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AN INTEGRAL REPRESENTATION FOR GENERAL OPERATORS IN BANACH SPACES

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Abstract: Results concerning spectral sets in von Neumann's sense are extended to operators on Banach spaces. In particular, there are generalized well-known Lebow's results concerning measure-theoretic characterization of spectral sets. Applications to harmonic spectral measures are also considered.

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1. Introduction

An important approach to develop an adequate spectral analysis for non-normal operators is based on the concept of spectral sets introduced by J. Neumann in [15]. There is a substantial literature on this topic. We mention relatively former works [9], [12], [14], [18], [20], [22], and some of more recent [2], [3], [4], [7]. We refer also to [16] and [17] for a relative complete surveys on the theory of spectral sets. Recall that a compact set Ω of the complex plane is called a spectral set for a given operator T on a Hilbert space $\mathcal{H}$ if Ω contains the spectrum $\sigma(T)$ of T and if

$$\| f(T) \| \leq \| f \|_\Omega := \sup\{| f(z) | : z \in \Omega\}$$

for each rational function f without poles in Ω. The collection of such rational

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functions further is denoted by $R(\Omega)$. It can be mentioned that an inequality of this type is in fact equivalent with the existence of a spectral representation of the given operator (see [11], [12]). In particular, in [12] it was shown that Ω is a spectral set for an operator T bounded on a Hilbert space $\mathcal{H}$ if and only if for every pair of elements x and y of $\mathcal{H}$ there exists a regular Borel measure $\mu_{x,y}$ with support on the boundary $\partial\Omega$ of Ω such that

$$\langle f(T)x, y\rangle = \int_{\partial\Omega} f d\mu_{x,y}, \quad f \in R(\Omega),$$

and

$$\mid \mu_{x,y} \mid (\Omega) \leq \| \, x \, \| \, \| \, y \, \|,$$

where $\mid \mu_{x,y} \mid$ means the total variation of $\mu_{x,y}$ ($\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denotes the inner product and the norm of $\mathcal{H}$, respectively). This measure-theoretic characterization of spectral sets was used in [12] as the basic tool in order to study spectral properties of non-normal operators.

Our purpose is to extend this mentioned result to the case of Banach spaces, concomitantly considering more general situations then those assumed in the above inequality. We let Ω to be a compact Hausdorff space, and consider an operator $\mathcal{K} : \mathcal{K}(\Omega) \longrightarrow \mathcal{B}(X)$ defined on an arbitrary linear manifold $\mathcal{K}(\Omega)$ of the space $C(\Omega)$ of all continuous complex-valued functions on Ω. $\mathcal{B}(X)$ stands for the set of all linear and bounded operators acting on a Banach space X. Assuming that the operator $\mathcal{K}$ is linear and bounded on $\mathcal{K}(\Omega)$ it can be derived a family of regular Borel measure $(\mu_{x,x^*})_{x \in X, x^* \in X^*}$ having similar properties as in mentioned case of Hilbert spaces and representing the operators $\mathcal{K}f$, $f \in \mathcal{K}(\Omega)$. This result is given in Section 2. For the case of a weakly complete Banach space X in Section 2 it is also described an extension procedure of the obtained representation to more larger class of functions. It should be mentioned that in case $\mathcal{K}(\Omega)$ is dense in $C(\Omega)$ it can be derived further information concerning representative measures by applying directly, after the extension of the operator $\mathcal{K}$ on $C(\Omega)$, the Riesz Representation Theorem for operators defined on the space of continuous functions on Ω (see [6], and also the references cited therein). In general, when $\mathcal{K}(\Omega)$ is not necessarily dense in $C(\Omega)$, the scalar measures μ_{x,x^*} are not determined uniquely. This fact provide complications in describing of representing operator-valued measures (if those exist). Despite of this fact, a construction of a representing measure can be derived from the results given by Theorem 1 of Section 2. This is shown in the Section 3. Applications of obtained results are given in Section 4. Spectral sets are discussed from the viewpoint of the results of Sections 2 and 3 in the first part. The second part deals with a description of the harmonic spectral measure for an operator defined on a

Hilbert space.

2. An Integral Representation Theorem

In this section we present an integral representation theorem for operators defined on linear manifolds of continuous functions. This representation result will be used as a basic tool in our main purposes.

2.1.

We consider the general situation of linear manifolds of the space $C(\Omega)$ of all continuous complex-valued functions defined on a compact Hausdorff space. As usual the space $C(\Omega)$ is equipped with the natural algebraic operations on functions and with the norm

$$\| f \|_\Omega := \sup\{|f(\omega)| : \omega \in \Omega\}, \;\; f \in C(\Omega).$$

In the following X will denote an arbitrary complex Banach space with a norm denoted by $\| \cdot \|$. The set of all linear and bounded operators on X is denoted by $\mathcal{B}(X)$. We define the adjoint space X^* of X as the set of all bounded linear functionals on X. It will be convenient to write $\langle x, x^* \rangle$ instead of $x^*(x)$.

Theorem 1. *Let Ω be a compact Hausdorff space, X be an arbitrary complex Banach space, and let $\mathcal{K} : \mathcal{K}(\Omega) \to \mathcal{B}(X)$ be a linear and bounded operator defined on a linear manifold $\mathcal{K}(\Omega)$ of $C(\Omega)$. Then for each pair $x \in X$ and $x^* \in X^*$ there exists a regular Borel measure μ_{x,x^*} on Ω such that the following properties are fulfilled:*

(i) $\langle (\mathcal{K}f)x, x^* \rangle = \int_\Omega f d\mu_{x,x^*}, \;\; f \in \mathcal{K}(\Omega)$;

(ii) $|\mu_{x,x^*}|(\Omega) \le c \| x \| \| x^* \|$, *where c is a bound of $\mathcal{K}$ on $\mathcal{K}(\Omega)$, i.e.*

$$\| \mathcal{K}f \| \le c \| f \|_\Omega, \;\; f \in \mathcal{K}(\Omega). \tag{1}$$

Conversely, if for a linear operator $\mathcal{K}$ defined on a given linear manifold $\mathcal{K}(\Omega)$ of $C(\Omega)$ and with values in $\mathcal{B}(X)$ there exists a family of scalar Borel measures $(\mu_{x,x^*})_{x\in X, x^*\in X^*}$ on Ω having the properties (i) and (ii), then the operator $\mathcal{K}$ is bounded on $\mathcal{K}(\Omega)$ and, moreover, an estimate (1) *holds with a constant c as in (ii).*

Proof. The proof will depend upon a modification of arguments traditionally used in proving Riesz type representation theorems for operators on spaces of continuous or bounded measurable functions [6].

Let $x \in X$ and $x^* \in X^*$ and consider on $\mathcal{K}(\Omega)$ the following functional

$$\psi_{x,x^*}(f) = \langle (\mathcal{K}f)x, x^* \rangle, \quad f \in \mathcal{K}(\Omega).$$

It is seen that ψ_{x,x^*} is a linear functional on $\mathcal{K}(\Omega)$ and, obviously, it is bounded on $\mathcal{K}(\Omega)$ with

$$\| \psi_{x,x^*} \| \leq c \| x \| \| x^* \| . \tag{2}$$

Recall that $\mathcal{K}(\Omega)$ is taken to be a linear manifold of $C(\Omega)$, therefore, by the Hahn-Banach Theorem, ψ_{x,x^*} can be extended to an linear and bounded functional Ψ_{x,x^*} on $C(\Omega)$ with the same bound. Then, according to the Riesz Representation Theorem there is a regular Borel measure μ_{x,x^*} such that

$$\Psi_{x,x^*}(f) = \int_\Omega f d\mu_{x,x^*}, \quad f \in C(\Omega), \tag{3}$$

and, moreover

$$\| \Psi_{x,x^*} \| = |\mu_{x,x^*}|(\Omega). \tag{4}$$

From (3), in particular for $f \in \mathcal{K}(\Omega)$, it follows the assertion (i). Due to (2) and (4), (ii) also follows, and the proof of the first part of the theorem is complete.

A proof of the second part can be given as follows. For given $x \in X$ and $f \in \mathcal{K}(\Omega)$, by a consequence of the Hahn-Banach Theorem, we can find an $x^* \in X^*$ such that

$$\langle (\mathcal{K}f)x, x^* \rangle = \| (\mathcal{K}f)x \|^2 \quad \text{and} \quad \| x^* \| = \| (\mathcal{K}f)x \| .$$

Then, we have

$$\begin{aligned} \| (\mathcal{K}f)x \|^2 = \langle (\mathcal{K}f)x, x^* \rangle = \left| \int_\Omega f d\mu_{x,x^*} \right| &\leq \| f \|_\Omega \, |\mu_{x,x^*}|(\Omega) \\ &\leq c \| f \|_\Omega \| x \| \| x^* \| = c \| f \|_\Omega \| x \| \| (\mathcal{K}f)x \| . \end{aligned}$$

Therefore

$$\| (\mathcal{K}f)x \| \leq c \| f \|_\Omega \| x \|$$

for all $f \in \mathcal{K}(\Omega)$ and all $x \in X$, and thus the estimate (1) holds true. □

Next we denote the identity element of the algebra $C(\Omega)$ by 1, i.e. we let $1(\omega) = 1$ for $\omega \in \Omega$. It will be also convenient to use the so-called duality map $\phi : X \to X^*$ defined by $\phi(x) = x^*$, where for $x \in X$ the element x^* is taken satisfying $\langle x, x^* \rangle = \| x \|^2$ and $\| x^* \| = \| x \|$ (the existence of which is guaranteed by the Hahn-Banach Theorem).

Corollary 2. *Under the hypotheses of Theorem 1 suppose that* $1 \in \mathcal{K}(\Omega)$, $\mathcal{K}1 = I$ *and that the estimation* (1) *holds with* $c = 1$. *Then there are true the* *following assertions:*

(i) $\mu_{x,\phi(x)}(\delta) \geq 0$ *for any Borel set* δ *of* Ω *and* $x \in X$*;*

(ii) $Re\langle(\mathcal{K}f)x, \phi(x)\rangle = \int_\Omega (Ref) d\mu_{x,\phi(x)}$ *for* $f \in \mathcal{K}(\Omega)$ *and* $x \in X$*;*

(iii) $\sup_{\|x\|=1} |Re\langle(\mathcal{K}f)x, \phi(x)\rangle| \leq \| Ref \|_\Omega, \;\; f \in \mathcal{K}(\Omega)$*;*

(iv) $Re\langle(\mathcal{K}f)x, \phi(x)\rangle \geq 0, \;\; x \in X,$ *whenever* $Ref(\omega) \geq 0$ *for* $\omega \in \Omega$.

Proof. (i) Since $c = 1$ and since $\mathcal{K}1 = I$ from (i) of Theorem 1 it follows

$$\langle x, \phi(x)\rangle = \int_\Omega d\mu_{x,\phi(x)} = \mu_{x,\phi(x)}(\Omega),$$

i.e.

$$\mu_{x,\phi(x)}(\Omega) = \| x \|^2 .$$

On the other hand, from the second property (ii) of Theorem 1, one has

$$|\mu_{x,\phi(x)}|(\Omega) \leq \| x \| \| \phi(x) \| = \| x \|^2 .$$

Thus

$$\mu_{x,\phi(x)}(\Omega) = |\mu_{x,\phi(x)}|(\Omega), \;\; x \in X.$$

But, a scalar measure with this property is non-negative.

(ii) Taking into account that $\mu_{x,\phi(x)}$ is a non-negative measure on Ω we can write

$$\overline{\langle(\mathcal{K}f)x, \phi(x)\rangle} = \overline{\int_\Omega f d\mu_{x,\phi(x)}} = \int_\Omega \overline{f} d\mu_{x,\phi(x)}.$$

Therefore

$$Re\langle(\mathcal{K}f)x, \phi(x)\rangle = \frac{\langle(\mathcal{K}f)x, \phi(x)\rangle + \overline{\langle(\mathcal{K}f)x, \phi(x)\rangle}}{2} = \int_\Omega (Ref) d\mu_{x,\phi(x)},$$

and the desired formula follows.

(iii) We have

$$\begin{aligned} |Re\langle(\mathcal{K}f)x, \phi(x)\rangle| = \left| \int_\Omega (Ref) d\mu_{x,\phi(x)} \right| &\leq \int_\Omega |Ref| d\mu_{x,\phi(x)} \\ &\leq \| Ref \|_\Omega \, \mu_{x,\phi(x)}(\Omega) = \| Ref \|_\Omega \| x \|^2, \end{aligned}$$

and thus the property (iii) is proved.

(iv) It follows from (ii) at once. □

Without any loss of generality, due to the extension principle, it can be considered from the beginning that the operator $\mathcal{K}$ is defined on a closed manifold (i.e. on a subspace) of $C(\Omega)$. We consider $\mathcal{K}$ on an arbitrary manifold (not necessarily closed) only for the sake of a certain formal convenience in applications. Of course, the assertions given by Theorem 1 and Corollary 2 can be extended to functions f belonging to $\overline{\mathcal{K}(\Omega)}$. For instance, using the same notation for $\mathcal{K}$ on $\overline{\mathcal{K}(\Omega)}$, there holds

$$\langle(\mathcal{K}f)x, x^*\rangle = \int_\Omega f d\mu_{x,x^*}, \quad f \in \overline{\mathcal{K}(\Omega)} \tag{5}$$

for each $x \in X$ and $x^* \in X^*$.

It should be noted that the set $\overline{\mathcal{K}(\Omega)}$ can be formed by rather general functions. For instance, if Ω is a compact set of the complex plane and if $\mathcal{K}(\Omega)$ is the set of all analytic functions on Ω, then $\overline{\mathcal{K}(\Omega)}$ coincides with the so-called Ω-analytic functions. For this concept we refer to [15] and [11]. Also, if additionally Ω does not separate the plane and if $\mathcal{K}(\Omega)$ is the set of all polynomial functions on Ω, then, due to of a theorem of Mergelyan [13], $\overline{\mathcal{K}(\Omega)}$ consists of continuous functions on Ω and analytic at the interior points of Ω. Clearly, the discussion on examples of other classes of functions important in this context can be continued.

In case that X is a weakly complete Banach space (or, in particular, reflexive) the integral representation of the operator $\mathcal{K}$ given by Theorem 1 (i) (or (5)) and, respectively, the assertions (ii), (iii) and (iv) of Corollary 2, can be extended to a broader class of functions on Ω, namely to those functions f on Ω which are limits of uniformly bounded sequences of functions in $\mathcal{K}(\Omega)$. This is the topic of the next subsection.

2.2. An Extension Procedure

Let X be a weakly complete (in particular, reflexive) Banach space, and, as in the previous subsection, let $\mathcal{K}$ be a linear operator defined and bounded on a linear manifold $\mathcal{K}(\Omega)$ of $C(\Omega)$. By $\widehat{\mathcal{K}}(\Omega)$ we denote the set of all complex-valued functions f defined on Ω for which there is a sequence (f_n) with f_n in $\mathcal{K}(\Omega)$ such that $\sup_n \| f_n \|_\Omega < \infty$ and $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. For a convenience we call then that the sequence (f_n) is b-convergent to $f \in \widehat{\mathcal{K}}(\Omega)$ and write $f_n \xrightarrow{b} f$. $\widehat{\mathcal{K}}(\Omega)$ is in turn a linear manifold of the space $B(\Omega)$ of all bounded complex-valued functions on Ω, and obviously $\widehat{\mathcal{K}}(\Omega) \supset \mathcal{K}(\Omega)$.

Let $f \in \widehat{\mathcal{K}}(\Omega)$ and let (f_n) be a sequence b-convergent to f. By the bounded convergence theorem, it follows

$$\lim_{n\to\infty} \langle(\mathcal{K}f_n)x, x^*\rangle = \lim_{n\to\infty} \int_\Omega f_n d\mu_{x,x^*} = \int_\Omega f d\mu_{x,x^*} \tag{6}$$

for each $x \in X$ and $x^* \in X^*$. Thus the sequence of operators $(\mathcal{K}f_n)$ converges weakly (recall that X is supposed to be a weakly complete Banach space) to an operator which we denote by $\widehat{\mathcal{K}}f$. Clearly, $\widehat{\mathcal{K}}f$ depends only on the limit function f and does not depend of the chosen sequence of functions (f_n).

In view of (6) the representation formula presented in Theorem 1 (i) is extended to functions $f \in \widehat{\mathcal{K}}(\Omega)$, i.e.

$$\langle(\widehat{\mathcal{K}}f)x, x^*\rangle = \int_\Omega f d\mu_{x,x^*}, \;\; f \in \widehat{\mathcal{K}}(\Omega), \tag{7}$$

for $x \in X$ and $x^* \in X^*$.

Substituting $\lambda f + \mu g$ with $f, g \in \widehat{\mathcal{K}}(\Omega)$ and $\lambda, \mu \in \mathbb{C}$ into (7) we obtain

$$\begin{aligned}\langle\widehat{\mathcal{K}}(\lambda f + \mu g)x, x^*\rangle &= \int_\Omega (\lambda f + \mu g) d\mu_{x,x^*} = \lambda \int_\Omega f d\mu_{x,x^*} + \mu \int_\Omega g d\mu_{x,x^*} \\ &= \lambda\langle(\widehat{\mathcal{K}})x, x^*\rangle + \mu\langle(\widehat{\mathcal{K}}g)x, x^*\rangle = \langle(\lambda\widehat{\mathcal{K}}f + \mu\widehat{\mathcal{K}}g)x, x^*\rangle,\end{aligned}$$

i.e.

$$\langle\widehat{\mathcal{K}}(\lambda f + \mu g)x, x^*\rangle = \langle(\lambda\widehat{\mathcal{K}}f + \mu\widehat{\mathcal{K}}g)x, x^*\rangle \;\; (x \in X, x^* \in X^*)$$

from which we derived that $\widehat{\mathcal{K}}$ is a linear operator on $\widehat{\mathcal{K}}(\Omega)$.

It is also seen from (7) that $\widehat{\mathcal{K}}$ is a bounded operator on $\widehat{\mathcal{K}}(\Omega)$. Moreover,

$$|\langle(\widehat{\mathcal{K}}f)x, x^*\rangle| = \left|\int_\Omega f d\mu_{x,x^*}\right| = \| f \|_\Omega \, |\mu_{x,x^*}|(\Omega) \leq c \, \| f \|_\Omega \| x \| \| x^* \|$$

for each $f \in \widehat{\mathcal{K}}(\Omega)$ and each $x \in X$, $x^* \in X^*$. Hence

$$\| \widehat{\mathcal{K}}f \| \leq c \, \| f \|_\Omega, \;\; f \in \widehat{\mathcal{K}}(\Omega),$$

and thus c is a bound for both operators $\mathcal{K}$ and $\widehat{\mathcal{K}}$.

Notice that the notion of the b-convergence can be extended to sequences of functions in $\widehat{\mathcal{K}}(\Omega)$. So, for (f_n), $f_n \in \widehat{\mathcal{K}}(\Omega)$, we will write $f_n \xrightarrow{b} f$ if $\sup_n \| f_n \|_\Omega < \infty$ and $f_n(\omega) \to f(\omega)$ for $\omega \in \Omega$. It is also clear that

$$\langle(\widehat{\mathcal{K}}f_n)x, x^*\rangle = \int_\Omega f_n d\mu_{x,x^*} \longrightarrow \int_\Omega f d\mu_{x,x^*} = \langle(\widehat{\mathcal{K}}f)x, x^*\rangle$$

for every $x \in X$ and $x^* \in X^*$, and thus $(\widehat{\mathcal{K}}f_n)$ converges weakly to $\widehat{\mathcal{K}}f$.

Further, let us suppose that $\mathcal{K}(\Omega)$ is contained in $C(\Omega)$ as a subalgebra of $C(\Omega)$ and that the operator $\mathcal{K}$ is multiplicative. Then $\widehat{\mathcal{K}}(\Omega)$ is a subalgebra of $B(\Omega)$ and the extended operator $\widehat{\mathcal{K}}$ is also multiplicative. Indeed, let $f, g \in \widehat{\mathcal{K}}(\Omega)$ and let $f_n \xrightarrow{b} f$, $g_n \xrightarrow{b} g$, where $f_n, g_n \in \mathcal{K}(\Omega)$. By a straightforward verification, it can be shown that $fg \in \widehat{\mathcal{K}}(\Omega)$. Since it was supposed that $\mathcal{K}$ is multiplicative we can write

$$\langle\mathcal{K}(f_n g_m)x, x^*\rangle = \langle(\mathcal{K}f_n)(\mathcal{K}g_m)x, x^*\rangle \;\; (m, n = 1, 2, ...) \tag{8}$$

for each $x \in X$ and $x^* \in X^*$. Passing to the limit in (8) when $n \to \infty$ and then

$m \to \infty$, we obtain the desired affirmation.

We collect the above discussed properties in the following proposition.

Proposition 3. *Let X be a weakly complete Banach space, and let $\mathcal{K}$ be an operator as in Theorem 1. Then $\mathcal{K}$ can be extended uniquely to all functions $f \in \widehat{\mathcal{K}}(\Omega)$ that are limits of uniformly bounded sequences of functions in $\mathcal{K}(\Omega)$ so that the integral representation* (7) *holds true. Moreover, if $\mathcal{K}(\Omega)$ is a subalgebra of $C(\Omega)$ and $\mathcal{K}$ is multiplicative, then $\widehat{\mathcal{K}}(\Omega)$ is a subalgebra of $B(\Omega)$ and the extension $\widehat{\mathcal{K}}$ is also a multiplicative operator.*

3. Representing Operator-Valued Measures

Let X be a weakly complete Banach space, and, as in the previous section, consider an linear operator $\mathcal{K} : \mathcal{K}(\Omega) \to \mathcal{B}X$ defined and bounded on a linear manifold $\mathcal{K}(\Omega)$ of $C(\Omega)$ (recall that Ω is a compact Hausdorff space). In this section we describe an operator-valued measure representing the operator $\widehat{\mathcal{K}}$ and, in particular, $\mathcal{K}$. It will be seen that if $\widehat{\mathcal{K}}(\Omega)$ contains characteristic functions χ_δ of the Borel sets δ in Ω, then the values $\widehat{\mathcal{K}}(\chi_\delta)$ determines a representing operator-valued measure of the operator $\mathcal{K}$. This result will be derived from the results given by Theorem 1.

We begin with the following observation. If for a Borel set $\delta \in \Omega$ there is a function $f \in \mathcal{K}(\Omega)$ such that $f(\omega) = 1$ for $\omega \in \delta$ and $\mid f(\omega) \mid < 1$ for $\omega \in \Omega \setminus \delta$, then the characteristic function χ_δ of δ is contained in $\widehat{\mathcal{K}}(\Omega)$. In fact, the sequence (f_n), where $f_n(\omega) = (f(\omega))^n$, $\omega \in \Omega$, is uniformly bounded and converges pointwise to χ_δ. We let

$$F(\delta) = \widehat{\mathcal{K}}(\chi_\delta)$$

and, by (7), we have

$$\langle F(\delta)x, x^* \rangle = \int_\Omega \chi_\delta d\mu_{x,x^*} = \int_\delta \mu_{x,x^*} = \mu_{x,x^*}(\delta),$$

i.e.

$$\langle F(\delta)x, x^* \rangle = \mu_{x,x^*}(\delta) \tag{9}$$

for each $x \in X$ and $x^* \in X^*$.

Consequently, if $\chi_\delta \in \widehat{\mathcal{K}}(\Omega)$ for every Borel set δ in Ω, one can obtain a family operators $F(\delta)$ related with scalar measures μ_{x,x^*} by (9). In other words we obtain a mapping F from the algebra of Borel set in Ω into the space $\mathcal{B}(X)$ which, as follows from (9), is weakly and hence strongly countable additive.

According to Theorem 1 (ii) F represents therefore a bounded $\mathcal{B}(X)$-valued measure. From (7) and (9) it follows

$$\langle(\widehat{\mathcal{K}}f)x, x^*\rangle = \left\langle \left(\int_\Omega f dF\right) x, x^* \right\rangle$$

for each $f \in \widehat{\mathcal{K}}(\Omega)$ and each $x \in X$ and $x^* \in X^*$, from which we obtain the following integral representation

$$\widehat{\mathcal{K}}f = \int_\Omega f dF, \quad f \in \widehat{\mathcal{K}}(\Omega). \tag{10}$$

Notice that $\| F \| = \| \widehat{\mathcal{K}} \|$ $(= \| \mathcal{K} \|)$ that also follows from (7), (9) and (10) by straightforward calculations (for the concept of the norm of an operator-valued measure we refer [5] and [6]). Besides, in accordance with the results given by Theorem 1, μ_{x,x^*}, which due to of (9) are associated to F, are regular Borel measures on Ω. Thus the operator-valued measure F satisfies all conditions to be termed as a representing measure of $\mathcal{K}$.

Remark 4. If F is a bounded $\mathcal{B}(X)$-valued Borel measure on Ω for which the integral representation (10), or even

$$\mathcal{K}f = \int_\Omega f df, \quad f \in \mathcal{K}(\Omega), \tag{11}$$

holds true, then (11) defines a bounded linear operator from $\mathcal{K}(\Omega)$ to $\mathcal{B}(X)$ and $\| \mathcal{K} \| = \| F \|$.

The following theorem summarizes the above discussion.

Theorem 5. *Let Ω, X and $\mathcal{K}$ be as above, and let $\widehat{\mathcal{K}}(\Omega)$ contains the characteristic functions of the Borel sets in Ω. Then there is a bounded $\mathcal{B}(X)$-valued Borel measure F on Ω such that:*

i) $\| F \| = \| \mathcal{K} \|$;

ii) the associated scalar measures defined by $\mu^F_{x,x^*}(\delta) = \langle F(\delta)x, x^*\rangle$, where $x \in X$, $x^* \in X^*$ and δ is a Borel set in Ω, are regular on Ω;

iii) the integral representation (10) *(and hence* (11)*) holds true.*

Conversely, if F a bounded $\mathcal{B}(X)$-valued measure defined on the Borel sets in Ω for which (ii) and (iii) hold, then the representation (11) *defines a linear operator from $\mathcal{K}(\Omega)$ to $\mathcal{B}(X)$ such that $\| \mathcal{K} \| = \| F \|$.*

Specific properties of the measure F are presented in the following proposition.

Proposition 6. *Under the hypotheses of Theorem 5 if, in addition, $\mathcal{K}$*

is multiplicative operator, then each $F(\delta)$ commutes with each operator $\widehat{\mathcal{K}}f$, $F(\partial \cap \delta) = F(\partial)F(\delta)$ for every Borel set ∂ and δ in Ω, so, in particular, $F(\delta)$ is a projection in Ω. If also $1 \in \mathcal{K}(\Omega)$ and $\mathcal{K}1 = I$, then the completeness property $F(\Omega) = I$ holds.

Proof. The details follow at once from the fact that $\widehat{\mathcal{K}}$ is also a multiplicative operator. □

Remark 7. In case the linear manifold $\mathcal{K}(\Omega)$ is dense in $C(\Omega)$, i.e. $\overline{\mathcal{K}(\Omega)} = C(\Omega)$, then the functional ψ_{x,x^*} in the proof of Theorem 1 can be extended uniquely by continuity to $C(\Omega)$. The corresponding measures μ_{x,x^*} are also determined uniquely by the Riesz Representation Theorem. Besides, for any Borel set δ in Ω the functional $\mu_{x,x^*}(\delta)$ is linear in x and in x^*. Taking into account these facts the operator-valued measure F corresponding to μ_{x,x^*} is in turn determined uniquely.

Remark 8. Having an operator-valued measure F with prescribing above properties, by using standard arguments, the integral representation (10) can be extended to all F-bounded F-measurable functions $f \in L_\infty(\Omega, F)$.

4. Applications

In this section we apply the previous results to the theory of spectral sets in the von Neumann's sense and also to a description of the so-called harmonic spectral measure for an operator on a Hilbert space.

4.1. Spectral Sets

Let X be an arbitrary complex Banach space, and let T be a linear and bounded operator on X, i.e. $T \in \mathcal{B}(X)$.

In what follows Ω is a compact set of the complex plane. Denote by $R(\Omega)$ the collection of all rational functions bounded on Ω. The set Ω is called a c-spectral set for T, c being a positive constant, if $\sigma(T) \subset \Omega$ and

$$\| f(T) \| \le c \| f \|_\Omega, \quad f \in R(\Omega). \tag{12}$$

1-spectral sets are called simply spectral sets for T.

The mapping $\mathcal{K} : R(\Omega) \to \mathcal{B}(X)$ defined by

$$\mathcal{K}f = f(T), \quad f \in R(\Omega), \tag{13}$$

is an algebraic homomorphism of $R(\Omega)$ into $\mathcal{B}(X)$, and, due to (12), it is bounded on $R(\Omega)$. Therefore, the arguments used in Section 2 and 3 can be applied in order to obtain integral representation for $f(T)$. One need only to emphasize that, due to the fact that $R(\Omega)$ can be reviewed as a subspace of $C(\partial\Omega)$ ($\partial\Omega$ denotes the boundary of Ω), the supports of the regular Borel measures μ_{x,x^*} constructed as in the proof of Theorem 1 can be taken in $\partial\Omega$. Taking into account this fact, the results presented by Theorem 1 and also by Corollary 2 can be adjusted easily for the case considered in the present section. The corresponding results are summarized in the following theorem.

Theorem 9. *Let Ω be a compact set of the complex plane, and let T be a linear and bounded operator on a Banach space X. Ω is a c-spectral set for T if and only if for each pair $x \in X$ and $x^* \in X^*$ there exists a regular Borel measure μ_{x,x^*} with support in $\partial\Omega$ such that:*

(i) $\langle f(T)x, x_* \rangle = \int_{\partial\Omega} f d\mu_{x,x^*}, \ \ f \in R(\Omega)$;

(ii) $| \ \mu_{x,x^*} \ | \ (\Omega) \leq c \parallel x \parallel \ \parallel x^* \parallel$.

In particular, if Ω is a spectral set of T, then:

(iii) the measures $\mu_{x,\varphi(x)}$ are non-negative, that is $\mu_{x,\varphi(x)}(\delta) \geq 0$ for any Borel set in Ω and $x \in X$;

(iv) $Re\langle f(T)x, \varphi(x) \rangle = \int_{\partial\Omega} (Ref) d\mu_{x,\varphi(x)}$ *for $f \in R(\Omega)$ and $x \in X$;*

(v) $\sup_{\|x\|=1} | \ Re\langle f(T)x, \varphi(x) \rangle \ | \leq \parallel Ref \parallel_{\partial\Omega}, \ \ f \in R(\Omega)$;

(vi) $Re\langle f(T)x, \varphi(x) \rangle \geq 0, \ \ x \in X$, *whenever $Ref(z) \geq 0$ for $z \in \partial\Omega$.*

Here, as in Section 2, φ denotes the duality map.

Remark 10. A result analogous to that presented by Theorem 9 was obtained by Lebow [12] (see Theorem 1 and Corollary 1 in [12]) for the case when Ω is a spectral set for an operator T considered acting on a Hilbert space.

The results just presented in Theorem 9 can be extended by continuity to all functions f (so-called Ω-analytic functions) belonging to the closure of $R(\Omega)$ in $C(\partial\Omega)$ (see Subsection 2.1). The map $\mathcal{K}$ defined by (13) is a continuous homomorphism of the algebra $\overline{R(\Omega)}$ into $\mathcal{B}(X)$. In this way we have a functional calculus which in fact is an extension of the Riesz-Dunford functional calculus. We refer [15] and [11] for systematically treatments of the functional calculus involving Ω-analytic functions.

In case that X is a weakly complete Banach space this functional calculus can be extended further to a more larger class of functions, namely to those which are limits of uniformly bounded sequences of functions in $R(\Omega)$ (cf.

Proposition 3). In accord with notations made in Section 2 we denote this class of functions by $\widehat{R}(\Omega)$. In particular, we have

$$\langle f(T)x, x^* \rangle = \int_\Omega f d\mu_{x,x^*}, \quad f \in \widehat{R}(\Omega).$$

If $\widehat{R}(\Omega)$ is sufficiently rich class of functions (cf. Theorem 5) or, for instance, when $R(\Omega)$ is dense in $C(\Omega)$ (cf. Remark 7), then it can be derived a countable additive in the strong operator topology spectral measure F in X commuting with each operator $f(T)$ for $f \in \widehat{R}(\Omega)$ and for which the corresponding associated scalar measures μ_{x,x^*} $(x \in X, x^* \in X^*)$ of F are regular on Ω, $\| F \| \leq c$, and

$$f(T) = \int_\Omega f dF, \quad f \in \widehat{R}(\Omega). \tag{14}$$

In particular, it is seen that T is a scalar type spectral operator in X. This last result generalizes the J. von Neumann's Theorem concerning normal operators on Hilbert spaces; it is contained implicitely in [8] and [1] (cf. also [19]; Theorem 2). According to those mentioned in Remark 8, under above assumptions, the operational calculus given by (14) can be in turn extended to all F-bounded F-measurable functions f on Ω.

4.2. Harmonic Spectral Measures

In this subsection the ideas and approaches developed in Sections 2 and 3 are taken back in order to describe the so-called harmonic spectral measure for an operator T considering acting in a Hilbert space $\mathcal{H}$. The notion of the harmonic spectral measure together with a functional calculus for harmonic real valued functions defined on a spectral set for the operator have been studied by C. Foias in [10], [11]. The concept of spectral sets interspersed with the representation theory for commutative Banach algebras was substantially exploited in already mentioned works of C. Foias. In this aspect our starting-points are the same, but we emphasize that the corresponding constructions which we give are obtained by applying directly the results presented in Sections 2 and 3.

Let $\mathcal{H}$ be a complex Hilbert space, and let T be an arbitrary bounded operator T on $\mathcal{H}$. Suppose that Ω is a spectral set for T, so that Ω is a compact set in the complex plane, $\Omega \supset \sigma(T)$ and

$$\| f(T) \| \leq \| f \|_\Omega, \quad f \in R(\Omega). \tag{15}$$

Due to Runge Theorem one can be taken functions analytic on Ω. Denote by $A(\Omega)$ the algebra of all analytic functions on Ω. The map $f \to f(T)$ is a homomorphism of $A(\Omega)$ into $\mathcal{B}(\mathcal{H})$, and endowing $A(\Omega)$ with the supremum norm $\|\cdot\|_\Omega$, due to (15) it is a continuous homomorphism. For a harmonic (real-valued) function $u(z) = Ref(z)$, $z \in \Omega$, where $f \in A(\Omega)$, we let

$$u(T) := Ref(T) \ \ (:= 1/2(f(T) + f(T)^*)).$$

By virtue of Theorem 9 (v) for any harmonic function u on Ω, one has

$$\| u(T) \| \leq \| u \|_\Omega,$$

and also by Theorem 9 (vi) $u(T) \geq 0$ whenever $u(z) \geq 0$ for $z \in \partial\Omega$.

Next, it will be assumed that the set Ω does not separate the plane (the complement $\mathbb{C} \setminus \Omega$ forms a connected open set). According to a theorem of J.L. Walsh [21], in this case, $A(\Omega)$ is a Dirichlet algebra, i.e. the real parts of functions in $A(\Omega)$ are dense in the set $C_{\mathbb{R}}(\partial\Omega)$ of all continuous real-valued functions on $\partial\Omega$. So, for any $f \in C_{\mathbb{R}}(\partial\Omega)$ there is a Ω-harmonic function u_f, coinciding with f on the boundary $\partial\Omega$. Notice that, since for Ω-harmonic functions the maximum modulus principle is also valid, u_f is uniquely determined by f. So, to f it corresponds a unique (self-adjoint) bounded operator $u_f(T)$ on $\mathcal{H}$. Taking into account this fact, for any continuous complex-valued function $f \in C(\partial\Omega)$ we let

$$u_f(T) = u_{Ref}(T) + iu_{Imf}(T),$$

where the operators on the right-hand side are well defined provided that u_{Ref} and u_{Imf} being Ω-harmonic functions.

Now, we consider the operator $\mathcal{K} : C(\partial\Omega) \to \mathcal{B}(\mathcal{H})$ defined by

$$\mathcal{K}f = u_f(T), \ \ f \in C(\partial\Omega).$$

It is easy to verify that $\mathcal{K}$ is a linear and a bounded operator, and thus the results of Sections 2 and 3 can be applied. There holds the following theorem.

Theorem 11. *Let T be a linear and bounded operator on a Hilbert space $\mathcal{H}$, and let Ω be a spectral set for T which does not separate the plane. Then there is a $\mathcal{B}(\mathcal{H})$-valued Borel measure F on $\partial\Omega$ such that*

$$u_f(T) = \int_\Omega f dF, \ \ f \in C(\partial\Omega). \tag{16}$$

The operator-valued measure F given by Theorem 11 is called (following the terminology in [10] and [11]) the harmonic spectral measure of T. The integral representation defines an operational calculus $f \to u_f(T)$ for $f \in C(\partial\Omega)$, which certainly can be extended to all F-bounded F-measurable functions on Ω. It should be mentioned that this operational calculus in general is not multiplicative.

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