International Journal of Pure and Applied Mathematics
Volume 54 No. 3 2009, 437-450

VOTING MATRICES AND TIE-BREAKING

Jeffrey L. Stuart[1 §], James R. Weaver[2]

[1]Department of Mathematics
Pacific Lutheran University
Tacoma, WA 98447, USA
e-mail: jeffrey.stuart@plu.edu

[2]The University of West Florida
Pensacola, FL 32514, USA
e-mail: jweaver@uwf.edu

Abstract: Consider an election among m candidates in which each of n judges casts nonnegative, weighted votes with weights totaling $k > 0$ subject to certain rules. The weight of the vote cast by judge j for candidate i is recorded in the i, j entry of the $m \times n$ voting matrix A. The row sums of the candidate matrix $C = AA^T$ are used to rank the candidates. Under simple restrictions on A, the Perron vector for C is used to help break ties that occur among the row sums. Ranking of the judges via the judge matrix $J = A^T A$ is also examined.

AMS Subject Classification: 15A18
Key Words: Perron vector, ranking, tie-breaking, voting

1. Introduction

This paper is motivated by the problem of selecting a winning candidate when judges are allowed to allocate their weighted votes to multiple candidates. Specifically, we consider the following situation: there are n judges choosing among a list of m candidates. Each judge is granted a total of k votes, and each judge must divide those votes among the candidates according to preset rules. Note that $k > 0$, but k need not be an integer, and that the weight

Received: June 30, 2009

§Correspondence author

of a vote, while positive, need not be an integer. After all of the judges have voted, the votes awarded to each candidate are summed. The winner is the candidate possessing the largest sum of votes, if such a candidate exists. This paper is focussed on the situation in which there is a tie, that is, more than one candidate possesses the largest sum of votes.

Our broad approach encompasses a variety of well-known voting schemes. In the simplest case, k is a positive integer with $k < m$, and each judge must allocate exactly one vote to each of k candidates. Two common, more complicated voting schemes that are also encompassed are the Borda count and the modified Borda count. In the Borda count, there is a predetermined set of vote weights, $a_1 > a_2 > \cdots > a_m > 0$ with $k = a_1 + a_2 + \cdots + a_m$. Each judge ranks all m candidates, and then, for $i = 1, 2, \ldots, m$, awards a vote with weight a_i to the candidate with the i-th highest rank. In the modified Borda count, each judge ranks all m candidates, but only awards weights to the highest h candidates for some specified positive integer $h < m$, with the weights $a_1 > a_2 > \cdots > a_h > 0$ and $k = a_1 + a_2 + \cdots + a_h$. More generally, we allow any weighted voting scheme such that for $1 \leq j \leq n$, the j-th judge awards votes with weights $a_{1j} \geq a_{2j} \geq \cdots \geq a_{mj} \geq 0$ such that $a_{1j} + a_{2j} + \cdots + a_{mj} = k$. That is, each judge may have a different set of vote weights and may cast a different number of votes, but the sum of the weighted votes awarded by each judge is constant.

For convenience, votes can be recorded in a matrix that we will call a *voting matrix*. Specifically, we define a voting matrix A as follows: Label the candidates (rows) in any order with the integers 1 through m, and label the judges (columns) in any order with the integers 1 through n. For each pair i and j, set $A_{i,j} = a_{ij}$ if judge j awards a vote of weight a_{ij} to candidate i.

Throughout this paper, e will denote a column vector of ones of whatever size is required for matrix multiplication to be defined. When necessary for clarification, e_n will denote the $n \times 1$ vector of ones. If A is a voting matrix, then Ae will be called the *score vector for* A. The entries of Ae are precisely the vote totals for each of the candidates. Notice also that $A^T e = ke$ since the entries of $A^T e$ are the sums of the votes cast by each judge, and such sums are constant with value k.

If there is a unique maximum entry in the score vector, then there is a natural winner. Unfortunately, in many situations, there are multiple candidates tied for the maximum sum. It is the resolution of ties that we will explore in this paper. Clearly, it is desirable that our methods be independent of any relabeling of the judges or of the candidates, or equivalently, be independent of premultiplying or postmultiplying the voting matrix A by a permutation matrix of the appropriate size.

In considering the relative strengths of the candidates, it is natural to compare one candidate directly to another. We will employ an $m \times m$ matrix that we call the *candidate matrix* for this purpose. Specifically, if A is a voting matrix, then we call $C = AA^T$ the candidate matrix corresponding to A. We will also comment upon the related $n \times n$ matrix, $J = A^T A$, which we call the *judge matrix*.

The matrices C and J are positive semidefinite matrices obtained from a matrix A whose entries are nonnegative. Such matrices have been studied by T. Ando [1], and A. Berman and N. Shaked-Monderer [2] in the context of completely positive matrices; and by A. Berman and C. Xu [3] in the context of S-factorizable matrices and S-completely positive matrices. Candidate matrices and judge matrices are also closely related to hub matrices and authority matrices, which were introduced by Kleinberg as part of his HITS algorithm for ranking web pages [10, Chapter 3].

Recently, there has been substantial attention paid to ranking approaches [10], especially in the context of ranking the relevance of web pages to a particular search topic. Such approaches mostly bypass simple vote counts (numbers of web links in the context of web searches), seeking methods that capture the hierarchies of interconnection. These approaches often explicitly ignore issues of vote ties, breaking any ties that do occur with some sort of rule that takes the first "winner" encountered as the winner. In this paper, we are interested in the situation where actual vote totals are the primary means of ranking, as would be the case in voting on a committee or in some sort of election. For us, more sophisticated approaches serve the as a secondary method invoked to break ties. The availability of a secondary, tie-breaking approach is important because revoting often has political, temporal or financial costs. We will employ Perron vector techniques whose origins trace back to the ranking work of Kendall and Smith [8] and Wei [13] (see also the more recent work by Keener [7]).

Example 1. An example of a simple voting scheme where each judge votes for exactly three candidates with constant weighting:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$Ae = \begin{bmatrix} 6 & 7 & 7 & 4 & 6 \end{bmatrix}^T.$$

Here candidates 2 and 3 are tied for first place, each with 7 votes.

Example 2. An example of a modified Borda count scheme where each judge votes for his or her top three candidates using the weights of $3, 2$ and 1:

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 2 & 2 & 0 & 3 & 1 & 0 & 2 & 3 & 2 \\ 2 & 3 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 & 0 & 3 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 0 & 3 & 3 & 1 & 0 \end{bmatrix},$$

$$Ae = \begin{bmatrix} 11 & 15 & 12 & 10 & 12 \end{bmatrix}^T.$$

Here candidate 2 is the unique winner with 15 votes.

2. Symmetric Irreducible Matrices

We will ultimately employ certain eigenvectors whose existence for primitive, nonnegative matrices is guaranteed by Perron-Frobenius theory to break ties; consequently, we next investigate conditions that will guarantee the primitivity of the matrices C and J introduced in the the previous section.

Lemma 3. *If M is a symmetric matrix, then M is either irreducible, or else M is permutation similar to a direct sum of symmetric, irreducible matrices.*

Lemma 4. *If M is a nonzero, symmetric, irreducible matrix, then M is either primitive or imprimitive with index of imprimitivity 2. Further, if any diagonal entry of M is nonzero, then M is primitive.*

The proof of the first lemma follows from the Frobenius normal form of M since a matrix is symmetric if and only if its Frobenius normal form is symmetric. The proof of the index claim in the second lemma follows from the block circulant structure of the cyclic normal form for an irreducible, imprimitive matrix [4]. If a matrix is irreducible and there is a nonzero diagonal entry, then just look at paths in the corresponding digraph that pass through and loop as necessary at the vertex corresponding to the nonzero diagonal entry.

Lemma 5. *Let $C = AA^T$ and $J = A^T A$ for a voting matrix A. The matrix C has a zero entry on its diagonal if and only if A has a zero row; and all diagonal entries of J are positive.*

Proof. Note that $C_{ii} = \sum_{j=1}^{m} A_{ij}(A^T)_{ji} = \sum_{j=1}^{m} A_{ij}^2$. Thus $C_{ii} = 0$ if and only if $A_{ij} = 0$ for all j, which is to say, if and only if i-th row of A is a zero row. The proof for J has a zero on its diagonal if and only if A has a zero column is is analogous. Since every column of A has sum $k > 0$, the result follows. □

Remark 6. If A has a zero row then the corresponding candidate is awarded no votes. In such cases, it is natural to remove that candidate from consideration, and to delete the corresponding zero row of A.

Example 7. Suppose that the first three judges are members of one political party, and that party proposes a single candidate. Suppose that the remaining four judges are members of an opposing party, and that party has two candidates, each of whom is equally regarded within the party. Using $k = 1$, one possible voting matrix would be

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

with score vector $Ae = \begin{bmatrix} 3 & 2 & 2 \end{bmatrix}^T$. Observe that the candidate matrix is completely reducible:

$$C = AA^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

In this case the minority party's candidate would be the winner, a situation that the majority party would find unacceptable. Further, this situation is unresolvable without replacing the voting matrix with one in which the majority party changes its vote allocation to favor a single candidate.

The previous example motivates our interest in the following results.

Lemma 8. *Let A be a voting matrix that has no zero rows. If there exists a reordering of the rows and columns of A such that the resulting matrix $\widehat{A}$ has the form*

$$\widehat{A} = \begin{bmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{bmatrix}, \tag{1}$$

where A_1 and A_2 are nonempty, then $C = AA^T$ and $J = A^TA$ are completely reducible. Further, applying the row reordering to the rows and columns of C yields the matrix $\widehat{C}$ given below; and applying the column reordering to the rows and columns of J yields the matrix $\widehat{J}$ given below:

$$\widehat{C} = \begin{bmatrix} A_1A_1^T & \mathbf{0} \\ \mathbf{0} & A_2A_2^T \end{bmatrix} \text{ and } \widehat{J} = \begin{bmatrix} A_1^TA_1 & \mathbf{0} \\ \mathbf{0} & A_2^TA_2 \end{bmatrix},$$

where $C_1 = A_1A_1^T$ and $C_2 = A_2A_2^T$ are the candidate matrices corresponding *to the partitioned sets of candidates and judges, and where* $J_1 = A_1^T A_1$, *and* $J_2 = A_2^T A_2$ *are the judge matrices corresponding to the partitioned sets of candidates and judges.*

Proof. This follows from the fact that $\widehat{A} = PAQ^T$ for appropriate permutation matrices P and Q, so $\widehat{A}$ is a voting matrix. Each column of A_1 and of A_2 contains the same set of nonzero entries (including multiplicities) as a column of A. Thus A_1 and A_2 are themselves voting matrices for the same voting scheme that produced A. Clearly $\widehat{C}$ and $\widehat{J}$ have the specified forms with $\widehat{C} = PCP^T$ and $\widehat{J} = QJQ^T$. □

Lemma 9. *Let* A *be a voting matrix with no zero rows. If either* C *or* J *is not irreducible, then there is a reordering of the rows and columns of* A *such that the resulting matrix* $\widehat{A}$ *has form 1.*

Proof. Suppose that C is not irreducible. By Lemma 4, C must be completely reducible with at least two irreducible blocks. Since A has no zero rows, each irreducible, diagonal block of C must be nonzero. Partition the set of candidates into two nonempty sets $S^{(1)}$ and $S^{(2)}$ corresponding to those whose indices are in the first block and those whose indices are not, respectively. Thus $C_{\alpha\beta} \neq 0$ for $\alpha \in S^{(i)}$ and $\beta \in S^{(j)}$ implies $i = j$. Partition the set of judges into three (possibly empty) sets $T^{(1)}, T^{(2)}, T^{(12)}$ corresponding to those who vote only for candidates in $S^{(1)}$, those who vote only for candidates in $S^{(2)}$, and those who vote for candidates in both $S^{(1)}$ and $S^{(2)}$. Suppose that there exists a judge j is in $T^{(12)}$. Suppose that judge j casts votes for some candidate α in $S^{(1)}$ and some candidate β in $S^{(2)}$. Then $A_{\alpha j} \neq 0$ and $A_{\beta j} \neq 0$, so $A_{\alpha j} A_{\beta j} = A_{\alpha j} \left(A^T\right)_{j\beta} > 0$. Since A is nonnegative, and since $A_{\alpha j} \left(A^T\right)_{j\beta}$ is a summand in the α, β entry of $AA^T = C$, it follows that $C_{\alpha\beta} \neq 0$, a contradiction. Thus the sets $T^{(1)}$ and $T^{(2)}$ partition the judges. Since each irreducible, diagonal block of C is nonzero, there must be a judge that votes for candidates in $S^{(1)}$ and there must be a judge who votes for candidates in $S^{(2)}$; hence, $T^{(1)}$ and $T^{(2)}$ are both nonempty. Thus the sets $S^{(1)}$ and $S^{(2)}$ give rise to the row permutations needed to transform A to $\widehat{A}$, and $T^{(1)}$ and $T^{(2)}$ give rise to the column permutations needed.

The proof in the case that J is not irreducible is analogous, simply interchange the roles of rows and columns and the roles of candidates and judges. □

Example 10. The stipulation that the voting matrix has no zero row is needed. Let A be the voting matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then C is not irreducible, but A cannot be transformed into form 1 by row and column permutations.

If the rows of a voting matrix can be reordered to obtain a matrix A that has the form 1, then we can independently evaluate the candidates and judges corresponding to each of the diagonal blocks of A. Note that there is a finest reordering of the rows so that A is block-diagonal and no diagonal block can be subdivided into form 1.

Algorithm 11. Input: An $m \times n$ voting matrix A. Output: A partition of the judges $\{1, 2, \ldots, n\}$ into nonempty sets $U_1, U_2, \ldots, U_t$ and a partition of the candidates $\{1, 2, \ldots, m\}$ into nonempty sets $V_1, V_2, \ldots, V_t$ and a possibly empty set V_{t+1} such that: for $p = 1, 2, \ldots, t$, every judge in U_p only casts votes for candidates in V_p, such that every candidate in V_p receives at least one vote from some judge in U_p, and such that every candidate (if any) in V_{t+1} receives no votes from any judge.

1. Set $t = 1$. Set $U_1 = \{1\}$. Set $V_1 = \phi$

2. Let V_t consist of all candidates who received votes from judges in U_t.

3. Let U_t consist of all judges who voted for at least one candidate in V_t.

4. Return to step 2 unless neither U_t nor V_t have changed during the previous iteration of steps 2 and 3.

5. Set $t = t + 1$. If there is at least one judge not included in $\cup_{p<t} U_p$, let U_t be the set consisting of the index of the first judge not included $\cup_{p<t} U_p$, set $V_t = \phi$, and return to step 2. If every judge has already been included in $\cup_{p<t} U_p$, let V_t consist of all candidates who have not been included in any of the sets V_p for $p < t$; if there are no such candidates, set $V_{t+1} = \phi$.

6. Set $t = t - 1$. Stop.

Proof. Since every judge votes, the set V_t in step 2 is well-defined and nonempty. Since V_t is nonempty, the set U_t in step 3 is well-defined and nonempty. If step 4 returns to step 2, the set V_t must contain all candidates that were previously in V_t. Then the set U_t in step 3 must contain all judges that were previously in U_t. Since the number of judges and the number of candidates are finite, step 4 will return to step 2 a finite number of times and step 5 will return to step 2 only a finite number of times. Since every judge votes for at least one candidate, every judge will occur in exactly one of the nonempty sets U_p. Clearly the sets V_p are disjoint and partition the set of candidates, and V_{t+1} contains exactly those candidates who receive zero votes (if there are any such candidates). □

Remark 12. Given a voting matrix A, run the preceding algorithm on A. If V_{t+1} is nonempty, let B be the matrix obtained from A by deleting all rows of A whose indices are in V_{t+1}. If $V_{t+1} = \phi$, let $B = A$. Relabel the rows of B so that the rows are selected from V_1, then from V_2, and so on, and relabel the columns of A so that the columns are selected from U_1, then from U_2, and so on. After this relabelling, B is in block diagonal form with t nonzero blocks, and there is no relabeling of the rows and columns of B that results in a block diagonal matrix with more than t diagonal blocks.

Remark 13. The rows of a voting matrix cannot be reordered to obtain a matrix A in the form 1 precisely when Algorithm 11 returns $t = 1$ and $V_2 = \phi$.

A voting matrix A will be called a *proper voting matrix* if it has no zero rows and no relabeling of the rows will transform A to form 1.

Summarizing:

Theorem 14. *If A is a proper voting matrix then the nonnegative matrices $C = AA^t$ and $J = A^tA$ are irreducible and primitive.*

3. Perron-Frobenius Theory and Convergence of the Power Method

Let A be a proper voting matrix with rank r. We employ the compact form of the singular value decomposition [11] of the primitive matrix $C = AA^T$:

$$A = U\Sigma V^T\,,$$

where $U = [u_1, \cdots, u_r]$ and $V = [v_1, \cdots, v_r]$ are orthogonal matrices, and where Σ is the $r \times r$ matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_r \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Since $C = AA^T = U\Sigma^2U^T$ is nonnegative, irreducible and primitive by Theorem 14, it follows from the Perron-Frobenius Theorem that $(\sigma_1)^2 > (\sigma_2)^2$, that u_1 must be either strictly positive or strictly negative, and that u_1 is unique up to scalar multiplication. Hence $\sigma_1 > \sigma_2$. Further, since all powers of C are nonnegative, it follows that u_1 must be strictly positive. Applying the same arguments to $J = A^TA = V\Sigma^2V^T$, v_1 must be strictly positive. Since $\sigma_1 > \sigma_2$ and since C^pe is nonnegative for all positive integers p, it follows that the normalized power method applied to C with starting vector e converges to u_1. Similarly, the normalized power method applied to J with starting vector e converges to u_1.

Summarizing:

Theorem 15. *Let A be a proper voting matrix. Let ρ be the common Perron eigenvalue for $C = AA^T$ and $J = A^T A$, and let u and v be the unique positive eigenvectors of unit length for ρ for C and J, respectively. The normalized power method applied to C with any nonzero starting vector converges to u, and applied to J with any nonzero starting vector converges to v. Specifically, the sequence of vectors obtained by normalizing each vector in the sequence $Ce, C^2e, C^3e, \ldots$ converges to u; and the sequence of vectors obtained by normalizing each vector in the sequence $Je, J^2e, J^3e, \ldots$ converges to v.*

The vectors u and v in the previous theorem are called the Perron vectors for C and J, respectively.

4. Perron Vectors and Ranking Candidates

As was perhaps first noted by Kendall, the entries of u can be used to rank the candidates. Essentially, what u captures is the weighted interconnectivity of the candidates. Why? Consider the sequence $Ce, C^2e, C^3e, \ldots$, which, when normalized at each iteration, converges to u. The first vector in the sequence $Ce = AA^Te = A(ke) = k(Ae)$ is just k times the score vector for the voting matrix; that is, its entries effectively record the number of judges that voted for each candidate. What do the subsequent vectors record? It will be convenient to employ a special bipartite graph in analyzing this question. Our analysis will also tell us about how the entries of v can be used to evaluate the judges.

Let A be a proper voting matrix. Form a loop-free bipartite graph G whose vertex bipartition corresponds to the set of candidates and the set of judges, and whose edges arise from votes cast. Then G has edge-weighted adjacency matrix

$$B = \begin{bmatrix} \mathbf{0} & A \\ A^T & \mathbf{0} \end{bmatrix}.$$

Observe that

$$B^2 = \begin{bmatrix} AA^T & \mathbf{0} \\ \mathbf{0} & A^TA \end{bmatrix} = \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix},$$

and more generally,

$$B^{2p} = \begin{bmatrix} C^p & \mathbf{0} \\ \mathbf{0} & J^P \end{bmatrix}$$

for each positive integer p. The candidate matrix C, the first diagonal block of B^2, stores the sum of the weights of the loop-free walks in G between pairs of candidates. That is, C_{ii} is the sum of the weights of the walks of length two in G starting and ending at candidate i. Equivalently, $C_{ii} = \sum_{k=1}^{m} a_{ik}(A^T)_{ki} = \sum_{k=1}^{m} (a_{ik})^2$, the sum of the squares of the weighted votes earned by candidate i. Note that C_{ii} is relatively large when candidate i receives lots of votes and when those votes are highly weighted. For $i \neq j$, C_{ij} sums the weights of the walks of length two between candidates i and j. Equivalently, $C_{ij} = \sum_{k=1}^{m} a_{ik}(A^T)_{kj} = \sum_{k=1}^{m} a_{ik}a_{jk}$. Note that $a_{ik}a_{jk} \neq 0$ exactly when judge k votes for both candidates i and j, and that this product is larger when both of the candidates are highly ranked by judge k. It follows that C_{ij} is relatively large when candidates i and j both receive high weighted votes from multiple pairs of judges. Thus the i-th row sum of C is large when candidate i receives many votes or at least some high weighted votes, when candidate i receives high weighted votes from judges who also awarded high weighted votes to other candidates, or both. This is consistent with the fact that Ce is a multiple of the score vector. The analysis applied to C and Ce can also be applied to C^2 and C^2e. In $(C^2)_{ij}$, we are summing the weights of walks of length four in G. The summands are products associated with trios of not necessarily distinct candidates. $(C^2)_{ij}$ will be relatively large when the corresponding candidate(s) have high weighted votes; specifically, when $(a_{ij})^4$ is relatively large, and more generally, when nonzero products of the form $a_{ik_1}a_{kk_1}a_{kk_2}a_{jk_2}$ are relatively large or relatively frequent. Since more candidates are involved in the summations, one naturally expects that more of the interconnection between candidates is captured in C^2 than in C. In general, the entries of $(C^p)_{ij}$, which sum the weights of the length $2p$ walks from candidate i to candidate j in G, are relatively large when $(a_{ij})^{2p}$ is relatively large and when i and j are connected to other high vote weight candidates by multiple walks of length p. As p increases, the effect of interconnection between larger groups of candidates is included, and the importance of relatively larger weighted votes increases. Thus it is natural that candidates corresponding to the larger entries of C^pe should be ranked more highly than those corresponding to smaller entries. Since normalizing C^pe has no effect on the ranking induced by C^pe, and since as p increases, the normalized vectors converge to u, it is natural to use u to rank the candidates (for another discussion of why u is a good tool for ranking, see [5]).

Algorithm 16. Let A be a proper voting matrix. If Ae (or equivalently, Ce) has a unique largest entry, the corresponding candidate is the winner. If more than one candidate corresponds to the largest value in Ae, but the entry of u, the Perron vector for C, for one of those candidates is larger than the entries in u corresponding to the others, then that candidate is the winner. More generally, use Ae to rank the candidates, and use u to break ties when possible.

Example 17. Let A be the proper voting matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then $k = 3$, $Ae = \begin{bmatrix} 7 & 5 & 4 & 7 & 7 \end{bmatrix}^T$, and $Ce_5 = 3Ae$. Observe that based on Ae (or equivalently, on Ce), there are three candidates tied for first: $1, 4$ and 5; that 2 is ranked fourth; and that 3 is ranked last. The Perron vector for C is $u = \begin{bmatrix} 0.53280 & 0.33432 & 0.25153 & 0.50228 & 0.53740 \end{bmatrix}^T$. Thus u breaks the tie, ranking 5 first, followed by $1, 4, 2$ and then 3.

Remark 18. It is possible that two candidate tied for the largest value in Ce will also share the largest value in u. In fact, if $PAQ = A$, where P is a nontrivial permutation matrix and where Q is a permutation matrix, then neither the row sums for C^p (for any positive integer p) nor the entries of the Perron vector for C will distinguish between candidates whose indices lie on a common cycle of the permutation corresponding to P.

Example 19. Let A be the proper voting matrix given by

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $PAQ = A$ when

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigoplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that candidates 1 and 2 are on one cycle in the permutation corresponding to P, and that candidates 3 and 4 are on a second cycle. The vector $Ce = \begin{bmatrix} 6 & 6 & 4 & 4 \end{bmatrix}^T$ fails to distinguish between the stronger candidates 1 and 2; it also fails to distinguish between the weaker candidates 3 and 4. The Perron vector for C is $u = \begin{bmatrix} 0.60150 & 0.60150 & 0.37175 & 0.37175 \end{bmatrix}^T$, which fails to distinguish between the stronger candidates 1 and 2; it also fails to distinguish between the weaker candidates 3 and 4.

Remark 20. If each vote weight is scaled by the positive constant r, or equivalently, if A is replaced by rA, then all rankings of the candidates discussed in this paper remain unchanged. That is, replacing A by rA has no impact on the relative sizes of the entries in the rows sums vector of C^p for a positive integer p or on the relative sizes of the entries in the Perron vector for C. In addition, all rankings of the judges, as discussed in the next section, also remain unchanged.

5. Perron Vectors and Ranking Judges

If instead of focusing on the first diagonal block in B^{2p}, and hence on the candidates, we focus on the second diagonal block, and hence the judges, we can conclude that the sequence $Je, J^2e, J^3e, \dots$, which converges to v when normalized, can be used to rank the judges. What exactly does this ranking tell us? If we interchange the roles of candidates and judges in the previous section, we see that large values of J^pe correspond to judges who awarded their votes, especially any high weight votes, to candidates that received (high weight) votes from other judges. Again, as p increases, the entries of J^p reflect more of the interconnections between judges via G. Thus judges corresponding to relatively high values of J^pe can be thought of as consensus judges – those whose votes are similar to those of many other judges – and judges corresponding to relatively low values of J^pe can be thought of as contrarian judges – those whose votes are different from those of many other judges. If one were selecting tasters for product consistency in the food processing industry, it might be desirable to select for consensus judges. If, under other circumstances, one wanted judges who offered a variety of points of view, one might prefer contrarian judges. Finally, note that unlike the case for candidates, where Ce is a positive multiple of Ae, Je and A^Te are not generally parallel. Hence $A^Te = ke$ cannot be substituted for Je in ranking judges.

Algorithm 21. Let A be a proper voting matrix. If Je has a unique largest entry, the corresponding judge is the strongest. If more than one judge corresponds to the largest value in Je, but the entry of v, the Perron vector for J, for one of those candidates is larger than the entries in v corresponding to the others, then that candidate is the strongest. More generally, use the entries of Je to rank the judges, and use the entries of v to break ties when possible.

Example 22. Let A be the proper voting matrix in Example 17. Then $Je = \begin{bmatrix} 21 & 21 & 18 & 21 & 16 & 19 & 16 & 19 & 19 & 18 \end{bmatrix}^T$, which ranks the judges with $1, 2$ and 4 tie for first; $6, 8$ and 9 tie for fourth; 3 and 10 tie for seventh; and 5 and 7 tie for ninth. Using J^2e, we obtain judges $1, 2$ and 4 tie for first; 8 and 9 tie for fourth; 6 sixth; 10 seventh; 3 eighth; and 5 and 7 tie for ninth. Thus J^2e breaks several ties in Je. Using the Perron vector v for J produces the same ordering as J^2e.

Remark 23. It is possible that two judges tied for the largest (smallest) value in Je will also share the largest (smallest) value in v. In fact, if $PAQ = A$ where P is a permutation matrix and where Q is a nontrivial permutation matrix, then neither the column sums for J^p(for any positive integer p) nor the entries of the Perron vector for J will distinguish between judges whose indices lie on a common cycle of the permutation corresponding to Q. In Example 19, for instance, neither of the judges 1 and 5 can be distinguished using J^pe or v since both judges lie on a common cycle in the permutation corresponding to Q.

References

[1] T. Ando, *Completely Positive Matrices*, Lecture Notes, Sapporo, Japan (1991).

[2] A. Berman, N. Shaked-Monderer, *Completely Positive Matrices*, World Scientific Publishing Co. Pte. Ltd., New Jersey (2003).

[3] A. Berman, C. Xu, $\{0, 1\}$ completely positive matrices, *Linear Algebra and its Applications*, **399** (2005), 35-51.

[4] A. Berman, R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia (1994).

[5] C. Eschenbach, F. Hall, R. Hemasinha, S. Kirkland, Z. Li, B. Shader, J. Stuart, J. Weaver, Properties of tournaments among well-matched players, *The American Mathematical Monthly*, **107**, No. 10, 881-892.

[6] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, NY (1987).

[7] J.P. Keener, The Perron-Frobenius Theorem and the ranking of football teams, *SIAM Review*, **35**, No. 1 (1993).

[8] M.G. Kendall, B.B. Smith, On the method of paired comparisons, *Biometrica*, **31** (1939).

[9] P. Lancaster, M. Tismenetsky, *The Theory of Matrices with Applications*, Second Edition, Academic Press, New York, NY (1985).

[10] A. Langville, C. Meyer, *Google's PageRank and Beyond: The Science of Search Engine Rankings*, Princeton University Press, Princeton (2006).

[11] S. Leon, *Linear Algebra with Applications*, 6-th Edition, Prentice Hall, Upper Saddle River, New Jersey (2002).

[12] A.D. Taylor, *Mathematics and Politics: Strategy, Voting, Power and Proof*, Springer-Verlag, New York, NY (1995).

[13] T.H. Wei, *The Algebraic Foundations of Ranking Theory*, Ph.D. Thesis, Cambridge University (1952).