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ON THE EXISTENCE OF DEGREE g SPANNED RANK 1 SHEAVES ON INTEGRAL CURVES AND ON BINARY CURVES

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Abstract: Let X be either an integral projective curve curve with only ordinary nodes or ordinary cusps (resp. a binary curve). Set $g := p_a(X)$. Here we prove the existence on X of degree g spanned rank 1 torsion free sheaves (resp. ω_X-semistable line bundles).

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1. Introduction

Let X be an integral projective curve. For all integers r, d let $W^r_d(X)$ denote the set of all rank 1 torsion free sheaves F on X such that $\deg(F) = d$ and $h^0(X, F) \geq r+1$. Now we drop the assumption that X is integral, but we assume that X is stable. In this case let $W^r_d(X)$ be the set of all depth 1 sheaves on X with pure rank 1 which are ω_X-semistable. There is a bijection between $W^r_d(X)$ and the set of all balanced line bundles of degree d with at least $r+1$ linearly independent section on the quasi-stable curves with X as their stable model (see [3], [9], Theorem 10.3.1, [4], [5]). Set $g := p_a(X)$. For any scheme A let $\dim(A)$ denote the maximal dimension of one of the irreducible components of A_{red} with the convention $\dim(\emptyset) = -1$. The set $W^r_d(X)$ has a structure an algebraic schemes, but we will only consider their reduced structure. Here we prove the following characterizations of "hyperelliptic curves", which are classical for

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smooth curves. We first define the words "hyperelliptic curves". Let X be an integral projective curve. We recall that X is called *honestly hyperelliptic* if there is a degree 2 morphism $X \to \mathbb{P}^1$. If X is honestly hyperelliptic, then it is Gorenstein. A Gorenstein integral curve X is honestly hyperelliptic if and only if there is a rank 1 torsion free sheaf F on X such that $\deg(F) = 2$ and $h^0(X, F) \geq 2$; moreover, if $X \neq \mathbb{P}^1$, then for every degree 2 torsion free sheaf F on X such that $h^0(X, F) \geq 2$ we have $h^0(X, F) = 2$ and F is spanned and locally free. For the definition and study of the Brill-Noether theory of binary curves, see [4]. In [4], Lemma 15, the reader may find the equivalence of a few definition of hyperellipticity for binary curves.

Proposition 1. *Let X be an integral curve whose singularities are only ordinary nodes and ordinary cusps. Set $g := p_a(X)$ and assume $g \geq 2$. There is no spanned $F \in W^1_g(X)$ if and only if X is honestly hyperelliptic and g is odd.*

Theorem 1. *Let X be a binary curve of arithmetic genus $g \geq 3$. There is no spanned ω_X-semistable depth 1 sheaf with pure rank 1 and degree g if and only if X is honestly hyperelliptic and g is odd.*

2. The Proofs

For all integers $g, r, d)$ set $\rho(g, r, d) := g-(r+1)(g+r-d) = (r+1)d-rg-r(r+1)$ (the Brill-Noether number for g^r_d's on a genus g curve).

Remark 1. Let F be a rank 1 torsion free sheaf on X. Set $\mathrm{Sing}(F) = \{P \in X : F$ is not locally free at $P\}$. We have $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. Take any $P \in \mathrm{Sing}(F)$. The classification of depth 1 sheaves with pure rank 1 on an ordinary node (see [10], pp. 164–166) or an ordinary cusp (see [8]) shows that the germ F_P of F at P is an $\mathcal{O}_{X,P}$-module isomorphic to the maximal ideal $m_{X,P}$ of the local ring $\mathcal{O}_{X,P}$. Let $u : D \to X$ be the partial normalization of X in which we normalize only the points of $\mathrm{Sing}(F)$. We have $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X) + \sharp(\mathrm{Sing}(F))$. Set $L := u^*(F)/\mathrm{Tors}(u^*(F))$. Since F is locally free outside $\mathrm{Sing}(X)$, D is smooth at each point of $u^{-1}(D)$ and L has no torsion, L is locally free, i.e. $L \in \mathrm{Pic}(X)$. Since for every $P \in \mathrm{Sing}(X)$ the germ F_P of F at P is an $\mathcal{O}_{X,P}$-module isomorphic to the maximal ideal $m_{X,P}$, we see that the natural map $u_*(L) \to F$ is an isomorphism. Since u is finite, we get $h^i(D, L) = h^i(X, F)$, $i = 0, 1$. Hence Riemann-Roch on D and on Y gives $\deg(L) = \deg(F) - \sharp(\mathrm{Sing}(F))$. Since the tensor product is a right exact functor we also see that L is spanned if F is spanned.

Remark 2. Let Γ be any irreducible component of $W^r_d(X)$. Assume $r \ge d-g$. It is easy to check that $h^0(X,F) = r+1$ for a general $F \in \Gamma$.

Remark 3. Let X be an integral curve of genus g. For all integers r, d such that $r \ge 0$ and $d \le g+r$, aand every irreducible component G of $W^r_d(X)$ has a general member F with $h^0(X,F) = r+1$. If X is smoothable, then semicontinuity gives $W^r_d(X) \ne \emptyset$ if $\rho(g,r,d) \ge 0$ and that every irreducible component of $W^r_d(X)$ has dimension at least $\rho(g,r,d)$. If X is not integral, but it is stable, then the same is true with the same semicontinuity and non-emptiness proof (see [3], [4], [5]). In both cases it is essential to consider also the non-locally free sheaves.

We need the following well-known lemma.

Lemma 1. *Let X be an integral honestly hyperelliptic curve of genus $g \ge 2$. Let $R \in Pic^2(X)$ be the degree 2 spanned line bundle. Let F be any rank 1 torsion free sheaf on X such that $a := h^0(X,F) > 0$ and $h^1(X,F) > 0$. Let G be the subsheaf of F spanned by $H^0(X,F)$. Then $G \cong R^{\otimes(a-1)}$.*

Remark 4. Let Y be a reduced projective curve with only planar singularities. Then for every integer $b > 0$ the Hilbert scheme of all length b zero-dmensional subschemes of Y has dimensoion b (see [2]).

Proof of Proposition 1. The "if" part follows from Lemma 1. We only check the "only if" part. Assume that no $F \in W^1_g(X)$ is spanned. Remark 3 gives $\dim(W^1_g(X)) \ge \rho(g,1,g) = g-2$. Fix an irreducible component A of $W^1_g(X)$ such that $\dim(A) \ge g-2$. Since no $F \in W^1_g(X)$ is spanned, there is an integer $b > 0$ and a non-empty open subset A' of A such that for every $F \in A'$ the subsheaf G of F spanned by $H^0(X,F)$ has degree $g-b$. Remark 4 gives the existence of a non-empty open subset B of an irreducible component of $W^1_{g-b}(X)$ formed by these sheaves B and with the property $\dim(B) \ge \dim(A) - b \ge g-2-b$. Restricting if necessay B we may assume that the integer $c := \sharp(\mathrm{Sing}(G)) \ge 0$ is constant for all $G \in B$. Since $\mathrm{Sing}(X)$ is finite, we get the existence of $S \subseteq \mathrm{Sing}(X)$ such that $\sharp(S) = c$ and $\mathrm{Sing}(G) = S$ for all $G \in B$. Let $u : Y \to X$ be the partial normalization of X in which we only normalize the points of S. For any $G \in B$ set $L_G := u^*(G)/\mathrm{Tors}(G)$. Remark 1 gives that L_G is a degree $g-b-c$ line bundle and that $u_*(L_G) \cong G$, i.e. L_G uniquely determines G. Thus the family $B' := \{L_G\}_{G \in B}$ is a $\dim(B)$ algebraic family of degree $g-c-b$ line bundles on Y. Since $h^0(Y,L_G) = h^0(X,G)$ for all $G \in B$ (Remark 1), $B' \subseteq W^1_{g-c-b}(Y)$. Each $L_G \in B'$ is spanned (Remark 1). Y is an integral curve of genus $g-c$ with only ordinary nodes and ordinary cusps as singularities. If either $g-c=0$ or $g-c=1$, then the inequality $b > 0$ gives a contradiction.

Hence we may assume $g-c \ge 2$. Since each $L \in B'$ is a spanned line bundle, we may apply to it the Petri map $\mu_L : H^0(Y,L)\otimes H^0(Y,\omega_Y\otimes L^*) \to H^0(Y,\omega_Y)$ (the trouble for a singular curve Y is that if a line bundle is not spanned, that its subsheaf spanned by $H^0(Y,L)$ may be non-locally free). Hence the proof of H. Martens Theorem given in [1], pp. 191–192, gives the result for Y, i.e. Y is honestly hyperelliptic. Lemma 1, the first part of Remark 3 and the spannedness of each $L_G \in B'$ gives $g-c-b=2$ and that B' is formed by a unique element. Thus $\dim(A) \le \dim(B')+b = g-2-c$. Thus $c=0$, i.e. $Y=X$. Hence X is honestly hyperelliptic. □

Proof of Theorem 1. For the Brill-Noether theory of honestly hyperelliptic (or, equivalently, hyperelliptic or weakly hyperelliptic) binary curves, see [4], §3; for the non-locally free sheaves use induction on g and part (iii) of [4], Lemma 15. Now assume that X is not hyperelliptic and that there is no spanned $F \in W^1_g(X)$. Remark 3 gives $\dim(W^1_g(X)) \ge \rho(g,1,g-1) = g-2$. Fix an irreducible component A of $W^1_g(X)$ such that $\dim(A) \ge g-2$. Hence there is an integer $b>0$ and a non-empty open subset A' of A such that for every $F \in A'$ the subsheaf G of F spanned by $H^0(X,F)$ has degree $g-1-b$. Remark 4 gives the existence of a non-empty open subset B of an irreducible component of $W^1_{g-1-b}(X)$ formed by these sheaves B and with the property $\dim(B) \ge \dim(A)-b \ge g-2-b$. Restricting if necessary B we may assume that the integer $c := \sharp(\mathrm{Sing}(G)) \ge 0$ is constant for all $G \in B$.

First assume $c=0$. Since B is irreducible, the bidegree (d_1,d_2) of all $G \in B$ is the same. We have $d_1+d_2 = g-1-b < g$. Fix any $G \in B$ and call $F \in A$ the associated sheaf. Set $x := \sharp(\mathrm{Sing}(F))$ and let (u_1,u_2) the bidegree of the associated line bundle L_F on the binary curve X_F obtained normalizing only the points of $\mathrm{Sing}(F)$. Since the sheaf F is ω_X-semistable, L_F is semibalanced (use [9], Theorem 10.3.1). Hence $|u_1-u_2| \le p_a(X_F)+1 = g+1-x$. We have $u_1+u_2 = g-x$. Since G is a locally free subsheaf of F, the support of F/G contains $\mathrm{Sing}(F)$. Thus $b \ge x$. We also get $d_1 \le u_1$ and $d_2 \le u_2$. Since $h^0(X_F,L_F) = h^0(X,G) \ge 2$ and $|u_1-u_2| \le p_a(X_F)+1$, [4], ==, applied to X_F gives $u_1 \ge 0$ and $u_2 \ge 0$. Thus $u_i \le g-x$ for all i. We get $d_1 \ge 0$ and $d_2 \ge 0$. Since $d_1+d_2 \le g+1$, every element of B is balanced (see [4], Definition 2). Since X is not hyperelliptic, a strong form of H. Martens theorem works for line bundles (see [4], Proposition 22). Hence $\dim(B) \le g-3-b$, contradiction.

From now on we assume $c>0$. Since $\mathrm{Sing}(X)$ is finite, we get the existence of $S \subseteq \mathrm{Sing}(X)$ such that $\sharp(S)=c$ and $\mathrm{Sing}(G)=S$ for all $G \in B$. Let $u : Y \to X$ be the partial normalization of X in which we only normalize the points of S. For any $G \in B$ set $L_G := u^*(G)/\mathrm{Tors}(u^*(G))$. Remark 1 gives that L_G is a degree $g-1-b-c$ line bundle and that $u_*(L_G) \cong G$, i.e. L_G uniquely determines G. Thus the family $B' := \{L_G\}_{G \in B}$ is a $\dim(B)$ algebraic family of degre $g-1-c-b$ line bundles on Y. Since $h^0(Y, L_G) = h^0(X, G)$ for all $G \in B$ (Remark 1), $B' \subseteq W^1_{g-1-c-b}(Y)$. Each $L_G \in B'$ is spanned (Remark 1). Since $h^0(X, G) > 0$ for every $G \in B$, we have $c \le g-2-b$. Thus $g - c \ge b+2 \ge 3$. Hence Y is a binary curve of genus $g-c \ge b+2$. As in the first part we see that the bidegree of each L_F is non-negative and that each L_F is semibalanced. Since $c > 0$ and $b > 0$ we get $\dim(W^1_{g-1-c-b}(Y)) \le g-3$ even if Y is hyperelliptic. □

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