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OSCILLATORY BEHAVIOR OF THE SOLUTIONS OF NONLINEAR n-TH ORDER DIFFERENTIAL EQUATIONS WITH FORCING TERM AND RETARDED ARGUMENTS DEPENDING ON THE UNKNOWN FUNCTION

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Abstract: In this paper the n-th order differential equation

$$L_n x(t) + f\left(t, \tilde{x}\langle \Delta\left(t, x(t)\right)\rangle\right) = q(t) \qquad (E)$$

is concidered, where $n \geq 2$ and the retarded arguments $\Delta = (\Delta_1, \cdots, \Delta_m)$ depend on the independent variable t as well as on the unknown function x.

Some problems of the asymptotic and oscillatory behavior of the solutions of equation (E) are investigated.

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1. Introduction

We study the n-th order differential equation

$$L_n x(t) + f\left(t, \tilde{x}\langle \Delta\left(t, x(t)\right)\rangle\right) = q(t)\,, \qquad (1)$$

where the retarded arguments $\Delta = (\Delta_1, \cdots, \Delta_m)$ depend on the independent

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variable t as well as on the unknown function x.

Here $n \geq 2$ is an integer, $t \in J = [\alpha, +\infty) \subseteq \mathbb{R}_+ = [0, +\infty)$,

$$\tilde{x}\langle \Delta\,(t, x(t))\rangle = (x\,(\Delta_1(t, x(t)))\,, \cdots, x\,(\Delta_m(t, x(t))))$$

and $L_0 x(t) = x(t), \quad L_k x(t) = r_k(t)\,(L_{k-1}x(t))', \quad k = 1, \ldots, n.$

The domain $D(L_n)$ of L_n is defined to be the set of all functions $x\colon [t_0, +\infty) \to \mathbb{R}$ such that $L_k x(t), k = 1, \ldots, n$ exist and are continuous on some interval $[t_0, +\infty) \subseteq J$.

By a *proper* solution of equation (1) is meant a function $x \in D(L_n)$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq t_0$. We assume that equation (1) does possess proper solutions. A proper solution of equation (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*. Equation (1) is said to be oscillatory if all its proper solutions are oscillatory.

In this paper sufficient conditions are found under which equation (1) is oscillatory. The theorems generalize and improve analogous oscillatory results obtained by Onose [7] where $L_n x(t) = x^{(n)}(t)$ and the retarded arguments Δ_j do not depend on $x\colon \Delta_j = g_j(t)$.

Analogous results are given in the papers of Kartsatos [1], Kiguradze [3], Kusamo and Onose [4].

The oscillatory and asymptotic behavior of equations with deviating arguments depending on the unknown function have been considered in Markova and Simeonov [5], [6].

2. Preliminary Notes

Introduce the following conditions:

H1. $r_i \in C(J, (0, +\infty)),\ i = 1, \ldots, n-1, \quad r_n \equiv 1$ and

$$\int^{\infty} \frac{ds}{r_i(s)} = +\infty,\ i = 1, \ldots, n-1.$$

H2. $f \in C(J \times \mathbb{R}^m, \mathbb{R})$ and $x_1 f(t, x_1, \ldots, x_m) > 0$ for $t \in J$,

$$x_1 x_j > 0,\ j = 1, \ldots, m.$$

H3. $f(t, x_1, \ldots, x_m) \leq f(t, y_1, \ldots, y_m)$ provided that

$$0 < x_j \leq y_j, \quad j = 1, \ldots, m \text{ or } x_j \leq y_j < 0,\ j = 1, \ldots, m.$$

H4. $\Delta_j \in C(J \times \mathbb{R}, \mathbb{R}),\ j = 1, \ldots, m.$

H5. There exist functions $\tau_j \in C(J, \mathbb{R}),\ j = 1, \ldots, m$ and $T \geq \alpha$ such that

$$\lim_{t \to +\infty} \tau_j(t) = +\infty \text{ and } \tau_j(t) \leq \Delta_j(t,\ x) \leq t$$

for $j = 1,\ \ldots, m,\ t \geq T$ and $x \in \mathbb{R}$.

H6. $q \in C(J, \mathbb{R})$.

We need the following lemma which generalizes the well-known lemma of Kiguradze [2].

Lemma 1. *Suppose condition* ***H1*** *holds and the functions $L_n x$ and $x \in D(L_n)$ are of constant sign and not identically zero for $t \geq t_* \geq \alpha$. Then there exist a $t_k \geq t_*$ and an integer $k,\ 0 \leq k \leq n$ with $n + k$ even for $x(t)L_n(x(t))$ nonnegative or $n + k$ odd for $x(t)L_n(x(t))$ nonpositive and such that for every $t \geq t_k$*

$$x(t)L_i x(t) > 0,\ i = 0, 1, \ldots, k,$$
$$(-1)^{i-k}\, x(t)L_i x(t) > 0,\ i = k, k+1, \ldots, n.$$

3. Main Results

Consider the equation

$$L_n u(t) + f(t, \tilde{u}\langle\tau(t)\rangle) = 0 \tag{2}$$

and the inequality

$$(L_n x(t) + f(t, \tilde{x}\langle\Delta(t, x(t))\rangle))\, \operatorname{sign} x(t) \leq 0, \tag{3}$$

where $\tilde{u}\langle\tau(t)\rangle = (u(\tau_1(t)), \ldots, u(\tau_m(t)))$.

Theorem 1. *Assume that conditions* ***H1-H5*** *hold.*

(i) If n is even and equation (2) is oscillatory, then inequality (3) is oscillatory.

(ii) If n is odd and every solution $u(t)$ of equation (2) is oscillatory or $\lim_{t \to +\infty} u(t) = 0$, then every solution $x(t)$ of inequality (3) is oscillatory or $\lim_{t \to +\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of inequality (3). Without loss of generality we assume that $x(t) > 0,\ t \geq t_0 \geq \alpha$. By condition **H5** there exists $t_1 \geq t_0$ such that $\Delta_j(t, x(t)) \geq \tau_j(t) \geq t_0$ and $x\left(\Delta_j(t, x(t))\right) > 0$ for

$t \geq t_1,\ j = 1, \dots, m$. From inequality (3) we have

$$L_n x(t) \leq -f(t, \tilde{x}\langle\Delta(t, x(t))\rangle) < 0, \quad t \geq t_1. \tag{4}$$

Case 1. n is even. Then by Lemma 1 there exist $t_k \geq t_1$ and $k \in \{1, 3, \dots, n-1\}$ such that

$$L_i x(t) > 0, \quad i = 0, 1, \dots, k,\ t \geq t_k, \tag{5}$$

$$(-1)^{i-k} L_i x(t) > 0, \quad i = k, k+1, \dots, n,\ t \geq t_k.$$

In this case $x'(t) > 0,\ x(t)$ is increasing for $t \geq t_k$ and

$$f(t, \tilde{x}\langle\Delta(t, x(t))\rangle) \geq f(t, \tilde{x}\langle\tau(t)\rangle), \quad t \geq T \geq t_k. \tag{6}$$

Integrating (4) and using (5) and (6) we obtain

$$x(t) \geq x(T) + \Phi(t, \tilde{x}); \quad t \geq T,$$

where $x(T) > 0$ and

$$\Phi(t, \tilde{x}) = \int_T^t \frac{1}{r_1(u_1)} \cdots \int_T^{u_{k-1}} \frac{1}{r_k(u_k)} \int_{u_k}^{\infty} \frac{1}{r_{k+1}(u_{k+1})} \cdots \int_{u_{n-1}}^{\infty} f(s, \tilde{x}\langle\tau(s)\rangle) \times ds du_{n-1} \dots du_1.$$

We show that the equation

$$z(t) = x(T) + \Phi(t, \tilde{z}), \quad t \geq T$$

has a positive solution. To prove this, we define the sequence $\{z_n(t)\}$ such that

$$z_0(t) = x(t),\ t \geq t_1,$$

$$z_{n+1}(t) = \begin{cases} x(T) + \Phi(t, \tilde{z}_n), & t \geq T\,, \\ x(T), & t_1 \leq t \leq T\,. \end{cases} \tag{7}$$

We see that the sequence $\{z_n(t)\}$ is well-defined and by induction we obtain

$$0 < x(T) \leq z_{n+1}(t) \leq z_n(t) \leq x(t), \quad t \geq T. \tag{8}$$

We put $z(t) = \lim\limits_{n\to+\infty} z_n(t),\ t \geq t_1$. Then by (7), (8) and Lebesque's Dominated Convergence Theorem we have

$$z(t) = x(T) + \Phi(t, \tilde{z}),\ t \geq T. \tag{9}$$

From (9) it follows that the function $u = z(t) \geq x(T) > 0$ is a positive solution of equation (2) which leads to a contradiction.

Case 2. n is odd. Then it follows from (4) and Lemma 1 that there exist $t_k \geq t_1$ and $k \in \{0, 2, \dots, n-1\}$ such that (5) holds.

Let $k > 0$. Then $x'(t) > 0$ and the proof is the same as in the case of even n.

Let $k = 0$. Then $x'(t) < 0$, x is decreasing and there exists the limit $\lim_{t\to+\infty} x(t) = c \geq 0$ and $x(t) \geq c$. Suppose that $c > 0$. Then there exists $T \geq t_k$ such that $x(\Delta_j(t, x(t))) \geq c$ for $t \geq T$, $j = 1, \ldots, m$ and from (4) we have

$$L_n x(t) \leq -f(t, \tilde{c}), \quad t \geq T, \tag{10}$$

where $\tilde{c} = (c, \ldots, c) \in \mathbb{R}^m$. Integrating (10) and using (5) and (6) we obtain the inequality

$$x(t) \geq c + \Psi(t, \tilde{c}),\ t \geq T,$$

where

$$\Psi(t, \tilde{x}) = \int_t^\infty \frac{1}{r_1(u_1)} \int_{u_1}^\infty \frac{1}{r_2(u_2)} \cdots \int_{u_{n-1}}^\infty f(s, \tilde{x}\langle\tau(s)\rangle) ds du_{n-1} \ldots du_1 .$$

Consider the equation

$$z(t) = c + \Psi(t, \tilde{z}), \quad t \geq T. \tag{11}$$

We prove that (11) has a positive solution. Put

$$z_0(t) = c, \quad t \geq t_1,$$

$$z_{n+1}(t) = \begin{cases} c + \Psi(t, \tilde{z}_n), & t \geq T, \\ c + \Psi(T, \tilde{z}_n), & t_1 \leq t \leq T. \end{cases}$$

By induction we have

$$0 < c \leq z_n(t) \leq z_{n+1}(t) \leq x(t), \quad t \geq t_1.$$

Then the function $u = z(t) = \lim_{n\to\infty} z_n(t)$ is a positive solution of equation (2) and $z(t) \geq c > 0$, which is a contradiction. □

Consider the equation

$$L_n x(t) + f(t, \tilde{x}\langle\Delta(t, x(t))\rangle) = 0. \tag{12}$$

As a consequence of Theorem 1 we obtain the following theorem.

Theorem 2. *Assume that conditions* ***H1-H5*** *hold.*

(i) *If n is even and equation (2) is oscillatory, then equation (12) is oscillatory.*

(ii) *If n is odd and every solution $u(t)$ of equation (2) is oscillatory or $\lim_{t\to+\infty} u(t) = 0$, then every solution $x(t)$ of equation (12) is oscillatory or $\lim_{t\to+\infty} x(t) = 0$.*

Theorem 3. *Assume that conditions* ***H1-H6*** *hold and there exists $Q \in C(J, \mathbb{R})$ such that $L_n Q(t) = q(t)$, $\lim_{t\to+\infty} Q(t) = 0$ and $Q(t)$ has an unbounded set of zeros.*

(i) If n is even and equation (2) is oscillatory, then equation (1) is oscillatory.

(ii) If n is odd and every solution $u(t)$ of equation (2) is oscillatory or $\lim_{t\to+\infty} u(t) = 0$, then every solution $x(t)$ of equation (1) is oscillatory or $\lim_{t\to+\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Assume without loss of generality that $x(t) > 0,\ t \geq t_0 \geq \alpha$. Then there exists $t_1 \geq t_0$ such that $x\left(\Delta_j(t, x(t))\right) > 0$ for $t \geq t_1,\ j = 1, \ldots, m$. Put $y(t) = x(t) - Q(t)$; then

$$L_n y(t) + f\left(t, \tilde{x}\langle\Delta(t, x(t))\rangle\right) = 0$$

whence it follows that $L_n y(t) < 0,\ t \geq t_2 \geq t_1$. By Lemma 1 we have that

$$y \text{ and } y' \text{ are of constant sign for } t \geq T \geq t_2. \tag{13}$$

If $y(t) < 0$ for $t \geq T$, then $0 < x(t) < Q(t)$ and this contradicts the assumption that $Q(t)$ has an unbounded set of zeros. Therefore we must have

$$y(t) > 0,\ t \geq T. \tag{14}$$

From (13) and (14) it follows that there exists the limit

$$\lim_{t\to+\infty} y(t) = c \qquad (0 \leq c \leq +\infty). \tag{15}$$

The case $c = 0$ is possible only if n is odd. Then $\lim_{t\to+\infty} x(t) = 0$.

Let $0 < c \leq +\infty$ and $2d \in (0,\ c)$. Then we conclude from (13)-(15) that there exists $t_3 \geq T$ such that

$$y(t) > 2d,\ |Q(t)| \leq d \ \text{ for } t \geq t_3.$$

The function $z(t) = y(t) - d$ satisfies the inequalities

$$0 < d < z(t) \leq y(t) + Q(t) = x(t), \quad t \geq t_3,$$

and either

$$L_n z(t) + f(t, \tilde{z}\langle\tau(t)\rangle) \leq 0 \quad (\text{if } y' > 0),$$

or

$$L_n z(t) + f(t, \tilde{d}) \leq 0 \quad (\text{if } y' < 0).$$

In the both cases we conclude as in the proof of Theorem 1 that equation (2) has a positive solution $u(t)$ with $u(t) \geq d$, which is a contradiction. □

Theorem 4. *Assume that conditions **H1-H6** hold and there exists $Q \in C(J, \mathbb{R})$ such that $L_n Q(t) = q(t)$ and $\lim_{t\to+\infty} Q(t) = 0$.*

(i) If n is even and equation (2) is oscillatory, then every solution $x(t)$ of equation (1) is oscillatory or $\lim_{t\to+\infty} x(t) = 0$.

(ii) If n is odd and every solution $u(t)$ of equation (2) is oscillatory or

$\lim_{t\to+\infty} u(t) = 0$, *then every solution* $x(t)$ *of equation (1) is oscillatory or* $\lim_{t\to+\infty} x(t) = 0$.

The proof of Theorem 4 is contained in the proof of Theorem 3.

Theorem 5. *Assume that conditions **H1-H6** hold and there exist constants* q_1, q_2 *and sequences* $\{t'_s\}, \{t''_s\}$ *such that*

$$\lim_{s\to+\infty} t'_s = \lim_{s\to+\infty} t''_s = +\infty,\ Q(t'_s) = q_1,\ Q(t''_s) = q_2$$

and

$$q_1 \le Q(t) \le q_2.$$

(i) *If* n *is even and equation (2) is oscillatory, then equation (1) is oscillatory.*

(ii) *If* n *is odd and every solution* $u(t)$ *of equation (2) is oscillatory or* $\lim_{t\to+\infty} u(t) = 0$, *then every solution of equation (1) is oscillatory or*

$$\lim_{t\to+\infty} (x(t) - Q(t)) = -q_1 \qquad (\text{or } -q_2).$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Assume without loss of generality that $x(t) > 0,\ t \ge t_0 \ge \alpha$. Then there exists $t_1 \ge t_0$ such that $x\left(\Delta_j(t, x(t))\right) > 0$ for $t \ge t_1,\ j = 1, \dots, m$. Let $y(t) = x(t) - Q(t)$; then

$$L_n y(t) + f\left(t, \tilde{x}\langle\Delta(t, x(t))\rangle\right) = 0, \tag{16}$$

which implies $L_n y(t) < 0,\ t \ge t_2 \ge t_1$. By Lemma 1 it follows that (13) holds. Put $z(t) = y(t) + q_1$. Since $y(t)$ is monotone we have

$$\lim_{t\to+\infty} z(t) = c \quad (-\infty \le c \le +\infty).$$

Let $-\infty \le c < 0$. Then we have that $y(t) + q_1 < 0$ for all sufficiently large t. It follows that

$$0 > y(t'_s) + q_1 = y(t'_s) + Q(t'_s) = x(t'_s) > 0,$$

which is a contradiction.

Let $c = 0$. Then we have $\lim_{t\to+\infty} (x(t) - Q(t)) = -q_1$.

Let $0 < c \le +\infty$ and $d \in (0, c)$. Then we have

$$0 < d \le z(t) = y(t) + q_1 \le y(t) + Q(t) = x(t). \tag{17}$$

We conclude from (16) and (17) that

$$L_n z(t) + f\left(t, \tilde{z}\langle\Delta(t, x(t))\rangle\right) \le 0.$$

The following two cases are possible.

Case 1. If $y'(t) > 0$, then $z'(t) > 0$, $z(t)$ is increasing and

$$f\left(t, \tilde{z}\langle \Delta(t, x(t)) \rangle\right) \geq f\left(t, \tilde{z}\langle \tau(t) \rangle\right).$$

Then $z(t)$ satisfies the inequality

$$L_n z(t) + f\left(t, \tilde{z}\langle \tau(t) \rangle\right) \leq 0.$$

Case 2. If $y'(t) < 0$, then $z'(t) < 0$, $z(t)$ is decreasing and

$$f\left(t, \tilde{z}\langle \Delta(t, x(t)) \rangle\right) \geq f\left(t, \tilde{d}\right).$$

Then $z(t)$ satisfies the inequality

$$L_n z(t) + f\left(t, \tilde{d}\right) \leq 0.$$

In the both cases we conclude as in the proof of Theorem 1 that equation (2) has a positive solution $u(t)$ with $u(t) \geq d$, which is a contradiction. □

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