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SECTIONS OF THETA-CHARACTERISTICS ON STABLE CURVES

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Abstract: Here we classify the genus g stable curves with a theta-characteristic with g linearly independent sections.

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1. Introduction

As in [3] and [1] we work over an algebraically closed base field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. Let X be a stable curve of genus $g \geq 2$. For each $S \subseteq \mathrm{Sing}(X)$ let $u_S : X_S \to X$ be the blowing-up of S (i.e. X_S is a quasistable curve, u_S is the stable reduction of X_S and $u_S^{-1}(P)$, $P \in S$, is the set of all exceptional components of X_S). Let $A(X)$ denote the set of all theta-characteristics on X in the sense of [3]. We recall that $\sharp(A(X)) = 2^{2g}$ and that each element $\mathcal{L}$ of $A(X)$ is associated to a line bundle L on a uniquely determined quasistable curve X_S with total degree $g-1$ and whose restriction to each exceptional component of X_S has degree 1 (see [3]; see [4] for integral curves, [2] and references therein for r-roots). The pair (S, L) is uniquely determined by $\mathcal{L}$ and hence the integer $h^0(X_S, L_S)$ is well-defined. For any integer $r > 0$ let $A(X, S)$ be the set of all $\mathcal{L} \in A(X)$ defined by a line bundle on X_S. Let $A(X, S; r)$ the set of all element of $A(X, S)$ associated to a line bundle M on X_S such that $h^0(X_S, L_S) \geq r + 1$. Set $A(X; r) := \cup_{S \subseteq \mathrm{Sing}(X)} A(X, S; r)$. A nodal curve is said to be *quasicompact* if its dual graph is a tree. We need the following generalization of the words "quasi-compact with g irreducible components of arithmetic genus 1". We recall

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that a semistable curve with arithmetic genus 1 is either a smooth elliptic curve or a rational curve with a unique node or a cycle of $t \geq 2$ $\mathbb{P}^1$'s. The dualizing sheaf of any semistable genus 1 curve A is trivial and hence it has an effective square-root: the trivial line bundle. We will say that the stable genus g curve X is maximally split if there are g subcurves $A_1, \dots, A_g$ of X such that $A_i \cap A_j$ is finite (or empty) for all $i \neq j$, each A_i is a semistable, and $p_a(A_i) = 1$ for all i. Here we prove the following result.

Theorem 1. *Let X be a stable curve of genus g. Then:*

(a) $A(X; g) = \emptyset$.

(b) $A(X; g-1) \neq \emptyset$ if and only if X is maximally split. In this case we have $\sharp(A(X; g-1)) = 1$.

We also shows how to use disconnecting nodes of stable curves to construct theta-characteristics with many sections (see Propositions 1 and 2 and Corollary 1).

2. Proofs and Related Results

For any reduced projective curve Y let $\mathcal{B}(Y)$ denote the set of its irreducible components.

Lemma 1. *Let Y be a quasistable curve, S a non-empty subset of its exceptional conents and $L \in Pic(Y)$ such that $\deg(L|E) = 1$ for all $E \in S$. Set $C := \overline{Y \setminus \cup_{E \in S} E}$. Then $h^0(Y, L) = h^0(C, L|C)$.*

Proof. Using induction on the integer $\sharp(S)$ we reduce to the case $\sharp(S) = 1$, say $S = \{E\}$. Look at the following Mayer-Vietoris exact sequence

$$0 \to L \to L|C \oplus L|E \to L|E \cap C \to 0\,. \tag{1}$$

By assumption the restriction map $H^0(E, L|E) \to H^0(E \cap C, L|E \cap C)$ is bijective. Hence (1) shows that the restriction map $H^0(Y, L) \to H^0(C, L|C)$ is bijective. □

Proposition 1. *Let X be a stable curve with a disconnecting node. Then $A(X, 1) \neq \emptyset$.*

Proof. If $A(X, S) \neq \emptyset$, then S contains every disconnecting node (see [3], Example 3.1). Apply Lemma 1 and that every integral nodal curve D of arithmetic genus ≥ 1 has a line bundle L such that $h^0(D, L) > 0$ and $L^{\otimes 2} \cong \omega_D$ (take $L = \mathcal{O}_D$ if $p_a(D) = 1$). □

Remark 1. Let X be a general element of $\mathcal{M}_g$. It is well-known that $A(X,1) = \emptyset$ (see [6], Theorem 2.17, or [5] for much more). Proposition 1 shows that this is false for every element of all except one among the irreducible components of $\overline{\mathcal{M}}_g$.

In the case of curves with quasicompact type we may prove a more precise result.

Proposition 2. *Let X be a reducible stable curve of quasicompact type. Let s be the number of the irreducible components of X with arithmetic genus ≥ 1. Then $A(X; s-1) \neq \emptyset$.*

Proof. Set $z := \sharp(\mathcal{B}(X))$ and let B the subset of $\mathcal{B}(X)$ formed by the the irreducible components with arithmetic genus $g \geq 1$. Hence $\sharp(B) = s$. By [3], p. 7, we have $A(X,S) = \emptyset$ if $S \subsetneq \mathrm{Sing}(X)$. Set $A := \mathrm{Sing}(X)$. Let Y_A be the closure in X_A of the complement of its irreducible components. Set $v_A := u_A|Y_A$. Since X has compact type, Y_A is the disjoint union of z smooth curves and v_S is the normalization map. Each irreducible component D of Y_A with positive genus has a theta-characteristic L_D such that $h^0(D, L_D) > 0$. If $D \in \mathcal{B}(X)\backslash B$ (i.e. if $D \cong \mathbb{P}^1$) let L_D be the only degre -1 line bundle on D. The family $\{L_D\}_{D\in\mathcal{B}(X)} L_D$ defines a square-root M of ω_{Y_S}. Since $Y_A = \sqcup_{D\in\mathcal{B}(X)} D$, $h^0(Y, A, D) = \sum_{D\in\mathcal{B}(X)} h^0(D, L_D) \geq s$. There is a unique $L \in \mathrm{Pic}(X_A)$ such that $L|Y_A \cong M$ and $\deg(M|E) = 1$ for every exceptional component of X_A. L defines an element $\mathcal{L}$ of $A(X)$ (see [3]). Lemma 1 gives $h^0(X_A, L) = h^0(Y_A, M) \geq s$. Hence $\mathcal{L} \in A(X; s-1)$. □

Corollary 1. *Let X be a genus g stable curve of compact type with g irreducible components of genus $g \geq 1$, i.e. a curve of quasicompact type whose irreducible components are elliptic curves. Then $\sharp(A(X;r)\backslash A(X;r+1)) = \binom{g}{r+1} \cdot 3^{g-r-1}$ for every integer x such that $1 \leq x \leq g-1$ and in particular $\sharp(A(X; g-1)) = 1$ and $A(X; g) = \emptyset$.*

Proof. Let D_i, $1 \leq i \leq g$, be the irreducible components of X with arithmetic genus ≥ 1, and D_i, $s+1 \leq i \leq z$, the irreducible components of X isomorphic to $\mathbb{P}^1$. All elements of $A(X;r)\backslash A(X;r+1)$ are described in the following way. Fix any $S \subseteq \{1, \ldots, g\}$ such that $\sharp(S) = r+1$. For any $i \in S$ set $L_{D_i} := \mathcal{O}_{D_i}$. For each $i \in \{1, \ldots, g\}\backslash S$ take as L_{D_i} any of the 3 ineffective theta-characteristics of the elliptic curve L_{D_i}. For all $i \in \{s+1, \ldots, z\}$ let L_{D_i} be the only degree -1 line bundle on $D_i \cong \mathbb{P}^1$. Then apply the proof of Proposition 1 to get $\mathcal{L} \in A(X;r)\backslash A(X;r+1)$. □

It is easy to compute the integer $\sharp(A(X; g-1)) = 1$ if X is quasicompact, with e irreducible components which are smooth of genus 1 and with $g-e$ irreducible components which are singular of genus 1 ($e \in \{0, \ldots, g\}$).

Lemma 2. *Let X be a maximally split genus g stable curves and $A_1, \ldots, A_g$ its defining subcurves.*

(a) The subcurves $A_1, \ldots, A_g$ are uniquely determined (up to a permutation of the indices).

(b) Either $X = A_1 \cup \cdots \cup A_g$ or each connected component of

$$\overline{X \backslash (A_1 \cup \cdots \cup A_g)}$$

is a tree of $\mathbb{P}^1$'s.

Proof. Up to a permutation of the indices we may assume $A_1, \ldots, A_e$ integral and $A_{e+1}, \ldots, A_g$ reducible ($e \in \{0, 1, \ldots, g\}$). The curves $A_1, \ldots, A_e$ are the only irreducible components of X with genus at least 1. Hence these curves are uniquely determined by X. Let η be the marked graph obtained from $\|X\|$ contracting to a point marked with the integer 1 each subgraph associated to A_i, $1 \leq i \leq g$. The subcurves $A_{e+1}, \ldots, A_g$ are a basis for the cycles of the unmarked graph associated to η. These cycles are uniquely determined (up to the order) and statement (b) is true because $p_a(X) = g$. □

Proof of Theorem 1. Take X, $S \subseteq \operatorname{Sing}(X)$, and $L \in A(X, S; g-1)$. Set $M := L|C_S$. Let $E' := \cup_{E \in S}$ be the union of the exceptional components of X_S. Hence $C_S \cup E' = X_S$ and $\sharp(C_S \cap E') = 2\sharp(S)$ and $\omega_{X_S}|C_S \cong \omega_{C_S}(C_S \cup E')$. By Cornalba's definition of spin structure (more precisely, the condition $\deg(L|E) = 1$ for every exceptional component E of X_S) shows that natural map $L^{\otimes 2} \to \omega_{X_S}$ induces an injective map $M^{\otimes 2} \to \omega_{C_S}(C_S \cup E')$ with $C_S \cap E'$ in the support of its cokernel. Hence it induces an isomorphism $M^{\otimes 2} \to \omega_{C_S}$ (see [3], Section 2). We have $\deg(M) = g - 1 - \sharp(S)$. Lemma 1 gives $h^0(C_S, M) = h^0(X_S, L) \geq g$. Let F be the closure in C_S of the complements of the 1-dimensional part (if any) of the base locus of M. Since M is a square-root of ω_{C_S} and C_S is nodal, there are $R \in \operatorname{Pic}(F)$ and an inclusion $j : R \hookrightarrow M|F$ whose cokernel is the set $F \cap \overline{C_S \backslash F}$, which induces an isomorphism $j_* : H^0(F, R) \to H^0(C_S, M)$ and with $R^{\otimes 2}\omega_F$. Since R is spanned outside finitely many points, ω_F is spanned outside finitely many points. Hence each connected component of F has arithmetic genus ≥ 1 and it is semistable. We have $h^0(F, R) = h^0(C_S, M) = h^0(X_S, L)$.

(i) Let A be a connected component of F. Since $R^{\otimes 2} \cong F$, we have $\deg(F|A) = p_a(A) - 1$. Set $q := p_a(A)$. First assume $q = 1$. Since A is nodal and ω_A is generically spanned, it is semistable and $\omega_A \cong \mathcal{O}_A$. Since $R|A$ is spanned outside finitely many points and $(R|A)^{\otimes 2} \cong \omega_A$, we have $R \cong \omega_A$.

Hence $h^0(A, R|A) = 1 = q$. Now assume $q \geq 2$. Set $x := h^0(A, R|A)$. We have $\deg(R|A) = q - 1$. In order to obtain a contradiction we assume $x \geq q$. Since $R|A$ is spanned outside finitely many points, for every irreducible component T of A the vector space $H^0(A, \mathcal{I}_T \otimes (R|A))$ is a proper subspace of $H^0(A, R|A)$. Hence it is easy to construct points $P_1, \dots, P_{x-1} \in A_{reg}$ such that $h^0(A, \mathcal{I}_{\{P_1,\dots,P_{x-1}\}} \otimes (R|A)) = h^0(A, R|A) - x + 1$ and the line bundle $(R|A)(-P_1 - \cdots - P_{x-1})$ has finite base locus. Since $\deg(R|A) = q - 1 - x + 1 \leq 0$, we get a contradiction, unless $x = q$ and this line bundle is trivial. However, even the latter possibility is excluded if we move each point $P_1, \dots, P_{x-1}$ in the irreducible component in which it lives, because $R|A$ is a fixed line bundle, $x - 1 > 0$ and $h^1(A, \mathcal{O}_A) = q > 0$.

(ii) Let A_i, $1 \leq i \leq f$, be the connected components of F. Notice that $\sum_{i=1}^{f} p_a(A_i) \leq g$. Since $p_a(A_i) \geq 1$, for all i, we have $f \leq g$ and $p_a(A_i) = 1$ for all i if $f = g$. By step (i) we find $h^0(F, R) < g$, unless $f = g$ and $p_a(A_i) = 1$ for all i.

(iii) Here we assume that X is maximally split. By assumption there are g subcurves $A_1, \dots, A_g$ of X such that $A_i \cap A_j$ is finite (or empty) for all $i \neq j$, each A_i is a semistable and $p_a(A_i) = 1$ for all i. Let η be the marked graph obtained from $\|X\|$ contracting to a point marked with the integer 1 each subgraph associated to A_i, $e+1 \leq i \leq g$. Let Y be any nodal curve such that $\|Y\| = \eta$. Y is a quasicompact type with g irreducible components of arithmetic genus 1. If $X = Y$, then we may apply Corollary 1. Assume $\|X\| \neq \eta$, i.e. assume that X is not quasicompact. We use η to see which nodes we must blowing-up to have some theta-characteristic in the sense of [3]. Set $S := \mathrm{Sing}(X) \backslash (\cup_{i=1}^{g} \mathrm{Sing}(A_i))$. Let $v_S : C_S \to X$ be the partial normalization of X in which we only normalize the points of S. C_S has g connected components isomorphic to $A_1, \dots, A_g$ and (perhaps) other connected components with arithmetic genus 0. The latter components have no effective theta-characteristic. Hence Lemma 1 gives $A(X, S; g-1) \neq \emptyset$. Steps (i) and (ii) give $A(X; g) = \emptyset$. Lemma 2 and steps (i) and (ii) give $\sharp(A(X; g-1)) = 1$. $\square$

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