International Journal of Pure and Applied Mathematics
Volume 54 No. 3 2009, 383-386

IMPOSING NEW NODES TO A LINEAR SYSTEM

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Abstract: Let $\underline{C}$ be a disjoint union of smooth projective curves and $L \in \mathrm{Pic}(\underline{C})$ ample and spanned. Here we compute the maximal and the minimal dimension of a linear subsystems of $|L|$ obtained imposing some nodes to $\underline{C}$.

AMS Subject Classification: 14H20, 14H55, 14H51
Key Words: nodal curve, linear series, morphism to $\mathbb{P}^1$

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We start by introducing one of our players in the statement of the following result.

Proposition 1. *Fix integers $s \geq 1$, $a_{ij} \geq 0$, $1 \leq i \leq j \leq s$, $d_i \geq 2$, $1 \leq i \leq s$, and s smooth projective curves $C_1, \dots, C_s$. Assume the existence of spanned $L_i \in Pic^{d_i}(C_i)$ inducing a separable morphism $h_{L_i} : C_i \to \mathbb{P}^{n_i}$, $n_i := h^0(C_i, L_i)$. Set $X := \sqcup_{i=1}^{s} C_i$ and let L be the only line bundle on X such that $L|C_i \cong L_i$ for all i. Then there exist a nodal curve Y and a spanned $R \in Pic(Y)$ such that X is the normalization of Y (call $u : X \to Y$ the normalization map) and $u^*(R) \cong L$.*

The set Γ of pairs (Y, R) satifying the thesis of Proposition 1 (for the fixed data s, a_{ij}, d_i, C_i) is parametrized by an integral projective variety. We want to study properties of the general $(Y, R) \in \Gamma$ and study the pairs (Y, R) with some extremal property (roughly speaking, the more special elements of Γ). We also introduced the following parameter spaces Φ and Φ'.

Received: June 19, 2009

Φ': Fix integers $s \geq 1$, $a_{ij} \geq 0$, $1 \leq i,j \leq s$, $i \neq j$, $d_i \geq 2$, $1 \leq i \leq s$, and s integral projective curves $E_1, \dots, E_s$. Assume the existence of spanned $L_i \in \mathrm{Pic}^{d_i}(E_i)$ inducing a separable morphism $h_{L_i} : E_i \to \mathbb{P}^{n_i}$, $m_i := h^0(E_i, L_i)$. Set $E := \sqcup_{i=1}^s E_i$ and let L be the only line bundle on X such that $L|C_i \cong L_i$ for all i.

Φ: As in Φ', but we require $n_i = 2$ for all i.

Remark 1. In the set-up of Theorem 1 set $a_{ij} := a_{ji}$ if $1 \leq j < i \leq s$. The curve Y in the statement of Theorem 1 is connected if and only if there is no $J \subsetneq \{1, \dots, s\}$, $J \neq \emptyset$, such that $a_{ij} = 0$ for all $(i,j) \in J \times (\{1, \dots, s\} \backslash J)$. Obviously, Y is connected if and only if there is a permutation τ of the set $\{1, \dots, s\}$ such that for every $\in \{2, \dots, s\}$ there is $j \in \{1, \dots, i-1\}$ with $a_{\tau(j)\tau(i)} > 0$. In this case for every $1 \leq i \leq s$ the curve obtained from $\sqcup_{j=1}^i Y_{\tau(j)}$ making the gluing determined by the matrix $a_{\tau(i)\tau(j)}$ is connected.

Proof of Proposition 1. Since $d_i > 0$ for all i, L is ample. Since each L_i is spanned L is spanned. Hence it is easy to check that a general 2-dimensional linear subspace V of $H^0(X, L)$ induces a morphism $\phi : X \to \mathbb{P}^1$ such that $\phi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong L$, $\phi(C_i) = \mathbb{P}^1$ for all i. Since each h_{L_i} is separable and V is general, ϕ is unramified outside finitely many points, i.e. each $\phi|C_i$ is separable. For every $i \in \{1, \dots, s\}$ fix a_{ii} general points $P_{i,1}, \dots, P_{i,a_{ii}}$ and glue each $P_{i,h}$ with another point of $\phi^{-1}(P_{i,h}) \cap X_h$. Such a point $Q_{i,h}$ exists because $\sharp(\phi^{-1}(P_{i,h}) \cap X_h) = d_i$ by the separability of the morphism $\phi|C_i$ and the generality of the point P_i. Let Y be the nodal curve obtained making these gluing. By construction X is the normalization of Y and the morphism ϕ factors through a morphism $\psi : Y \to \mathbb{P}^1$. Set $R := \psi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. $\square$

Remark 2. In the set-up of Proposition 1 let $I \subseteq \{1, \dots, s\}$ be the set of all i such that $C_i \cong \mathbb{P}^1$. If $i \in I$ and $a_{ii} = 0$, then we may allow $d_i = 1$ in the statement of Proposition 1.

Remark 3. Take a general $(Y, R) \in \Phi'$. Here we show how to compute $h^0(Y, R)$ in terms of the matrix (a_{ij}). We recall that Y is obtained from the curve $X := \sqcup_{i=1}^s E_i$ making $\alpha := \sum_{1<j\leq s} a_{ij}$ gluing and that $m_i := h^0(E_i, L_i)$. We order the gluing prescribed bu the matrix (a_{ij}) and call $Y[i]$, $0 \leq i \leq \alpha$, the curve obtained making the first i gluing in a general way. Let $R[i]$ the associated line bundle. Set $n[i] := h^0(Y[i], R[i])$. Hence $Y[0] = X$, $R[0] = L$, $n[0] = \sum_{i=1}^s m_i$, $Y[\alpha] = Y$, $R[\alpha] = R$. The goal is to compute $n[\alpha]$. Fix $1 \leq i \leq \alpha$ and assume known $n[i-1]$. For all integers $a \in \{1, \dots, s\}$ and $b \in \{0, \dots, \alpha\}$ let $F_a[b]$ denote the image of the curve E_a in the projective space $|R[b]|$. Since each L_a is ample, every $F_a[b]$ is a curve. Suppose that the i-th

gluing involves the curve E_u and E_v, $u \neq v$. If $F_u[i-1]$ and $F_v[i-1]$ are the same line, then we set $n[i] := n[i-1]$ and the images of $Y[i-1]$ and $Y[i]$ inside $\mathbb{P}^{n[i]-1}$ are the same. In all other cases we set $n[i] = n[i-1]-1$. Here the image B' in $|R[i]|^*$ of any subcurve B of $X[i-1] \subset |R[i-1]|^*$ containing both $F_u[i-1]$ and $F_v[i-1]$ satisfies $\dim(\langle B' \rangle) = \dim(\langle B \rangle) - 1$. The curves $F_u[i]$ and $F_v[i]$ are the same line if and only if $\langle F_u[i-1] \cup F_v[i-1] \rangle$ is a plane. A curve $F_a[i]$ is a line if and only if either $F_a[i-1]$ is a line or $F_a[i-1]$ is a plane curve and a general line intersecting both $F_u[i-1]$ and $F_v[i-1]$ intersects the plane $\langle F_a[i-1] \rangle$. The latter condition occurs if and only if either $\langle F_u[i-1] \cup F_v[i-1] \rangle$ is a plane intersecting $\langle F_a[i-1] \rangle$ at least in a line or $\langle F_u[i-1] \cup F_v[i-1] \rangle$ is a 3-dimensional linear space containing the plane $\langle F_a[i-1] \rangle$.

Remark 4. Take a general $(Y, L) \in \Gamma$. Let E_i be the general curve obtained from C_i making $a_{[ii]}$ gluing preserving a general 2-dimensional linear subspace of $H^0(C_i, L_i)$. Call $R_i \in \mathrm{Pic}^{d_i}(E_i)$ the degree d_i spanned line induced by L_i. It is easy to check that $h^0(E_i, R_i) = \max\{2, h^0(C_i, L_i) - a_{ii}\}$. Apply Remark 3 to the element of ϕ' obtained by the pairs (E_i, R_i), $1 \le i \le s$, and the matrix (a'_{ij}) such that $a'_{ij} = a_{ij}$ if $i \neq j$, while $a'_{ii} = 0$ for all $1 \le i \le s$. In our set-up we have $m_i = \max\{2, h^0(C_i, L_i) - a_{ii}\}$.

Under certain assumptions we may give a sharp upper bound for $h^0(Y, R)$ for every element (Y, R) of Φ, Φ' or Γ. In characteristic zero in the next two statements we have $k_i = d_i$. In positive characteristic it seems that prescribing the integers $k_1, \dots, k_s$ is better than prescribing the integers $d_1, \dots, d_s$.

Proposition 2. *Assume that the data of Φ are connected. Fix $(Y, R) \in \Phi$. Let k_i denote the separable degree of the morphism ϕ_{R_i} induced by $|R_i|$. Assume $\sum_{j \neq i} a_{ij} \le k_i$ for all $i \in \{1, \dots, s\}$. Then*

$$h^0(Y, R) \le s + 1. \tag{1}$$

There are pairs (Y, R) for which (1) is an equality.

Proof. Since Y is connected, we may find an ordering $T_1, \dots, T_s$ of its irreducible components such that each curve $\cup_{j=1}^{i} T_i$, $i \in \{1, \dots, s\}$ is connected. Let U be a reduced projective curve, $M \in \mathrm{Pic}(U)$, and U_i, $i = 1, 2$, proper subcurves of U such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 \neq \emptyset$. Assume that $M|U_1$ is spanned at least at one of the points of $U_1 \cap U_2$. A Mayer-Vietoris exact sequence gives $h^0(U, M) \le h^0(U_1, M|U_1) + h^0(U_2, |M_2) - 1$. Thus $h^0(U, M) \le h^0(U_1, M|U_1) + 1$ if $h^0(U_2, M|U_2) = 1$. The inequality (1) follows applying $s-1$ times this observation, each time to a curve $\cup_{j=1}^{i} T_i$, $2 \le i \le s$, with $U_2 : +T_i$ and $M := R| \cup_{j=1}^{i} T_j$.

We may build pairs (Y, R) for which (1) is an equality in the following way. Fix a general $Q \in \mathbb{P}^1$. Hence $\sharp(\phi_{R_i}^{-1}(Q)) = k_i$. Fix any $\sum_{j \neq i} a_{ij}$ points of $\phi_{R_i}^{-1}(Q)$ and attach to a_{ij} of them the label j. Glue the points of E_i with label j with the points of E_j with label i. The same Mayer-Vietoris exact sequence gives $h^0(\cup_{j=1}^{i} T_j, R| \cup_{j=1}^{i} T_j) = h^0(\cup_{j=1}^{i-1} T_j, R| \cup_{j=1}^{i-1} T_j) + 1$ for all $I \in \{2, \dots, s\}$. □

Proposition 3. *Assume that the data of* Γ *are connected and that* $m_i = 2$ *for all* i. *Fix* $(Y, R) \in \Gamma$. *Let* k_i *denote the separable degree of the morphism* ϕ_{R_i} *induced by* $|R_i|$. *Assume* $\sum_{j \neq i} a_{ij} \leq k_i$ *and* $k_i \geq 2$ *for all* $i \in \{1, \dots, s\}$. *Then the inequality (1) is satisfied. There are pairs* $(Y, R) \in \Gamma$ *for which (1) is an equality.*

Proof. Since $h^0(C_i, L_i) = 2$ and $k_i \geq 2$, there are gluing of the pair (C_i, RL_i) for which the associated pair (E_i, R_i) has the same number of sections and only these gluing are admidsible to get the curve Y. Hence we may apply Proposition 2 to the associated datum of type Φ. □

The connectedness of the data in Φ or Γ required in Propositions 2 and 3 is not restrictive, because we may easily decompose any datum in its "connected components".

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).