curve C is said to have *maximal rank as a curve in* $\mathbb{P}^r$ if for every integer $x > 0$ the restiction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \to H^0(C, \mathcal{O}_C(x))$ has maximal rank, i.e. either it is injective or it is surjective. The curve C is said to have *maximal rank modulo* S if for every integer $x > 0$ the restriction map $\rho_{C,S,x} : H^0(W_S, \mathcal{L}_{x,S}) \to H^0(C_S, \mathcal{L}_{x,S}|C_S)$ has maximal rank. Notice

Received: May 25, 2009

that $H^0(W_S, \mathcal{L}_{x,S}) \cong H^0(\mathbb{P}^r, \mathcal{I}_S(x))$, $H^0(C_S, \mathcal{L}_{x,S}|C_S) \cong H^0(C, \mathcal{O}_C(x)(-S))$ and that, up to these identifications, the restriction map $\rho_{C,S,x}$ is identified with the restriction map $H^0(\mathbb{P}^r, \mathcal{I}_S(x)) \to H^0(C, \mathcal{O}_C(x)(-S))$.

Remark 1. We have $h^0(W_S, \mathcal{L}_{k,S,x}) = h^0(W_S, \mathcal{L}_{k,S,0})$ if $x < 0$. Set $s := \sharp(S)$. Since $\mathcal{O}_{\mathbb{P}^r}(1)$ is ample, here is an integer $x(S)$ such that $h^1(\mathbb{P}^r, \mathcal{I}_S(x)) = 0$ for all $x \geq x(S)$, i.e. $h^0(\mathbb{P}^r, \mathcal{I}_S(x)) = \binom{r+x}{r} - s$ for all $x \geq x(S)$, i.e. $h^1(W_S, \mathcal{L}_{x,S}) = 0$ for all $x \geq x(S)$, i.e. $h^0(W_S, \mathcal{L}_{x,S}) = \binom{r+x}{r} - s$ for all $x \geq x(S)$. For arbitrary S such that $\sharp(S) = s$ we may take $x(S) = s - 1$ and this bound is achieved if and only if S is contained in a line. If S is in linearly general position, then we may take $x(S) = \lceil (s-1)/r \rceil$. If S is general, then we may take as $x(S)$ the first integer z such that $\binom{r+z}{r} \geq s$.

Our main task is to define the family Θ. We first look at the case $S = 0$ and recall [1] and [4].

For all schemes $A \subset B$ let $N_{A,B}$ denote the normal sheaf of A in B. For any locally complete intersection curve $C \subset \mathbb{P}^r$ let N_C denote its normal bundle. If C is a locally complete intersection and reduced curve with degree d and arithmetic genus g, then N_C is a vector bundle with rank $r-1$ and degree $(r+1)d + 2 - 2g$.

Fix integers r, d, g such that $r \geq 3$, $g \geq 0$ and either $d \geq g + r$ or $d - r < g \leq d - r + \lfloor (d-r-2)/(r-2) \rfloor$. There is an irreducible component $W(d, g; r)$ of the Hilbert scheme of $\mathbb{P}^r$ which is generically smooth and of dimension $(r+1)d - (r-3)(g-1)$ such that a general $C \in W(d, g; r)$ has the following properties (see [1] for the case $r = 3$, [4] for the case $r \geq 4$):

(a) C is a smooth and connected non-degenerate curve with degree d, genus g and $h^1(C, N_C) = 0$;

(b) if $d \geq g + r$, then $h^1(C, \mathcal{O}_C(1)) = 0$;

(c) if $d < g + r$, then C is linearly normal and $h^1(C, \mathcal{O}_C(2)) = 0$;

(d) if $\rho(g, r, d) \geq 0$, then C has general moduli;

(e) if $\rho(g, r, d) < 0$, then the general fiber of the natural rational map $\gamma_{d,g,r} : W(d, g; r) \dashrightarrow \mathcal{M}_g$ has dimension $\dim(\mathrm{Aut}(\mathbb{P}^r)) = r^2 + 2r$, i.e. $W(d, g; r)$ has the right number of moduli in the sense of [9].

Now we consider the general case.

Fix integers $r \geq 3$ and $s \geq 0$. Set $\delta(r, 0) := 0$, $\delta(r, s) = r$ if $1 \leq s \leq 2$ and $\delta(r, s) := r(s-1)$ for all $s \geq 3$. Set $\gamma(r, s) := 0$ if $0 \leq s \leq 2$ and $\gamma(r, s) := (s-2)(r-1)$ if $s \geq 3$. Set $\delta'(r, s) := r \cdot \lceil s/2 \rceil$, $A'(r, 0) := \emptyset$ Fix $S \subset \mathbb{P}^r$ such that $\sharp(S) = s$. Set $W(d, g; r, \emptyset) := W(d, g; r)$. If $1 \leq s \leq 2$ let $A(r, s)$

be any rational normal curve of $\mathbb{P}^r$ containing S. If $s \geq 3$ define the curve $A(r,s)$ in the following way. Fix an ordering $P_1, \dots, P_s$ of S. Let $A(r,s)$ be the general union of $s-1$ rational normal curves $D_1, \dots, D_{s-1}$ with the restriction that $S \cap D_1 = \{P_1, P_2\}$, $S \cap D_i = P_{i+1}$ for all $2 \leq o \leq s-1$, $D_i \cap D_j = \emptyset$ for all i, j such that $|i-j| \geq 2$, $\sharp(D_i \cap D_{i+1}) = 1$ for all $1 \leq i \leq s-2$ and D_i intersects transversally D_{i+1} for all $1 \leq i \leq s-2$. Notice that $A(r,s)$, $s > 0$, is connected, nodal and $A(r,s) \in W(\delta(r,s), \gamma(r,s); r)$. As in [4], Corollary 1.3, Remark 2 easily gives $h^1(A(r,s), N_{A(r,s)}(-S)) = 0$. Now fix integers d, g such that $W(d-\alpha(r,s), g-\gamma(r,s); r)$ is defined, i.e. such that $g \geq \gamma(r,s)$ and either $d - \alpha(r,s) \geq g + r - \gamma(r,s)$ or $g - \gamma(r,s) \geq 2$ and $d - r - \alpha(r,s) < g \leq \lfloor (d-r-2)/(r-2) \rfloor$. If $d - \alpha(r,s) \geq g + r - \gamma(r,s)$ let $B(r,s)$ be the general union of $A(r,s)$ and a smooth elements of $W(d-\alpha(r,s), g-\gamma(r,s); r)$ intersecting $A(r,s)$ quasi-transversally and at a unique point which belong to $D_{\lceil s/2 \rceil}$. Now assume $d-r-\alpha(r,s) < g-\gamma(r,s) \leq \lfloor (d-r-\alpha(r,s)-2)/(r-2) \rfloor$ and set $t := g + r - \gamma(r,s) + \alpha(r,s) - d$, $z := d - \alpha(r,s) - r - 2 - t(n-2)$. Let $C \subset \mathbb{P}^r$ be a general normal rational curve with the only restriction that $\sharp(C \cap D_{\lceil s/2 \rceil}) = 1$. Hence C intersects quasi-transversally D_x, $x := \max\{1, s-1\}$. Fix general secant lines A, B of C. Fix $t(r-2)$ general points $P_{ij} \in C$, $1 \leq j \leq t$, $3 \leq i \leq r$. As in [4] let $V_j \subset \mathbb{P}^r$, $1 \leq j \leq t$, be the only linear subspace such that $V_j \cap A \neq \emptyset$, $V_j \cap B \neq \emptyset$, $P_{xy} \in V_j$ if and only if $y = j$ and $V_j \cap (A \cup B) \cup C = \emptyset$. Set $P_{1j} := A \cap V_j$ and $P_{2j} := B \cap V_j$. Fix a general rational normal curve C_j of V_j such that $P_{ij} \in C_j$ for all $i \in \{1, \dots, r\}$. Fix z general secant lines $T_1, \dots, T_z$ of C. For a general choices of these curves we have $S \cap C \cup A \cup B \cup C_1 \cup \dots \cup C_t \cup T_1 \cup T_z) = \emptyset$. Set $B(d,g;r,S) := A(r,S) \cup C \cup A \cup B \cup C_1 \cup \dots \cup C_t \cup T_1 \cup T_z$. For general choices of these additional data the curve $B(d,g;r,S)$ is nodal. In both cases (i.e. even if $d - \alpha'(r,s) \geq g+r$) it is easy to check that $B(d,g;r,S) \in W(d,g;r)$ (use [4], Lemmas 2.2 and 2.3), that $S \subset B(d,g;r,S)_{reg}$, and that $h^1(B(d,g;r,S), N_{B(d,g;r,S)}(-S)) = 0$. Hence $B(d,g;r,S)$ is a smooth point of $\mathrm{Hilb}(d,g,S)'$. Hence there is a unique irreducible component of $\mathrm{Hilb}(d,g,S)'$ containing $B(d,g;r,S)$ and we call it $W(d,g;r,S)$.

In Section 2 we will prove the following results.

Theorem 1. *Fix integers $r \geq 3$ and $s > 0$. Then there are an integer $\eta(r,s) \geq \gamma(r,s)$ and a function $u_{r,s} : \mathbb{N} \to \mathbb{N}$ such that $\lim_{g \to +\infty} u_{r,s}(g) = (r-2)/(r-1)$ and the following properties are true. Let $S \subset \mathbb{P}^r$ be a general subset with cardinality s. Fix integers d, g such that $g \geq \eta(r,s)$ and $d \geq g \cdot u_{r,s}(g)$. Then $W(d,g;r,S)$ is defined and a general $C \in W(d,g;r,S)$ has maximal rank both as a curve in $\mathbb{P}^r$ and modulo S, i.e. for all integers $x > 0$ the restriction* *maps* $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \to H^0(C, \mathcal{O}_C(x))$ *and* $H^0(W_S, \mathcal{L}_{x,S}) \to H^0(C_S, \mathcal{L}_{x,S}|C_S)$. *Moreover,* $h^0(C_S, \mathcal{L}_{x,S}|C_S) = h^0(C, \mathcal{O}_C(x)(-S)) = xd + 1 - g - s$ *for all* $x \geq 2$.

Proposition 1. *Fix integers* $r \geq 3$, $s \geq 0$ *and* $g \geq 0$. *There is an integer* $\delta(r, s, g) \geq g + r + s$ *such that for every finite* $S \subset \mathbb{P}^r$ *such that* $\sharp(S) = s$ *and every integer* $d \geq \delta(r, s, g)$ *the set* Γ *of all smooth and connected curves* $C \subset \mathbb{P}^r$ *such that* $S \subset C$, $\deg(C) = d$ *and* $p_a(C) = g$ *is non-empty and its general element* C *satisfies* $h^1(C, N_C(-S)) = 0$, C *has maximal rank as a curve in* $\mathbb{P}^r$ *and maximal rank modulo* S.

In section 3 we consider the general set-up for the low genus case.

2. Proof of Theorem 1 and Proposition 1

Remark 2. Let $D \subset \mathbb{P}^r$, $r \geq 2$, be a rational normal curve. Then N_D is a direct sum of $r - 1$ line bundles of degree $r + 2$ (see, e.g. [8] or [7]).

Using Remark 2 and [4], Lemmas 2.2 and 2.3, we immediately get the following result.

Lemma 1. *Fix a finite* $S \subset \mathbb{P}^r$ *and integers* $d, g, , b$ *such that* $1 \leq b \leq r-1$ *and* $W(d, g; r, S)$ *is defined. Fix an integral non-degenerate* $Y \in W(d, g; r, S)$ *such that* $h^1(Y, N_Y(-S)) = 0$. *Let* D *be a rational normal curve of a* b*-dimensional linear suspace of* $\mathbb{P}^r$ *such that* $1 \leq \sharp(D \cap Y) \leq b + 1$, D *intersects quasi-transversally* Y *and* $S \cap D = \emptyset$. *Let* F *be a rational normal curve of* $\mathbb{P}^r$ *such that* $F \cap S = \emptyset$, $1 \leq \sharp(F \cap Y) \leq r+2$ *and* F *intersects quasi-transversally* Y. *Then* $Y \cup D \in W(d+b, g+\sharp(D \cap Y)-1; r, S)$, $Y \cup F \in W(d+r, g+\sharp(F \cap Y)-1; r, S)$ *and* $h^1(Y \cup D, N_{Y \cup D}(-S)) = h^1(Y \cup F, N_{Y \cup F}(-S)) = 0$.

Proof of Theorem 1. By semicontinuity to get the statement of the theorem for the pair of integers (d, g) is sufficient to find a pair (C, S) such that $\sharp(S) = s$, $W(d, g; r, S)$ is defined, $C \in W(d, g; r, S)'$, C smooth, $h^1(C, N_C(-S)) = 0$ (and hence $C \in W(d, g; r, S)'_{reg}$), $h^0(C, \mathcal{O}_C(2)(-S)) = 0$ and C has maximal rank both as a curve in $\mathbb{P}^r$ and modulo S. For large $\eta(r, s)$ we are working in a range for which for all d, g such that $W(d, g; r, S)$ is defined we have $h^1(C, \mathcal{O}_C(2)(-S)) = 0$ and $h^0(\mathbb{P}^r, \mathcal{I}_C(x)) = 0$ if $x \leq s - 1$. Hence we are only looking at the restriction maps for line bundles $\mathcal{O}_{\mathbb{P}^r}(x)$ and $\mathcal{L}_{x,S}$ for integer $x \geq s$, i.e. for integers such that $h^0(W_S, \mathcal{L}_{x,S}) = \binom{r+x}{r} - x$ and $h^1(W_S, \mathcal{L}_{x,S}) = 0$, while $h^0(C, \mathcal{O}_C(x)) = xd + 1 - g$, $h^0(C, \mathcal{O}_C(x)(-S)) = xd + 1 - g - s$ and $h^1(C, \mathcal{O}_C(x)) = h^1(C, \mathcal{O}_C(x)(-S)) = 0$. The *critical value* $k_{d,g,r}$ of (d, g) it the minimal integer $k \geq 2$ such that $\binom{r+k}{r} \geq kd+1-g$. Notice that $k_{d,g,r}$ is it the minimal integer $k \geq 2$ such that $\binom{r+k}{r} - s \geq kd+1-g-s$. Notice that $C \in W(d,g;r)$ has maximal rank as a curve in $\mathbb{P}^r$ if and only if $h^0(\mathbb{P}^r, \mathcal{I}_C(k_{d,g,r}-1)) = 0$ and $h^1(\mathbb{P}^r, \mathcal{I}_C(k_{d,g,r})) = 0$. With these restrictions on the pair (d,g) a curve $C \in W(d,g;r,S)$ has maximal rank modulo S if and only if $h^0(W_S, \mathcal{I}_{C_S} \otimes \mathcal{L}_{k_{d,g,r}-1,S}) = 0$ and $h^1(W_S, \mathcal{I}_{C_S} \otimes \mathcal{L}_{k_{d,g,r}-1,S}) = 0$. Thus it has maximal rank as a curve in $\mathbb{P}^r$ if it has maximal rank modulo S. Since $W(d,g;r,S)$ is irreducible, semicontinuity shows that to prove Theorem 1 for the pair (d,g) it is sufficient to find $C_1, C_2 \in W(d,g;r,S)$ such that $h^0(W_S, \mathcal{I}_{C_{1S}} \otimes \mathcal{L}_{k_{d,g,r}-1,S}) = 0$ and $h^1(W_S, \mathcal{I}_{C_{2S}} \otimes \mathcal{L}_{k_{d,g,r}-1,S}) = 0$.

(a) Here we assume $r = 3$. Look at the assertion $H(t)$ of [1] and the proof gives in [1], §6, that for large t we have $H(t-2) \Longrightarrow H(t)$. To fince C_1 (resp. C_2) such that $h^0(W_S, \mathcal{I}_{C_{1S}} \otimes \mathcal{L}_{k_{d,g,r}-1,S}) = 0$ (resp. $h^1(W_S, \mathcal{I}_{C_{2S}} \otimes \mathcal{L}_{k_{d,g,r}-1,S}) = 0$) we start with an integer $w \equiv k_{d,g,r} - 1 \pmod 2$ (resp. $w \equiv k_{d,g,r} \pmod 2$) for which $H(w)$ is true. Hence $H(t)$ is true for all integers $t \geq w$ such that $t \equiv w \pmod 2$. Then we see for which pairs (d,g) we may prove the assertions $R(n,g)$, $n \equiv 2 \pmod 2$ so that to cover the maximal rank with respect to $\mathbb{P}^3$ and the pair (d,g). We also assume that in the last step from the critical value $k_{d,g,3} - 3$ to the critical value $k_{d,g,3} - 1$ (for C_1) or from the critical value $k_{d,g,3} - 2$ to the critical value $k_{d,g,3}$ (for C_2) we insert at least s lines in the same system of lines in a quadric surface Q. We insert S inside Q so that each of these lines intersects exactly one of the points of S. Then we apply Horace Lemma as in [1], but not with respect to the Cartier divisor Q of $\mathbb{P}^3$. Instead we use the strict transform Q_S of Q in W_S as Cartier divisor of W_S and the line bundle $\mathcal{L}_{k_{d,g,3}-1,S}$ (for C_1) or the line bundle $\mathcal{L}_{k_{d,g,3},S}$ (for C_2) instead respectively of the line bundles $\mathcal{O}_{\mathbb{P}^3}(k_{d,g,3}-1)$ or $\mathcal{O}_{\mathbb{P}^3}(k_{d,g,3})$. To get the condition $h^1(C, N_C(-S)) = 0$ we also impose that each of these s lines intersects the other part (outside the nilpotent considered in [1]) only in one point. The associated reducible and reduced curve is not linearly normal, but still the restriction map at degree $k_{d,g,3}$ or degree $k_{d,g,3} - 1$ is surjective or injective and this linear map is a map between vector spaces of the expected dimension $\binom{k_{d,g,3}+3}{3} - s$ and $k_{d,g,3}d + 1 - g - s$ or $\binom{k_{d,g,3}+2}{3} - s$ and $(k_{d,g,3}-1)d + 1 - g - s$. Incresing if necessary $\eta(3,s)$ we may assume $k_{d,g,3} \geq 6s$. Under this numerical condition to get the injectivity part we may always assume that in the last step we insert at least $2s$ lines. Hence it is sufficient to consider the surjectivity part, i.e. it is sufficient to prove the existence of C_1 when in the last step we insert at most $s+4$ lines. Look again to [1]. Call T the scheme obtained in degree $k_{d,g,3} - 2$. By assumption $T \cap S = \emptyset$ and $h^i(\mathbb{P}^3, \mathcal{I}_T(k_{d,g,3}-2)) = 0$, $i = 0, 1$. By assumption we have $d - \deg(T) \leq k_{d,g,3}/3$. Hence in the last step we get inside a smooth quadric Q the set S, a one-dimensional scheme Y, say of type (a, b), and a zero-dimensional set $Z := T \cap (Q \backslash Y)$. Under this numerical assumptions $(k_{d,g,3} - a + 1)(k_{d,g,3} - b + 1) - s - \mathrm{length}(Z) \geq s \cdot (k_{d,g,3} + 1)$. Hence in the previous step we add $s + 4$ lines less than in [1] and we add these $s + 4$ lines in the last step.

(b) Here we assume $r \geq 4$ and that the result is true for the integer $r' := r - 1$. Fix a hyperplane $H \subset \mathbb{P}^r$. We use [4] instead of [1] and H instead of the quadric surface. Again, we add S inside H only at the last step and we modify the last step in the surjectivity part in the last step we add a low degree scheme. □

Proof of Proposition 1. Define the critical value as in the proof of Theorem 1. Now we start with an arbitrary S and with add then piece by piece a reducible curve to obtain at the end a reducible and curve $T \subset \mathbb{P}^r$ such that $S \subset T_{reg}$, $\deg(T) = d$, $h^1(T, N_T(-S)) = 0$ and T is smoothable keeping S fixed and with a smoothing family of curves containing T. For the smoothing part we apply [5], Theorem 4.1, to the blowing-up W_S instead of $\mathbb{P}^r$. For the inductive procedure and the control of the intersection with a smooth quadric surface (case $r = 3$) or a hyperplane (case $r \geq 4$), see [2] and [3]. □

3. The Low Genus Case

In this section we consider the case $0 \leq g < \gamma(r, s)$. Fix integers $r \geq 3$ and $s \geq 0$. Set $\gamma'(r, s) := 0$, $\delta'(r, s) := r \cdot \lceil s/2 \rceil$, $A'(r, 0) := \emptyset$ and $W(d, g; r, \emptyset)' := W(d, g; r)$. If $s > 0$ define the curve $A'(r, s)$ in the following way. Fix an ordering $P_1, \dots, P_s$ of S. Let $A'(r, s)$ be the general union of $\lceil s/2 \rceil$ rational normal curves $D_1, \dots, D_{\lceil s/2 \rceil}$ with the restriction that $S \cap D_i = \{P_{2i-1}, P_{2i}\}$ for all i, $D_i \cap D_j = \emptyset$ for all i, j such that $|i - j| \geq 2$, $\sharp(D_i \cap D_{i+1}) = 1$ for all $1 \leq i \leq \lceil s/2 \rceil - 1$ and D_i intersects transversally D_{i+1} for all $1 \leq i \leq \lceil s/2 \rceil - 1$. Notice that $A'(r, s)$, $s > 0$, is connected, nodal and $A'(r, s) \in W(\delta'(r, s), 0; r)$. As in [4], Corollary 1.3, we get $h^1(A'(r, s), N_{A'(r,s)}(-S)) = 0$. Now fix integers d, g such that $W(d - \alpha'(r, s), g; r)$ is defined, i.e. such that $g \geq 0$ and either $d - \alpha'(r, s) \geq g + r$ or $g \geq 2$ and $d - r - \alpha'(r, s) < g \leq \lfloor (d - r - 2)/(r - 2) \rfloor$. If $d - \alpha'(r, s) \geq g + r$ let $B'(r, s)$ be the general union of $A'(r, s)$ and a smooth elements of $W(d - \alpha'(r, s), g; r)$ intersecting $A'(r, s)$ quasi-transversally and at a unique point which belong to $D_{\lceil s/2 \rceil}$. Now assume $d - r - \alpha'(r, s) < g \leq \lfloor (d - \alpha'(r, s) - r - 2)/(r - 2) \rfloor$ and set $t := g + r - d + \alpha'(r, s)$, $z := d - \alpha'(r, s) - r - 2 - t(n - 2)$. Let $C \subset$ be a general normal rational curve with the only restriction that $\sharp(C \cap D_{\lceil s/2 \rceil}) = 1$. Hence C intersects quasi-transversally $D_{\lceil s/2 \rceil}$. Fix general secant lines A, B of C. Fix $t(r-2)$ general points $P_{ij} \in C$, $1 \le j \le t$, $3 \le i \le r$. As in [4] let $V_j \subset \mathbb{P}^r$, $1 \le j \le t$, be the only linear subspace such that $V_j \cap A \ne \emptyset$, $V_j \cap B \ne \emptyset$, $P_{xy} \in V_j$ if and only if $y = j$ and $V_j \cap (A \cup B) \cup C = \emptyset$. Set $P_{1j} := A \cap V_j$ and $P_{2j} := B \cap V_j$. Fix a general rational normal curve C_j of V_j such that $P_{ij} \in C_j$ for all $i \in \{1, \dots, r\}$. Fix z general secant lines $T_1, \dots, T_z$ of C. For a general choices of these curves we have $S \cap C \cup A \cup B \cup C_1 \cup \dots \cup C_t \cup T_1 \cup T_z) = \emptyset$. Set $B'(d,g;r,S) := A'(r,S) \cup C \cup A \cup B \cup C_1 \cup \dots \cup C_t \cup T_1 \cup T_z$. For general choives the curve $B'(d,g;r,S)$ is nodal. In both cases (i.e. even if $d - \alpha'(r,s) \ge g + r$) it is easy to check that $B'(d,g;r,S) \in W(d,g;r)$ (use [4], Lemmas 2.2 and 2.3), that $S \subset B'(d,g;r,S)_{reg}$, and that $h^1(B'(d,g;r,S), N_{B'(d,g;r,S)}(-S)) = 0$. Hence $B'(d,g;r,S)$ is a smooth point of $\mathrm{Hilb}(d,g,S)'$. Hence there is a unique irreducible component of $\mathrm{Hilb}(d,g,S)'$ containing $B'(d,g;r,S)$ and we call it $W(d,g;r,S)'$. Using Remark 2 and [4], Lemmas 2.2 and 2.3, we immediately get the following result.

Lemma 2. *Fix a finite $S \subset \mathbb{P}^r$ and integers $d, g, , b$ such that $1 \le b \le r-1$ and $W(d,g;r,S)'$ is defined. Fix an integral non-degenerate $Y \in W(d,g;r,S)'$ such that $h^1(Y, N_Y(-S)) = 0$. Let D be a rational normal curve of a b-dimensional linear suspace of $\mathbb{P}^r$ such that $1 \le \sharp(D \cap Y) \le b+1$ and D intersects quasi-transversally Y and $S \cap D = \emptyset$. Let F be a rational normal curve of $\mathbb{P}^r$ such that $F \cap S = \emptyset$, $1 \le \sharp(F \cap Y) \le r+2$ and F intersects quasi-transversally Y. Then $Y \cup D \in W(d+b, g+\sharp(D \cap Y) - 1; r, S)'$, $Y \cup F \in W(d+r, g+\sharp(F \cap Y) - 1; r, S)'$ and $h^1(Y \cup D, N_{Y \cup D}(-S)) = h^1(Y \cup F, N_{Y \cup F}(-S)) = 0$.*

Theorem 2. *Fix integers $r \ge 3$ and $s > 0$. Then there are an integer $\theta(r,s) \ge \delta(r,s)$ and a function $v_{r,s} : \mathbb{N} \to \mathbb{N}$ such that $\lim_{g \to +\infty} v_{r,s}(g) = (r-2)/(r-1)$ and the following properties are true. Let $S \subset \mathbb{P}^r$ be a general subset with cardinality s. Fix integers d, g such that $g \ge 0$ and $d - \delta(r,s) \ge g \cdot v_{r,s}(g)$. Then $W(d,g;r,S)'$ is defined and a general $C \in W(d,g;r,S)'$ has maximal rank both as a curve in $\mathbb{P}^r$ and modulo S, i.e. for all integers $x > 0$ the restriction maps $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \to H^0(C, \mathcal{O}_C(x))$ and $H^0(W_S, \mathcal{L}_{x,S}) \to H^0(C_S, \mathcal{L}_{x,S}|C_S)$. Moreover, $h^0(C_S, \mathcal{L}_{x,S}|C_S) = h^0(C, \mathcal{O}_C(x)(-S)) = xd + 1 - g - s$ for all $x \ge 2$.*

Proof of Theorem 2. By semicontinuity to get the statement of the theorem for the pair of integers (d,g) is sufficient to find a pair (C,S) such that $\sharp(S) = s$, $W(d,g;r,S)'$ is defined, $C \in W(d,g;r,S)'$, C smooth, $h^1(C, N_C(-S)) = 0$ (and hence $C \in W(d,g;r,S)'_{reg}$, $h^0(C, \mathcal{O}_C(2)(-S)) = 0$ and C has maximal rank both as a curve in $\mathbb{P}^r$ and a modulo S. For large $\theta(r,s)$ we are working in a range for which for all d, g such that $W(d, g; r, S)'$ is defined we have $h^1(C, \mathcal{O}_C(2)(-S)) = 0$ and $h^0(\mathbb{P}^r, \mathcal{I}_C(x)) = 0$ if $x \le s-1$. Hence we may define the critical value $k_{d,g,r}$ of the pair (d, g) as in the proof of Theorem 1 and use the discussion made their. Hence it is sufficient to get $C \in W(d, g; r, S)'$ with maximal rank modulo S. By Theorem 1 for fixed r, s we only need to check finitely many genera g. Hence it is sufficient to prove Theorem 2 in $\mathbb{P}^r$ for a fixed genus g. Hence we may apply Proposition 1. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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