International Journal of Pure and Applied Mathematics
Volume 54 No. 3 2009, 375-381

GOOD COMPONENTS OF THE BRILL-NOETHER SCHEME FOR GENERAL STABLE CURVES WITH FIXED TOPOLOGICAL TYPE

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let τ be a topological type for nodal projective curves and $\mathcal{M}[\tau]$ the variety of all nodal curves with τ as topological type. Here we prove that for many τ and many multidegrees a general $X \in \mathcal{M}[\tau]$ has a a generically smooth component with the expected dimension of the Brill-Noether scheme of morphisms $X \to \mathbb{P}^n$, $n \geq 3$, with the prescribed multidegree.

AMS Subject Classification: 14H10, 14H50, 14H51
Key Words: stable curve, Brill-Noether theory, restricted tangent bundle, multidegree

1. Introduction

Let X be a nodal and connected projective curve with arithmetic genus g. Let $\mathcal{B}(X)$ denote the set of all irreducible components of X. A set of integers $\underline{d} = \{d_T\}_{T \in \mathcal{B}(X)}$ is called a multidegree for X or for the set $\mathcal{B}(X)$. Every line bundle L on X has a multidegree $\{\deg(L|T)\}_{T \in \mathcal{B}(X)}$. Hence it is reasonable to study the Brill-Noether theory of the line bundles of X using their multidegree. For every integer $n \geq 1$ and any multidegree $\underline{d}$ let $A^n_{\underline{d}}(X)$ denote the set of all pairs (L, V) such that L is a line bundle on X with multidegree $\underline{d}$ and V is an $(n+1)$-dimensional linear subspace of $H^0(X, L)$. This is not the usual Brill-Noether set for two reasons. In the Brill-Noether theory one should also consider L which are not locally free, but only sheaves with depth 1 and pure rank 1. For

Received: June 18, 2009

nodal curves it is possible to consider only line bundles if we consider them for other curves related to X (see [1] for the details). The second reason is that if X is semistable it is important to restrict the line bundles to the ones satisfying the so-called Basic Inequality (see [1],p. 611; called semibalanced or balanced in [2], [4] and [5], Definition 1.1). Since the truth of Basic Inequality for a line bundle L only depends from the multidegree of L, to restrict to semibalanced line bundles it is sufficient to avoid some (many) multidegrees in the statements of all our results. Here we want to prove that for many integers $n \ge 3$, many multidegrees $\underline{d}$ and many "sufficiently general" X there is a generically smooth irreducible component Γ of $A^n_{\underline{d}}(X)$ with the expected dimension, i.e. with dimension $\rho(g, r, \delta(\underline{d})) = g - (r+1)(r+g-\delta(\underline{d})) = (r+1)\delta(\underline{d}) - rg - r(r+1)$ (the Brill-Noether number), where $\delta(\underline{d}) := \sum_{T \in \mathcal{B}(X)} d_T$ is the total degree of the multidegree $\underline{d}$. We will do this proving the existence of $(L, V) \in A^n_{\underline{d}}(X)$ such that V induces an embedding $j : X \hookrightarrow \mathbb{P}^n$ such that $h^1(j(X), T\mathbb{P}^n|X) = 0$ (see Remarks 1 and 2). Of course, not all X have such a property for special line bundles (see [3] for binary curves). Hence the main part of the introduction is a description of our meaning of "sufficiently general". For any nodal curve X let $\mathrm{Sing}(X)'$ (resp. $\mathrm{Sing}(X)''$) be the set of all singular points of X lying on exactly one (resp. two) irreducible components of X. To any nodal projective curve X we may associate the following non-oriented marked graph $\|X\|$. The vertices of $\|X\|$ are the irreducible components of X. For any $T \in \mathcal{B}(X)$ let $[T]$ denote the associated vertex of $\|X\|$. For each $T \in \mathcal{B}(X)$ we give as a marking the non-negative integer q_T, where q_T is the geometric genus of T. $\|X\|$ contains $\sharp(\mathrm{Sing}(X)' \cap T)$ loops with $[T]$ as their vertex. For all $T, T' \in \mathcal{B}(X)$ such that $T \ne T'$ the vertices $[T]$ and $[T']$ of $\|X\|$ are joined by $\sharp(T \cap T')$ edges. Call τ the abstract marked graph $\|X\|$. If we forget the marking, i.e. if we forget the integers q_T, $T \in \mathcal{B}(X)$, then $\|X\|$ becomes the classical dual graph of the nodal curve X. If $\mathbb{K} = \mathbb{C}$, then the topological type of the complex analytic space $X(\mathbb{C})$ is uniquely determined by the marked graph τ and two non-isomorphic marked graphs give topologically different complex analytic spaces. The set of all nodal projective curves Y such that $\|Y\| \cong \tau$ (as marked graphs) is parametrized (not finite-to-one if Y is not stable) by an irreducible algebraic variety $\mathcal{M}[\tau]$. If Y is stable and $p_a(Y) = g$, then $\mathcal{M}[\tau]$ is a locally closed and irreducible subset of $\overline{\mathcal{M}}_g$. Here is our main result.

Theorem 1. *Fix integers $n \ge 3$ and $k \ge 1$. For each integer i such that $1 \le i \le k$, fix integers $s_i > 0$, e_i, g_i, q_i such that $0 \le e_i \le n-1$ and $0 \le q_i \le g_i \le (s_i - 2)(n+1) + \lfloor (n+1)(n+e_i)/n \rfloor$. For all integers i, j such that $1 \le j < i \le k$ fix an integer $w_{i,j} \ge 0$ such that $1 \le \sum_{j=1}^{i-1} w_{i,j} \le n+1$.*

Then there exists a nodal curve $X \subset \mathbb{P}^n$ such that $h^1(X, T\mathbb{P}^n|X) = 0$, X has k irreducible components $T_1, \dots, T_k$, $\deg(T_i) = s_i n + e$ for all i, $\sharp(T_i \cap T_j) = w_{i,j}$ and each T_i has geometric genus q_i and arithmetic genus g_i.

Here we collect two topological types to which we may apply Theorem 1.

(a) The topological type of any nodal and connected curve Y of compact type, i.e. such that the connected component of the identity of $\mathrm{Pic}(Y)$ is an abelian variety. This equivalent to assuming that unmarked graph associated to $\|Y\|$ is a tree.

(b) Quasi-compact type, i.e. as in (a), but we drop the assumption that the irreducible components are smooth, i.e. after deleting the loops the unmarked graph is a tree.

2. Proofs

Remark 1. Let X be a nodal and connected projective curve and $j : X \to \mathbb{P}^n$ a non-degenerate embedding, i.e. assume that $j(X)$ spans $\mathbb{P}^n$. Hence the pull-back map $j^* : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(X, j^*(\mathcal{O}_{\mathbb{P}^n}(1)))$ is injective. Let V denote its image and set $L := j^*(\mathcal{O}_{\mathbb{P}^n}(1))$. The pair (L, V) is an element of $A^n_{\underline{d}}(X)$, where $\underline{d}$ is the multidegree of $j(X)$. Set $g := p_a(X)$ and $d := \deg(j(X))$. Let $u : X \to \mathbb{P}^n$ be any morphism. By [11], §3.4.1, $H^1(X, u^*(T\mathbb{P}^n))$ is an obstruction space for the deformation functor $\mathcal{D}_{X,\mathbb{P}^n}$ of all morphisms $X \to \mathbb{P}^n$ with fixed domain X and fixed target $\mathbb{P}^n$, while $H^0(X, u^*(T\mathbb{P}^n))$ is the tangent space at u to the same deformation functor. Now assume $h^1(j(X), T\mathbb{P}^n|j(X)) = 0$,i.e. $h^1(X, L) = 0$. Take $u := j$. We get that $\mathcal{D}_{X,\mathbb{P}^n}$ is smooth and of dimension $h^0(X, L) = \rho(g, n, d) + \dim(\mathrm{Aut}(\mathbb{P}^n))$. Now we use the universal property of $\mathbb{P}^n$ and the openness of spannedness in a flat family of line bundles with constant cohomology groups to get that (up to a quotient by $\mathrm{Aut}(\mathbb{P}^n)$) the scheme $A^n_{\underline{d}}(X)$ is like the deformation functor $\mathcal{D}_{X,\mathbb{P}^n}$ in a neighborhood of (L, V).

Remark 2. Fix a flat and projective family $\{X_t\}_{t\in U}$ of reduced curves parametrized by an integral quasi-projective variety U and $o \in U$. Assume the existence of an embedding $j : X_o \to \mathbb{P}^n$, $n \geq 2$, such that $h^1(j(X_o), T\mathbb{P}^n|j(X_o)) = 0$. Since $\dim(X_o) = 1$, $h^2(X_o, \mathcal{F}) = 0$ for every coeherent sheaf $\mathcal{F}$ on X_o. Hence the vanishing of $h^1(X_o, j^*(T\mathbb{P}^n))$ implies $h^1(X_o, \mathcal{G}) = 0$ for every coherent sheaf $\mathcal{G}$ on X_o which is a quotient of $j^*(T\mathbb{P}^n)$. We apply this observation to the sheaf denote with N_j in [11], §3.4.2; (often N_j' or $N'_{j(X_o}$ denote this sheaf (see [10], p. 29)). By [11], §3.4.1, the deformation functor $\mathcal{D}_{-,\mathbb{P}^n}$ of all deformations of j

with target fixed, but domain X_o not fixed is smooth at the pair (X_o, j). Hence there exist an open neighborhood W of o in U and a flat family $\{j_t\}_{t\in W}$ of embeddings $j_t : X_t \to \mathbb{P}^n$ such that $j_t = j_o$ and $h^1(j(X_t), T\mathbb{P}^n|j(X_t)) = 0$ for all $t \in W$.

Remark 3. Fix an integer $n \geq 1$ and a finite set $S \subset \mathbb{P}^n$, in linearly general position, i.e. such that every $S' \subseteq S$ with $\sharp(S') \leq n+1$ spans a $(\sharp(S')-1)$-dimensional linear subspace $\langle S'\rangle$ of $\mathbb{P}^n$. Assume $\sharp(S) \leq n+2$. Then the restriction map $H^0(\mathbb{P}^n, T\mathbb{P}^n) \to H^0(S, T\mathbb{P}^n|S)$ is surjective (e.g. use the Euler's sequence to see the vector space $H^0(\mathbb{P}^n, T\mathbb{P}^n)$ as a quotient of the set of all $(n+1)\times(n+1)$ matrices of linear forms in $n+1$ variables). Hence the restriction map $H^0(Y, T\mathbb{P}^n|Y) \to H^0(S, T\mathbb{P}^n|S)$ is surjective for any closed subscheme $Y \subset \mathbb{P}^n$ such that $S \subset Y$.

Lemma 1. *Let $Y \subset \mathbb{P}^n$ be a nodal curve. Fix a proper subcurve C of X and set $D := \overline{X\backslash C}$ and $S := C \cap D$. Assume $h^1(C, T\mathbb{P}^n|C) = 0$, $h^1(D, T\mathbb{P}^n|D) = 0$, $\sharp(S) \leq n+2$ and that S is in linearly general position. Then $h^1(Y, T\mathbb{P}^n|Y) = 0$.*

Proof. Look at the long cohomology exact sequence of the Mayer-Vietoris exact sequence

$$0 \to T\mathbb{P}^n|Y \to T\mathbb{P}^n|C \oplus T\mathbb{P}^n|D \to T\mathbb{P}^n|S \to 0. \tag{1}$$

Remark 3 gives the surjectivity of the restriction map

$$H^0(C, T\mathbb{P}^n|C) \to H^0(S, T\mathbb{P}^n|S).$$

Use this observation and the assumption $h^1(C, T\mathbb{P}^n|C) = h^1(D, T\mathbb{P}^n|D) = 0$. □

Lemma 2. *Let $X \subset \mathbb{P}^n$ be a nodal curve. Assume $h^1(T, T\mathbb{P}^n|T) = 0$ for all $T \in \mathcal{B}(X)$ and the existence of $J \subseteq \mathcal{B}(X)$ such that $Sing(X)'' \subset \cup_{T\in J} T$ and the restriction map $H^0(T, T\mathbb{P}^n|T) \to T\mathbb{P}^n|(Sing(X)'' \cap T)$ is surjective for all $T \in J$. Then $h^1(X, T\mathbb{P}^n|X) = 0$.*

Proof. Use the Mayer-Vietoris exact sequence

$$0 \to T\mathbb{P}^n|T \to \oplus_{T\in\mathcal{B}(X)} T\mathbb{P}^n|T \to T\mathbb{P}^n|\mathrm{Sing}(X)'' \to 0 \tag{2}$$

and Lemma 1. □

Remark 4. Let $D \subset \mathbb{P}^n$, $n \geq 3$, be a general degree d smooth rational curve. First assume $d \geq n$. The generality of D implies that D spans $\mathbb{P}^n$. The vector bundle $T\mathbb{P}^n|D$ is rigid (see [7]), i.e. every indecomposable rank 1 factor of it has either degree $\lfloor (n+1)d/n \rfloor$ or degree $\lceil (n+1)d/n \rceil$. Now assume $d < n$. The generality of D implies that D spans a d-dimensional linear subspace M

of $\mathbb{P}^n$ and D is a rational normal curve of M. We just saw that $TM|D$ is the direct sum of m line bundles of degree $m+1$. Thus $T\mathbb{P}^n|D$ is the direct sum of m line bundles of degree $m+1$ and $n-m$ line bundles of degree m.

Remark 5. Fix integers $d \geq n \geq 3$. Let $D \subset \mathbb{P}^n$ be a general smooth rational curve with degree d. Its normal bundle N_D is balanced (see [9] or [8]), i.e. it is a direct sum of line bundles of degree either $\lfloor((n+1)d-2)/(n-1)\rfloor$ or $\lceil((n+1)d-2)/(n-1)\rceil$. Fix an integer k such that $1 \leq k \leq \lfloor((n+1)d-2)/(n-1)\rfloor$ and a general $S \subset \mathbb{P}^n$ such that $\sharp(S) = k$. Then there exists a smooth rational curve $C \subset \mathbb{P}^n$ such that $\deg(C) = d$ and $S \subset C$ (see [6], Theorem 1.5). For a general pair (S, C) we may also assume that N_C is balanced. If $k \leq n+2$ the condition "a general S" is equivalent to "any S in linearly general position", because if $k \leq n+2$ any two such sets are projectively equivalent.

Theorem 2. *Fix an integer $n \geq 3$, a topological type $\tau = \|Y\|$ for nodal curves and integers d_T, $T \in \mathcal{B}(Y)$, such that $T \cong \mathbb{P}^1$ and $d_T \geq n$ for all $T \in \mathcal{B}(Y)$. Assume the existence of an ordering $T_1, \dots, T_s$ of the irreducible components of Y such that $\lfloor(n+1)d_{T_i}/n\rfloor + 1 \geq \sharp((T_1 \cup \cdots \cup T_{i-1}) \cap T_i)$ for every $i \in \{2, \dots, s\}$. Then there exists $X \subset \mathbb{P}^n$ such that $X \in \mathcal{M}[\tau]$, $h^1(X, T\mathbb{P}^n|X) = 0$ and $\deg(T) = d_T$ (up to the identification of the irreducible components of X with the ones of Y).*

Proof. Apply Remark 4 to each T_i. Then apply Remark 5 $s-1$ times to T_i, $2 \leq i \leq s$, and the set $(T_1 \cup \cdots \cup T_{i-1}) \cap T_i$, $i \in \{2, \dots, s\}$. Then apply $s-1$ times Lemma 1 to the pairs $(T_1 \cup \cdots \cup T_{i-1}, T_i)$, $i \in \{2, \dots, s\}$. □

Corollary 1. *Fix an integer $n \geq 3$, a topological type $\tau = \|Y\|$ for nodal curves and integers d_T, $T \in \mathcal{B}(Y)$, such that $d_T \geq n$ and $T \cong \mathbb{P}^1$ for all $T \in \mathcal{B}(Y)$. Assume $\lfloor(n+1)d_T/n\rfloor \geq \sharp(T \cap \overline{X \backslash T}) - 1$, except at most for one irreducible component of Y. Then there exists $X \subset \mathbb{P}^n$ such that $X \in \mathcal{M}[\tau]$, $h^1(X, T\mathbb{P}^n|X) = 0$ and $\deg(T) = d_T$ (up to the identification of the irreducible components of X with the ones of Y).*

Proof. Call T_1 the only irreducible component of X for which we do not assume the inequality $\lfloor(n+1)d_T/n\rfloor \geq w_T - 1$. Take any ordering of $\mathcal{B}(Y)$ such that T_1 is its first element. Apply Theorem 2. □

We give the following explicit example of the application of Theorem 2, because we will use it to get a result for integral nodal curves (see Theorem 3).

Proposition 1. *Fix integers $n \geq 3$, $s \geq 1$ and e such that $0 \leq e \leq n-1$. Set $d_{T_i} := n$ if $1 \leq i \leq s-1$ and $d_{T_s} = n+e$. For every integer i such that $2 \leq i \leq s$ fix an integer a_i such that $1 \leq a_i \leq n+2$ if $i \neq s$ and*

$1 \le a_s \le 1 + \lfloor (n+1)(n+e)/n \rfloor$. *For each integer i such that $2 \le i \le s$ fix non-negative integers $a_{i,j}$, $1 \le j \le i-1$, such that $\sum_{j=1}^{i} a_{i,j} = a_i$. Then there exists a nodal curve $X = T_1 \cup \cdots \cup T_s \subset \mathbb{P}^n$ such that $T_i \cong \mathbb{P}^1$ and $\deg(T_i) = d_{T_i}$ for all i, $\sharp(T_i \cap T_j) = a_{i,j}$ for all $1 \le j < i \le s$, and $h^1(X, T\mathbb{P}^n|X) = 0$.*

Proof. We impose that the components T_i, $1 \le i \le s-1$, are rational normal curves, while we take as T_s a general smooth rational curve of degree $n+s$ (general, but not general after fixing the other components). Remark 4 gives $h^1(T_i, T\mathbb{P}^n|T_i) = 0$ for all i. Fix a finite $S_i \subset T_i$. Remark 4 shows that the restriction map $H^0(T_i, T\mathbb{P}^n|T_i) \to H^0(S_i, T\mathbb{P}^n|S_i)$ is surjective if and only if either $\sharp(S_i) \le n+2$ (case $i \le s$) or $i = s$ and $\sharp(S_s) \le 1 + \lfloor (n+1)(n+e)/n \rfloor$. This is a description of the single components seen as abstract curves in $\mathbb{P}^n$. Now we will see why we can take $X = T_1 \cup \cdots T_s$ nodal and with $\sharp(T_i \cap T_j) = a_{i,j}$ for all $1 \le j < i \le s$. Any finite subset of a rational normal curve is in linearly general position. Any $n+3$ points in linear general position in $\mathbb{P}^n$ are contained in a unique rational normal curve. Thus we easily see using Remark 5 the following observation. Fix a nodal curve $Y \subset \mathbb{P}^n$ and $S \subset Y_{reg}$ in general position. If $\sharp(S) \le n+2$, then there is a rational normal curve $T \subset \mathbb{P}^n$ such that $S = Y \cap T$ and $Y \cup T$ is nodal. If $\sharp(S) \le 1 + \lfloor (n+1)(n+e)/n \rfloor$, then there is a smooth degree $n+e$ rational curve $C \subset \mathbb{P}^n$ such that $S = Y \cap C$ and $Y \cup C$ is nodal. To obtain $h^1(\mathbb{P}^n, T\mathbb{P}^n|X) = 0$ use that $h^1(T_i, T\mathbb{P}^n|T_i) = 0$, that the restriction map $H^0(T_i, T\mathbb{P}^n|T_i) \to H^0(S_i, T\mathbb{P}^n|S_i)$ is surjective if either $\sharp(S_i) \le n+2$ $(2 \le i \le s-1)$ or $i = s$ and $\sharp(S_s) \le 1 + \lfloor (n+1)(n+e)/n \rfloor$ and apply Lemma 2. □

Theorem 3. *Fix integers $n \ge 3$, $s \ge 1$ and e, g, q such that $0 \le e \le n-1$ and $0 \le q \le g \le (s-2)(n+1) + \lfloor (n+1)(n+e)/n \rfloor$. Then there exists an integral nodal curve $X \subset \mathbb{P}^n$ such that $\deg(X) = sn + e$, $p_a(X) = g$, X has geometric genus q and $h^1(\mathbb{P}^n, T\mathbb{P}^n|X) = 0$.*

Proof. The curve is obtained from the reducible curve $\widetilde{X}$ of the statement of Proposition 1 for the integers a_i with $\sum_{i=1}^{s} a_i = g + s - 1$ smoothing all nodes, except $g - q$ of them, with the prescription that for each irreducible component T of $\widetilde{X}$ we smooth at least one of the singular points of X contained in T (a condition equivalent to the irreducibility of a nearby smoothing). The smoothing is possible as an abstract (non-embedded curve). To get the smoothing inside $\mathbb{P}^n$ it is sufficient to apply Remark 2, because $h^1(\widetilde{X}, T\mathbb{P}^n|\widetilde{X}) = 0$. □

Proof of Theorem 1. Apply Theorem 3 and then use Lemma 1 $k-1$ times. At each inductive step use that for any integer x such that $1 \le x \le n+2$ any two sets of x points in linearly general position of $\mathbb{P}^n$ are projectively equivalent. This observation allows us to find T_i such that $\sharp(T_i \cap T_j) = w_{i,j}$ for all $1 \leq j \leq i-1$ and T_i intersects quasi-transversally $T_1 \cup \cdots \cup T_{i-1}$. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] L. Caporaso, A compactification of the universal Picard variety over the moduli space of stable curves, *J. Amer. Math. Soc.*, **7**, No. 3 (1994), 589-660.

[2] L. Caporaso, Néron models and compactified Picard schemes over the moduli stack of stable curves, *Amer. J. Math.*, **130**, No. 1 (2008), 1-47.

[3] L. Caporaso, Brill-Noether theory of binary curves, *ArXiv:* math/0807.1484.

[4] L. Caporaso, Linear series on semistable curves, *ArXiv:* math/0812.1682.

[5] M. Melo, Compactified Picard stacks over $\overline{\mathcal{M}}_g$, *ArXiv:* math/0710.3008; *Math. Z.*, To Appear.

[6] D. Perrin, Courbes passant par m points g'enéraux de $\mathbb{P}^3$, *Bull. Soc. Math. France*, Mémoire 28/29 (1987).

[7] L. Ramella, La stratification du schèma de Hilbert des courbes rationnelles, de $\mathbb{P}^n$ par le fibré tangent restreint, *C.R. Acad. Sci. Paris Sèr.*, I Math., **311**, No. 3 (1990), 181-184.

[8] Z. Ran, Normal bundles of rational curves in projective spaces, *Asian J. Math.*, **11**, No. 4 (2007), 567-608.

[9] G. Sacchiero, Normal bundles of rational curves in projective space, *Ann. Univ. Ferrara Sez. VII* (N.S.), **26** (1980), 33-40.

[10] E. Sernesi, On the existence of certain families of curves, *Invent. Math.*, **75**, No. 1 (1984), 25-57.

[11] E. Sernesi, *Deformations of Algebraic Schemes*, Springer, Berlin (2006).