International Journal of Pure and Applied Mathematics
Volume 54 No. 3 2009, 429-435

TINKERBELL CHAOS IN A RING PHASE-CONJUGATED RESONATOR

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Abstract: The dynamics of an optical ring phase conjugated resonator with Tinkerbell chaos is presented. It is shown that a non-linear behavior takes places when a chaos generating element is introduced on the resonator. The matrix formalism is used in order to carry out the analysis of the resonator in his full-round trip when all the optical resonator elements are involved. Assuming ray optics inside the cavity with parameters $y(z)$ and $\theta(z)$ for the effective distance to the optical axis and the angle to the same axis respectively, the expressions for the n-th trip are obtained. The matrix of an optical chaos generating element $[a, b, c, e]$ able to produce Tinkerbell chaos in an ideal ring phase conjugation resonator are obtained for the first time to our knowledge.

AMS Subject Classification: 37N20, 78A05, 78A60, 37E99
Key Words: chaos, Tinkerbell cahos, resonator

1. Introduction

A chaotic map is a map that exhibits some sort of chaotic behavior. In general maps may be parameterized by a discrete-time or a continuous-time parameter. Discrete maps usually take the form of iterated functions. Chaotic maps often occur in the study of dynamical systems. Chaos has been found and studied in many different systems, e.g. mechanical [10], electrical [9], chemical and

Received: June 27, 2009
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biological [8], climatic and many others [2]. Chaos has also been widely studied in laser physics [7]. Lasers as dynamical systems, specially when compared to other systems such as, for example global climate models, are easy to experiment with, to study and to control. It is generally agreed that lasers are a powerful tool to test and experiment complex dynamical systems models. Therefore the study of chaotic physical systems using lasers allows for the search and better understanding of theoretical and computational models to describe them.

In this work a laser optical phase conjugated ring resonator is taken to a chaotic regime described by a theoretical Tinkerbell-chaos map [5, 4], this is done following the original idea proposed in [1], by introducing in the optical phase conjugated ring resonator an appropriate Tinkerbell chaos generating matrix element. The system obeys the conditions given by the Tinkerbell map, the basic features and the conditions involved in this happening are discussed.

2. Tinkerbell Map

The Tinkerbell map [3], [6] is a discrete-time dynamical system given by the equations:

$$y_{n+1} = y_n^2 - \theta_n^2 + \alpha y_n + \beta \theta_n \,, \tag{1}$$

$$\theta_{n+1} = 2 y_n \theta_n + \gamma y_n + \delta \theta_n \,. \tag{2}$$

As an example, Figure 1 shows the map obtained for particular values of the four parameters involved ($\alpha = 0.9$, $\beta = -0.6013$, $\gamma = 2.0$ and $\delta = 0.5$.).

3. Matrix Description of a Ring Resonator

As it is well known any optical element may be described by a matrix $[A, B, C, D]$. Having radial symmetry around the optical axis and defining the perpendicular distance of any ray to the optical axis and the angle to the same axis at a given position z as: $y(z)$ and $\theta(z)$, then within the paraxial approximation the parameters y and at two points along the optical axis are related by:

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} . \tag{3}$$

The total matrix of an optical system is obtained by taking the matrix product of each one of the optical elements of the system. Figure 2 shows an optical ring phase conjugated resonator made up by a phase conjugated cell

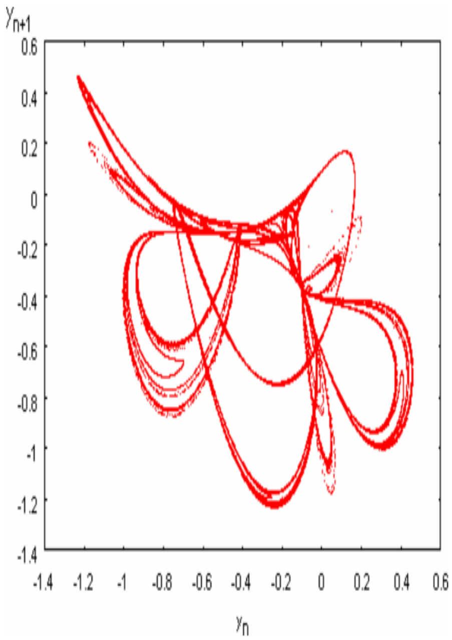

Figure 1: Tinkerbell map with $\alpha = 0.9$, $\beta = -0.6013$, $\gamma = 2.0$ and $\delta = 0.5$

PC and two flat mirrors separated by a distance d. Between the two mirrors an unknown chaos generating element represented by the matrix $[a, b, c, e]$ is introduced.

The total transformation matrix $[A, B, C, D]$ for a complete round trip of the above phase conjugating ring resonator is given by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d/2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} 1 & d/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} . \quad (4)$$

On the right hand side of expression (4), from left to right, the first matrix describes an ideal phase conjugation element, followed by a matrix describing the propagation of a beam in vacuum along a distance d, then the matrix of an ideal flat mirror is present followed by the propagation of the beam a distance $d/2$ where a chaos generating element $[a, b, c, e]$ is found, followed by a matrix describing the propagation along a distance $d/2$, a matrix describing

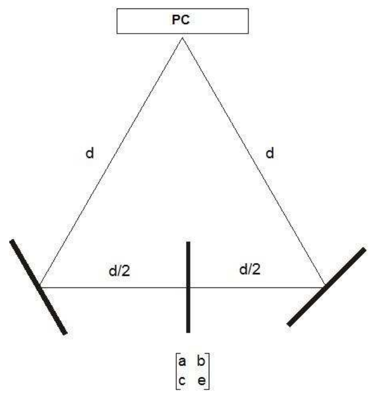

Figure 2: Ring resonator containing a phase conjugated mirror (PC) and a chaos generating element described by the matrix $[a, b, c, e]$

the reflection on a flat mirror and finally a matrix describing the propagation along a distance d. The elements of the above total transformation matrix of the ring phase conjugated resonator are therefore:

$$A = a + \frac{3cd}{2}, \tag{5}$$

$$B = b + \frac{3d}{4}(2a + 3cd + 2e), \tag{6}$$

$$C = -c, \tag{7}$$

$$D = -\frac{3cd}{2} - e. \tag{8}$$

Assuming a symmetric ring beam inside the resonator with parameters $y(z)$ and $\theta(z)$ confined in the resonator, and taking y_0 as the initial radius of the beam at an arbitrary position $z = 0$, we may obtain using equations (5)-(8), expressions for the n-th trip $y(z)_n$ and $\theta(z)_n$ at any position z along the resonator path.

4. Chaos Generating Matrix

Using the above round trip transformation matrix $[A, B, C, D]$ we can find an expression to obtain y_{n+1} and θ_{n+1} from y_n and θ_n. From (3) the following general expression holds:

$$\begin{pmatrix} y_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_n \\ \theta_n \end{pmatrix}. \tag{9}$$

Since we are looking for a chaos-generating mapping of the form shown in equations (1) and (2) the matrix $[a, b, c, e]$ of the chaos generating element needs to be found, such that equation (9) is able to generate the required equations for y_{n+1} and θ_{n+1}. From (9) we obtain,

$$y_{n+1} = Ay_n + B\theta_n \,, \tag{10}$$

$$\theta_{n+1} = Cy_n + D\theta_n \,. \tag{11}$$

In order to obtain the Tinkerbell map equations as given by (1) and (2) we must have in the above system the following values for the coefficients A, B, C and D. It should be noted that these coefficients are not constants but depend of y_n and θ_n and the Tinkerbell map parameters α, β, γ and δ.

$$A = y_n + \alpha \,, \tag{12}$$

$$B = -\theta_n + \beta \,, \tag{13}$$

$$C = 2\theta_n + \gamma \,, \tag{14}$$

$$D = \delta \,. \tag{15}$$

Substituting equations (5)-(8) into equations (12)-(15) we obtain a system of equations for a, b, c and e of the following form:

$$a + \frac{3cd}{2} = y_n + \alpha \,, \tag{16}$$

$$b + \frac{3d}{4}(2a + 3cd + 2e) = -\theta_n + \beta \,, \tag{17}$$

$$-c = 2\theta_n + \gamma \,, \tag{18}$$

$$-\frac{3cd}{2} - e = \delta \,. \tag{19}$$

When the equations system (16)-(19) is solved for a, b, c and e, the following solutions are obtained:

$$a = \alpha + \frac{3d}{2}(\gamma + 2\theta_n) + y_n \,, \tag{20}$$

$$b = -\frac{1}{4}(-4\beta + 6\alpha d - 6d\delta + 9d^2\delta + 4\theta_n + 18d^2\theta_n + 6dy_n) \,, \tag{21}$$

$$c = -\gamma - 2\theta_n \,, \tag{22}$$

$$e = -\delta + \frac{3d}{2}(\gamma + 2\theta_n) \,. \tag{23}$$

Taking the solutions (20) to (23) for the elements of the matrix $[a, b, c, e]$

into equation (4), the total transformation matrix $[A, B, C, D]$ for a round trip now has elements given by (12)-(15). Substituting in equation (9), we obtain the equations for y_{n+1} and θ_{n+1} as the Tinkerbell map (1) and (2). As can be seen and expected, the elements of the chaos generating matrix given by (20)-(23), depend on y_n and θ_n and also on the Tinkerbell map parameters α, β, γ and δ.

5. Conclusions

A ring resonator with a phase conjugated mirror and a chaos generating element is studied. The conditions that the elements of the chaos generating element must satisfy in order to obtain Tinkerbell chaos for the iteration y_{n+1} and θ_{n+1} are obtained. It is found that the elements of the chaos generating element are dynamic and depend not only on the previous y_n and θ_n value but also on the parameters of the Tinkerbell maping as well as on the constant resonator parameter d. The matrix elements of a chaos generating element $[a, b, c, e]$ able to produce Tinkerbell chaos in an ideal ring phase conjugation resonator are found for the first time to our knowledge.

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