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# SYZYGIES OF REDUCIBLE CURVES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

**Abstract:** Here we use works by Schreyer and Aprodu to check the Green conjecture for certain reducible curves with a nice nodal model inside $\mathbb{P}^1 \times \mathbb{P}^1$.

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For any scheme $X$ and any spanned line bundle $L$ on $X$ and all non-negative integers $a, b$ let $K_{u,v}(X, L)$ denote the Koszul cohomology groups (see [8], [4]).

We first use [11] to get several stable reducible curves $X$ with $\omega_X$ very ample and for which Green's conjecture holds.

**Theorem 1.** *Fix integers $q, p, \delta$ such that $q \geq p \geq 3$, $0 \leq \delta \leq p-2$. If $(p, q, \delta) = (3, 3, 1)$, then assume $Y$ irreducible. If $(p, \delta) = (3, 1)$ and $q > 3$, then assume that $Y$ has no component of type $(0, 1)$. Fix a general $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $\sharp(S) = \delta$. Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be any reduced curve of type $(p, q)$ such that $S \subset Y$ and $Y$ has an ordinary node at each point of $S$. Let $u : X \to Y$ be the partial normalization of $Y$ in which we normalize only the points of $S$. Then $X$ is connected and Gorenstein, $p_a(X) = pq - p - q + 1 - \delta$, $\omega_X$ is very ample and $K_{x,1}(X, \omega_X) = 0$ for every $x \geq g - p + 1$.*

See Lemma 5 for a list of reducible curves to which Theorem 1 may be applied.

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**Proposition 1.** *Fix integers $p, g, c$ such that $p \geq 2$, $g > p(p-1)$ and $0 \leq c \leq g$. There is an integral nodal curve $X$ such that:*

*(i) $p_a(X) = g$, $\sharp(Sing(X)) = c$, there is $M \in Pic^p(X)$ such that $h^0(X, M) = 2$, while there is no $N \in Pic^{p-1}(X)$ with $h^0(X, N) \geq 2$;*

*(ii) there are an integer $y \geq 2g$ and $R \in Pic^y(X)$ such that $K_{h^0(X,R)-p,1}(X, R) = 0$;*

*(iii) $K_{x,1}(X, L) = 0$ for every integer $d \geq y + 2g$, every integer $x \geq d + 1 - g - p$, and every $L \in Pic^d(X)$, i.e. $L$ has property $M_{k-1}$.*

*In particular (iii) implies that $X$ satsifies the Green-Lazarsfeld conjecture.*

For any scheme $A$ and any $P \in A_{reg}$ let $\{2P, A\}$ denote the closed subscheme of $A$ with ${\mathcal{I}_P}^2$ as its ideal sheaf. For any finite $S \subset A_{reg}$ set $\{2S, A\} := \cup_{P \in S}\{2P, A\}$.

**Lemma 1.** *Fix integers $q, p, \delta$ such that $q \geq p \geq 2$ and $0 \leq \delta \leq p - 2$. Fix a general $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that $\sharp(S) = \delta$. Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be any reduced curve of type $(p, q)$ such that $S \subset Y$ and $Y$ has an ordinary node at each point of $S$. Let $u : X \to Y$ be the partial normalization of $Y$ in which we normalize only the points of $S$. Then $X$ is connected.*

*Proof.* Assume that $X$ is not connected and write $X = A \sqcup B$ with $A$ one of its connected components. Set $A' := u(A)$ and $B' := u(B)$. Call $(a, b)$ the bidegree of $A'$. Thus $B'$ has bidegree $(p - a, q - b)$. Without loosing generality we may assume $0 \leq a \leq p/2$. Since $Y$ is nodal, $A' \cap B' \subseteq S$ and $A'$ intersects transversally $B'$, i.e. $\sharp(A' \cap B') = a(q - b) + b(p - a)$. Thus

$$a(q - b) + b(p - a) \leq p - 2. \tag{1}$$

Since $a \leq p/2$, the second term of the left hand side of (1) gives $b \in \{0, 1\}$. If $b \in \{0, 1\}$, then the first term of the left hand side of (1) gives $a = 0$. Even in the remaining case $(a, b) = (0, 1)$, the inequality (1) fails. □

**Lemma 2.** *Fix integers $u \geq 2$ and $v \geq 2$ and a general $S \subset A := \mathbb{P}^1 \times \mathbb{P}^1$ such that $\sharp(S) = uv - u - v + 1$. There is an integral $B \in |\mathcal{I}_{\{2S,A\}}(u, v)|$ if and only if $(u, v) = (2, 2)$.*

*Proof.* The "if" part is obvious. We only check the "only if" part. Fix an integer $t \geq 0$ and let $B \subset A$ be a general subset such that $\sharp(A) = t$. Since $u \geq 2$ and $v \geq 2$, [5], Corollary 4.6, gives that either $h^0(A, \mathcal{I}_{\{2B,A\}}(u, v)) = 0$ (case $3t \geq (u+1)(v+1)$) or $h^1(A, \mathcal{I}_{\{2B,A\}}(u, v)) = 0$ (case $3t \leq (u+1)(v+1)$), unless $(u, v, t) = (4, 4, 8)$ (in the exceptional case $|\mathcal{I}_{\{2B,A\}}(4, 4)| = \{2\Gamma\}$, where $\Gamma$ is the only smooth elliptic curve of bidegree $(2, 2)$ containg $B$). If $uv + u + v \geq$

$3uv - 3u - 3v + 3$, i.e. if $2u + 2v \geq 2uv + 3$, then $u = v = 2$. No other case may arise. □

*Proof of Theorem 1.* Obviously $X$ is Gorenstein. Lemma 1 gives the connectedness of $X$. The adjunction formula on $\mathbb{P}^1 \times \mathbb{P}^1$ gives the value of $p_a(Y)$ and hence of $p_a(X)$.

(a) Here we check that $X$ is very ample. Assume that $X$ is not very ample. By [6], Theorem 3.6, $X$ is either honestly hyperelliptic or $X$ is not numerically 3-connected.

(a1) Here show that $X$ is 3-connected. Let $A$ be a proper subcurve of $X$. Set $B := \overline{X \backslash A}$, $A' := u(A)$ and $B' := u(B)$. In order to obtain a contradiction we assume length$(A \cap B) \leq 2$. Let $(a, b)$ the bidegree of $A'$. Hence $B'$ has bidegree $(p - a, q - b)$. Without losing generality we may assume $0 \leq a \leq p/2$. Assume length$(A \cap B) \leq 2$. Since each point of $S$ is an ordinary node of $Y$, the scheme $A' \cap B'$ is the disjoint union of a scheme isomorphic to $A \cap B$ and a subset $S'$ of $S$. Set $z := \sharp(S')$. We get

$$a(q - b) + b(p - a) \leq z + 2 \leq \delta + 2 \leq p. \tag{2}$$

Since $a \leq p - 2$, (2) implies $b \leq 1$ (the case $(a, b, z, \delta) = (p/2, 2, p - 2, p - 2)$ is escluded, because $q \geq p \geq 3$). First assume $b = 0$. Since $(a, b) \neq (0, 0)$ and $q \geq p$, we get $a = 1$ and $z = \delta = p-2$. However, the generality of $S$ implies that each curve of bidegree $(1, 0)$ contains at most one point of $S$. Hence $z = \delta = 1$ and $q = 3$. Hence $p = 3$. Thus we are in the excluded case $(p, q, \delta) = (3, 3, 1)$. Now assume $b = 1$. Since $q \geq b$, (2) gives $a = 0$ and $z = \delta = p - 2$. However, the generality of $S$ implies that every curve of bidegree $(0, 1)$ contains at most one point of $S$. Thus $z = \delta = 1$ and $p = 3$. Hence we are in the excluded case $(p, \delta) = 1$ with $Y$ having a component of type $(0, 1)$.

(a2) Here we assume that $X$ is honestly hyperelliptic, i.e. assume the existence of a finite and flat morphism $f : X \to \mathbb{P}^1$ such that $\deg(f) = 2$. Set $R := f^*(\mathcal{O}_{\mathbb{P}^1}(1))$. $R$ is an ample and spanned line bundle and $\deg(R) = 2$. The existence of $f$ shows that either $X$ is irreducible or it has two irreducible components, each of them isomorphic to $\mathbb{P}^1$. First assume that $X$ is reducible and call $U, V$ the irreducible components of $X$. Set $U' := u(U)$ and $V' := u(V)$. Let $(c, d)$ be the bidegree of $U'$. Hence $V'$ has bidegree $(p - c, q - d)$. Without loosing generality we may assume $0 \leq c \leq p/2$. We have $p_a(U') = cd - c - d + 1$ and $p_a(V') = (p-c)(q-d) - p + c - q + d + 1$. Since $U \cong V \cong \mathbb{P}^1$ and each point of $S$ is an ordinary node of $Y$, we get $cd - c - d + 1 + (p-c)(q-d) - p + c - q + d + 1 \leq \delta$, i.e.

$$pq + 2cd - cq - dp + 2 \leq \delta \leq p - 2. \tag{3}$$

If $c = 0$, then $d = 1$, because $U$ is irreducible. In this case (3) fails. Now assume $c = 1$. In this case (3) gives $pq + 2d - q - dp \le p - 4$. To get a contradiction we see that the last inequality fails if either $d \ge q - 2$ (just because $q \ge p \ge 3$) or $d \le q - 3$ and $2d \ge q$ (because $3p > p - 4$) or $d \le q - 3$ and $2d < q$ (because $p \ge 3$). Now assume $d = 0$. Since $V$ is irreducible, we get $c = 0$. Hence in this case (3) fails. Now assume $d = 1$. In this case (3) gives $pq+2c-cq-p+2 \le p-2$. To get a contradiction we see that the last inequality fails if either $c \ge p - 2$ (obvious) or $c \le p-3$ (because $3q \ge 3p > 2p-4$). If $d = q$, then $c = 1$, because $V$ is connected; we excluded this case. Now assume $d = q - 1$. In this case (3) gives $p+(2q-2)c \le 2c+p-4$, which is obviously false. Hence from now on we may assume $2 \le c \le p/2$ and $2 \le d \le q-2$. Since $p_a(U) = p_a(V) = 0$ and $Y$ is nodal at each point of $S$, we also get $\sharp(S \cap (U'\backslash(U' \cap V'))) = cd - c - d + 1$ and $\sharp(S \cap (V'\backslash(U' \cap V'))) = (p-c)(q-d)-p-q+c+d+1$. The contradiction comes from Lemma 2 applied to the integers $(u, v) = (c, d)$ and $(u, v) = (p - c, p - c)$, even if $p = q = 4$ and $c = d = 2$.

Now assume that $X$ is irreducible. If $\delta = 0$, then $Y = X$ and hence $R$ is a degree 2 spanned line bundle on the irreducible curve $Y$ of bidegree $(p, q)$, contradicting [10] and the assumption $q \ge p \ge 3$. Hence we may assume $\delta > 0$. Since $f$ is finite, $h^i(Y, f_*(R)) = h^i(X, R)$ for all $i$. Since $p_a(Y) - p_a(X) = \delta$, Riemann-Roch applied to $R$ on $X$ and to $f_*(R)$ on $Y$ shows that the rank 1 torsion free sheaf $f_*(R)$ has degree $2 + \delta$. By [7] there is a smooth curve $C$ of bidegree $(p, q)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and a degree $2 + \delta$ line bundle $L$ on $C$ such that $h^0(C, L) \ge h^0(Y, f_*(R))$. Since $h^0(Y, f_*(R)) = h^0(X, R) \ge 2$, the smooth case of [10] gives $\delta = p - 2$. Let $C'$ be any smooth curve of bidegree $(p, q)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Every $g^1_p$ on $C'$ is induced by a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence $C'$ has exactly one (resp. two) $g^1_p$ if and only if $q > p$ (resp. $q = p$). Let $\{C_t\}_{t\in\Delta}$ be a smoothing of $Y$ with $\Delta$ a connected affine curve, $Y = C_o$ for some $o \in \Delta$, $C = C_{t_0}$ for some $t_0 \in \Delta\backslash\{o\}$, and $C_t$ smooth for all $t \in \Delta\backslash\{o\}$. A ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ induces a flat family of spanned and locally free $g^1_p$ and $g^1_q$ on the family $\{C_t\}_{t\in\Delta}$. The quoted uniqueness part for $C_t$, $t \ne o$, implies that $f_*(R)$ is one of the limits over $C_o$ of the restriction to $\Delta\backslash\{o\}$ of this relative $g^1_p$. Since the relative compactified Jacobian for integral curves contained in a smooth surface is relatively projective (and hence relatively separated) (see [1], Theorem 9; one can also quote [2] Corollary 6.7 (i), or [2], Theorem 8.5), we get that the sheaf $f_*(R)$ is locally free, i.e. $\delta = 0$, contradiction.

(b) The proof of [11], Proposition 6, shows that the canonical model of our curve $X$ has the same betti numbers as the curve $C$ of [11], Proposition 6 (the case in which $X$ is smooth). □

**Remark 1.** The same proof may be adapted for the linear systems on the Hirzebruch surface $F_e$ used in [4], Corollary 5.

**Lemma 3.** *Fix integers $u, v, z, w$ such that $v \geq u \geq 4$, $0 \leq z \leq u$ and $0 \leq w \leq uv - u - v + 1 - z$. Let $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a general subset such that $\sharp(S) = z$. There is a nodal and integral $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ with bidegree $(u, v)$, $S \subseteq Sing(Y)$ and $w$ further ordinary nodes as its only singularities.*

*Proof.* Since $z \leq p \leq q$, we may find a union $T$ of $p$ distinct curves of bidegree $(1, 0)$ and $q$ distinct curves of bidegree $(0, 1)$ such that each point of $S$ is contained in one of these components of bidegree $(1, 0)$ and in a component of bidegree $(0, 1)$, i.e. such that $S \subset \mathrm{Sing}(T)$. Since $\omega^*_{\mathbb{P}^1 \times \mathbb{P}^1}$ is ample, we may smooth $pq - z - w$ of the points of $\mathrm{Sing}(T) \backslash S$ appling [12], Lemma 2.2 and Corollary 2.14. □

**Lemma 4.** *Fix integers $q \geq p \geq 3$, $0 \leq z \leq p - 2$, $s > 0$ and $a_i \geq 0$, $b_i \geq 0$, $1 \leq i \leq s$, such that $(a_i, b_i) \neq (0, 0)$ for all $i$, $\sum_{i=1}^{s} a_i = p$ and $\sum_{i=1}^{s} b_i = q$. Set $\epsilon_i := 0$ if $(a_i, b_i) \neq (4, 4)$ and $\epsilon_i := 1$ if $(a_i, b_i) = (4, 4)$. Fix integers $\alpha_i \geq 0$, $\eta_i \geq 0$, $1 \leq i \leq s$, such that $\alpha_i = 0$ if either $a_i \leq 1$ or $b_i \leq 1$, $3\alpha_i + \max\{\eta_i, \epsilon_i\} \leq (a_i + 1)(b_i + 1) - 1$ and $2(\sum_{i=1}^{s} \alpha_i) + \eta_i = 2z$. Fix a general $S \subset A := \mathbb{P}^1 \times \mathbb{P}^1$ such that $\sharp(S) = z$. There is a nodal $Y \subset A$ such that $Y$ has $s$ irreducible components $Y_1, \dots, Y_s$, each $Y_i$ has bidegree $(a_i, b_i)$, $S \subseteq Sing(Y)$ and each $Y_i$ contains exactly $\alpha_i$ singular points of $Y$ lying in a unique irreducible component of $Y$ and $\eta_i$ singular points of $Y$ lying in two irreducible components of $Y$. Moreover, each finite set $S \cap Y_i \cap Y_j$, $1 \leq i < j \leq s$, may be an arbitrary subset $S_{i,j}$ of $S$, with the only restriction that with these choices $\sum_{j=i+1}^{s} \sharp(S_{i,j}) + \sum_{j=1}^{i-1} \sharp(S_{J,i}) = \eta_i$.*

*Proof.* Apply Lemma 5 $s$ times. □

The following lemma says that the bound for the vanishing of $K_{x,1}(X, \omega_X) = 0$, i.e. that $K_{g-p,1}(X, \omega_X) \neq 0$. Indeed, it allows to apply the proof of [9]; alternatively, if $q > p$ we may use [11], part (c) of Proposition 5.

**Lemma 5.** *Take $X$ as in in the proof of Theorem 1 and let $R \in Pic^p(X)$ be the spanned line bundle induced by a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mu_R : H^0(X, R) \otimes H^0(X, \omega_X \otimes R^*)$ is separately injective, i.e. $\mu_R(\alpha \otimes \beta) \neq 0$ for all $\alpha \in H^0(X, R) \backslash \{0\}$ and all $\beta \in H^0(X, \omega_X \otimes R^*) \backslash \{0\}$.*

*Proof.* Set $S := \mathrm{Sing}(Y)$. Thus $0 \leq \delta := \sharp(S) \leq p - 2$ and $S$ is general in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}) = 0$, adjuction theory gives $H^0(X, \omega_X \otimes R^*) \cong H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{I}_S(p-2, q-2)$. Since the restriction map $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0))$ is an isomorphism, it is sufficient to use the separate injectivity of the multiplication map for any two line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$, which is true because this is true on an arbitrary integral projective variety. □

The proof of [3], Lemma 4.1 (i.e. the case of a smooth curve) gives the following result.

**Lemma 6.** *Let $X$ be an integral projective curve. Fix an integer $k > 0$. Fix $R \in Pic(X)$ such that $h^1(X, R) = 0$ and assume $K_{h^0(X,R)-k,1}(X, R) = 0$. Let $E \subset X_{reg}$ be a zero-dimensional scheme. Then $K_{h^0(X,R(E))-k,1}(X, R(E)) = 0$ and $h^1(X, R(E)) = 0$.*

**Lemma 7.** *Let $X$ be an integral projective curve. Set $g := p_a(X)$. Fix an integer $k > 0$. Assume the existence of $R \in Pic(X)$ such that $h^1(X, L) = 0$ and $K_{h^0(X,L)-k,1}(XL) = 0$. Fix an integer $d \geq \deg(L)+2g$. Then $K_{h^0(X,L)-k,1}(X, L) = 0$ for every $L \in Pic^d(X)$.*

*Proof.* Fix $L \in \mathrm{Pic}^d(X)$. Since $d - \deg(R) \geq 2g$, the line bundle $L \otimes R^*$ is spanned. Hence the zero-locus of a general section of $L \otimes R^*$ is an effective divisor $E \subset X_{reg}$. Apply Lemma 6. □

*Proof of Proposition 1.* Set $q :=$ and $z :=$. Take $Y$ as in the statement of Lemma 2 and let $X$ be the partial normalization of $Y$ in which we normalize only the points of $S$. Part (ii) follows from the proof of the case $e = 0$ of [4]. Lemma 7 shows that part (ii) implies part (iii) and hence that $X$ satisfies the Green-Lazarsfeld conjecture. □

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