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GROUP DIVISIBLE DESIGNS WITH TWO ASSOCIATE CLASSES AND $\lambda_1 - \lambda_2 = 1$

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Abstract: A *group divisible design* GDD$(v, g, k, \lambda_1, \lambda_2)$ is a collection of k-subsets (called blocks) of a v-set of symbols where: the v-set is divided into g groups; each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair symbols from different groups occurs in exactly λ_2 blocks. Pairs of symbols occurring in the same group are known to statisticians as *first associates*, and pairs occurring in different groups are called *second associates*. The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs. Recently, such an existence problem when $g = 2$ was solved in the case where the groups have the same size and the blocks have size 3. In this paper, we continue to focus on blocks of size 3, solving the problem when the required designs having two groups of unequal sizes and $\lambda_1 - \lambda_2 = 1$ and prove that the conditions are sufficient.

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1. Introduction

A *pairwise balanced design* is an ordered pair $(S, \mathcal{B})$, denoted PBD$(S, \mathcal{B})$, where S is a finite set of symbols and $\mathcal{B}$ is a collection of subsets of S called *blocks*,

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such that each pair of distinct elements of S occurs together in exactly one block of $\mathcal{B}$. Here $|S| = v$ is called the *order* of the PBD. Note that there is no condition on the size of the blocks in $\mathcal{B}$. If all blocks are of the same size k, then we have a *Steiner system* $\mathrm{S}(v,k)$. A PBD with index λ can be defined similarly: each pair of distinct elements occurs in λ blocks. If all blocks are of the same size, say k, then we get a balanced incomplete block design BIBD(v,b,r,k,λ). In other words, a BIBD(v,b,r,k,λ) is a set S of v elements together with a collection of b k-subsets of S, called blocks, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks (see [1], [2], [3]).

Note that in a BIBD(v,b,r,k,λ) the parameters must satisfy the necessary conditions:

1. $vr = bk$ and
2. $\lambda(v-1) = r(k-1)$.

With these conditions a BIBD(v,b,r,k,λ) is usually written as BIBD(v,k,λ).

A *group divisible design* GDD$(v = v_1+v_2+\cdots+v_g, g, k, \lambda_1, \lambda_2)$ is a collection of k-subsets (called blocks) of a v-set of symbols, where the v-set is divided into g groups of size $v_1, v_2, \ldots, v_g$; each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks (see [1], [2]).

In this paper we consider the problem of determining necessary conditions for an existence of GDD$(v = m+n, 2, 3, \lambda_1, \lambda_2)$ and prove that the conditions are sufficient for some infinite families. Since we are dealing on GDDs with two groups and block of size 3, we will use GDD$(m,n,\lambda_1,\lambda_2)$ for GDD$(v = m+n, 2, 3, \lambda_1, \lambda_2)$ from now on, and we refer to the blocks as triples. We denote $(X;Y,\mathcal{B})$ for a GDD$(m,n,\lambda_1,\lambda_2)$ if X and Y are m-set and n-set, respectively. Punnim and Sarvate [5] have written the first paper in this direction. In particular they have completely solved the problem of determining all pairs of integers (n,λ) in which a GDD$(1,n,1,\lambda)$ exists. We continue to investigate in this paper all triples of integers (m,n,λ) in which a GDD$(m,n,\lambda,\lambda-1)$ exists. We will see that necessary conditions on the existence of a GDD$(m,n,\lambda_1,\lambda_2)$ can be easily obtained by describing it graphically as follows. Let λK_v denote the graph on v vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A *G-decomposition* of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Thus the existence of a GDD$(m,n,\lambda_1,\lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. The graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ is of order $m+n$ and size $\lambda_1[\binom{m}{2}+\binom{n}{2}]+\lambda_2 mn$. It contains m vertices of degree $\lambda_1(m-1)+\lambda_2 n$ and n vertices of degree $\lambda_1(n-1)+\lambda_2 m$. Thus the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ implies:

1. $3 \mid \lambda_1[\binom{m}{2}+\binom{n}{2}]+\lambda_2 mn$, and
2. $2 \mid \lambda_1(m-1)+\lambda_2 n$ and $2 \mid \lambda_1(n-1)+\lambda_2 m$.

2. Preliminary

We will review some known results concerning triple designs that will be used in the sequel, most of which are taken from [3].

Theorem 2.1. *Let v be a positive integer. Then there exists a* BIBD$(v, 3, 1)$ *if and only if* $v \equiv 1$ *or* $3 \pmod 6$.

A BIBD$(v, 3, 1)$ is usually called *Steiner triple system* and is denoted by STS(v). Let $(V, \mathcal{B})$ be an STS(v). Then the number of triples $b = |\mathcal{B}| = v(v-1)/6$. A *parallel class* in an STS(v) is a set of disjoint triples whose union is the set V. A parallel class contains $v/3$ triples, and hence an STS(v) having a parallel class can exist only when $v \equiv 3 \pmod 6$. When the set $\mathcal{B}$ can be partitioned into parallel classes, such a partition $\mathcal{R}$ is called a *resolution* of the STS(v), and the STS(v) is called *resolvable*. If $(V, \mathcal{B})$ is an STS(v) and $\mathcal{R}$ is a resolution of it, then $(V, \mathcal{B}, \mathcal{R})$ is called a *Kirkman triple system*, denoted by KTS(v), with $(V, \mathcal{B})$ as its *underlying* STS. It is well known that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$. Thus if $(V, \mathcal{B}, \mathcal{R})$ is a KTS(v), then $\mathcal{R}$ contains $(v-1)/2$ parallel classes.

The following results on existence of λ-fold triple systems are well known (see, e.g. [3]).

Theorem 2.2. *Let n be a positive integer. Then a* BIBD$(n, 3, \lambda)$ *exists if and only if λ and n are in one of the following cases.*

(a) $\lambda \equiv 0 \pmod 6$ *and for all positive integers* $n \neq 2$,

(b) $\lambda \equiv 1$ *or* $5 \pmod 6$ *and for all n with* $n \equiv 1$ *or* $3 \pmod 6$,

(c) $\lambda \equiv 2$ *or* $4 \pmod 6$ *and for all n with* $n \equiv 0$ *or* $1 \pmod 3$, *and*

(d) $\lambda \equiv 3 \pmod 6$ *and for all odd integers n.*

The following notation will be used for our constructions.

1. Let $T = \{x, y, z\}$ be a triple and $a \notin T$. We use $a * T$ for three triples of the form $\{a, x, y\}, \{a, x, z\}, \{a, y, z\}$. If $\mathcal{T}$ is a set of triples, then $a * \mathcal{T}$ is defined

as $\{a * T : T \in \mathcal{T}\}$.

2. Let V be a v-set. Then there may be many different STS(v)s that can be constructed on the set V. Let STS(V) be defined as

$$\text{STS}(V) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is an STS}(v)\}.$$

KTS(V) and BIBD($V, 3, \lambda$) can be defined similarly, That is:

$$\text{KTS}(V) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is a KTS}(v)\}, \text{ and}$$

$$\text{BIBD}(V, 3, \lambda) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is a BIBD}(v, 3, \lambda)\}.$$

Let X and Y be disjoint sets of cardinality m and n, respectively. We define GDD($X, Y, \lambda_1, \lambda_2$) as

$$\text{GDD}(X, Y, \lambda_1, \lambda_2) = \{\mathcal{B} : (X; Y, \mathcal{B}) \text{ is a GDD}(m, n, \lambda_1, \lambda_2)\}.$$

3. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus "$\cup$" in our construction will be used for the union of multi-sets.

3. GDD($m, n, \lambda, \lambda - 1$)

Let λ be a positive integer. We consider in this section the problem of determining all pairs of integers (m, n) in which a GDD($m, n, \lambda, \lambda - 1$) exists. Recall that the existence of GDD($m, n, \lambda, \lambda - 1$) implies $3 \mid \lambda[m(m-1) + n(n-1)] + 2(\lambda - 1)mn$, $2 \mid \lambda(m-1) + (\lambda - 1)n$ and $2 \mid \lambda(n-1) + (\lambda - 1)m$.

Let λ_1 and λ_2 be positive integers. Define

$$S(\lambda_1, \lambda_2) := \{(m, n) : \text{ a GDD}(m, n, \lambda_1, \lambda_2) \text{ exists}\}.$$

The next lemma summarizes the necessary conditions for an existence of a GDD($m, n, \lambda, \lambda - 1$).

Lemma 3.1. *Let t be a non-negative integer:*

(a) If $(m, n) \in S(6t + 2, 6t + 1)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k + 6, 6h + 6\}, \{6k + 4, 6h + 6\}\}$.

(b) If $(m, n) \in S(6t + 3, 6t + 2)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k + 1, 6h + 3\}, \{6k + 3, 6h + 3\}, \{6h + 3, 6k + 5\}\}$.

(c) If $(m, n) \in S(6t + 4, 6t + 3)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k + 6, 6h + 6\}, \{6k + 4, 6h + 6\}\}$.

(d) If $(m, n) \in S(6t + 5, 6t + 4)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k + 1, 6h + 3\}, \{6k + 3, 6h + 3\}\}$.

(e) If $(m, n) \in S(6t + 6, 6t + 5)$, then there exist non-negative integers h and

k such that $\{m,n\} \in \{\{6k+6, 6h+6\}, \{6k+2, 6h+6\}, \{6k+4, 6h+6\}\}$.

(f) If $(m,n) \in S(6t+7, 6t+6)$, *then there exist non-negative integers h and k such that* $\{m,n\} \in \{\{6k+1, 6h+1\}, \{6k+1, 6h+3\}, \{6k+3, 6h+3\}\}$.

Proof. The proof follows from solving the corresponding systems of congruences. □

Before proving the main result we shall need some auxiliary lemmas.

Lemma 3.2. *Let h and k be non-negative integers. Then:*

(a) $(6k+6, 6h+6) \in S(2,1)$, *and*

(b) $(6k+4, 6h+6), (6k+6, 6h+4) \in S(2,1)$.

Proof. The proofs have been shown in [4]. □

Let m and n be positive integers with $m+n \geq 3$. Then the existence of a BIBD$(m+n, 3, \lambda)$ is equivalence to the existence of a GDD(m, n, λ, λ). Thus we have the following results.

Lemma 3.3. *Let h and k be non-negative integers. Then:*

(a) $(6k+1, 6h+3), (6k+3, 6h+1), (6k+3, 6h+3) \in S(2,2)$,

(b) $(6k+1, 6h+3), (6k+3, 6h+1), (6k+3, 6h+3) \in S(4,4)$, *and*

(c) $(6k+1, 6h+3), (6k+3, 6h+1), (6k+3, 6h+3) \in S(6,6)$.

Lemma 3.4. *Let h and k be non-negative integers. Then:*

(a) $(6k+1, 6h+3), (6k+3, 6h+1) \in S(3,2)$,

(b) $(6k+3, 6h+3) \in S(3,2)$, *and*

(c) $(6k+3, 6h+5), (6k+5, 6h+3) \in S(3,2)$.

Proof. The proofs of (a) and (b) follow from the results of Lemma 3.3(a) and Theorem 2.2(b).

(c) Let X and Y be two sets of size $6k+3$ and $6h+5$, respectively, $a \in Y$ and put $Y' = Y - \{a\}$. Let $\mathcal{B}_1 \in$ KTS(X) with $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3k+1}$ as its parallel classes. It was shown in [5] that GDD$(\{a\}; Y', 1, 3) \neq \emptyset$. Choose $B_2 \in$ GDD$(\{a\}; Y', 1, 3)$ and $\mathcal{B}_3 \in$ BIBD$(X \cup Y', 3, 2)$. We now put $\mathcal{B}$ as

$$\mathcal{B}_2 \cup \mathcal{B}_3 \cup (a * \mathcal{P}_1) \cup (\bigcup_{i=2}^{3k+1} \mathcal{P}_i).$$

Thus $(X; Y, \mathcal{B})$ forms a GDD$(6k+3, 6h+5, 3, 2)$ and $(6k+3, 6h+5) \in S(3,2)$. Therefore the proof is complete. □

Lemma 3.5. *Let h and k be non-negative integers. Then:*

(a) $(6k+6, 6h+6) \in S(4,3)$, *and*

(b) $(6k+4, 6h+6), (6k+6, 6h+4) \in S(4,3)$.

Proof. Since the proofs for the two parts (a) and (b) are conceptually the same, we only provide that of part (b).

Let X and Y be two sets of size $6k+4$ and $6h+6$, respectively. By Lemma 3.2(b), we have $\mathrm{GDD}(X, Y, 2, 1) \neq \emptyset$ and let $B_1 \in \mathrm{GDD}(X, Y, 2, 1)$. Since $X \cup Y$ is a set of size $6j+4$, it follows from Theorem 2.2(c), we have $\mathrm{BIBD}(X \cup Y, 3, 2) \neq \emptyset$. We now choose $B_2 \in \mathrm{BIBD}(X \cup Y, 3, 2)$ and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then $(X; Y, \mathcal{B})$ forms a $\mathrm{GDD}(6k+4, 6h+6, 4, 3)$ and $(6k+4, 6h+6) \in S(4,3)$. □

Lemma 3.6. *Let h and k be non-negative integers. Then*

$$(6k+1, 6h+3), (6k+3, 6h+1), (6k+3, 6h+3) \in S(5,4).$$

Proof. The proofs follow from the results of Lemma 3.3(b) and Theorem 2.2(b). □

Lemma 3.7. *Let h and k be non-negative integers. Then:*

(a) $(6k+2, 6h+6) \in S(6,5)$,

(b) $(6k+4, 6h+6) \in S(6,5)$, *and*

(c) $(6k+6, 6h+6) \in S(6,5)$.

Proof. (a) Let X and Y be two sets of size $6k+2$ and $6h+6$, respectively, $a \in X$ and put $X' = X - \{a\}$. It was shown in [5] that $\mathrm{GDD}(\{a\}, Y, 1, 5)$ and $\mathrm{GDD}(\{a\}, X', 1, 6)$ are not empty and let $B_1 \in \mathrm{GDD}(\{a\}, Y, 1, 5), B_2 \in \mathrm{GDD}(\{a\}, X', 1, 6)$. Since $X' \cup Y$ is a set of size $6j+1$, it follows from Theorem 2.2(b), we have $\mathrm{BIBD}(X' \cup Y, 3, 5) \neq \emptyset$. We now choose $B_2 \in \mathrm{BIBD}(X' \cup Y, 3, 5)$ and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then $(X; Y, \mathcal{B})$ forms a $\mathrm{GDD}(6k+2, 6h+6, 6, 5)$ and $(6k+2, 6h+6) \in S(6,5)$.

Since the proofs for the two parts (b) and (c) are conceptually the same, we only provide that of part (b).

Let X and Y be two sets of size $6k+4$ and $6h+6$, respectively. By Lemma 3.2(b), we have $\mathrm{GDD}(X, Y, 2, 1) \neq \emptyset$ and let $B_1 \in \mathrm{GDD}(X, Y, 2, 1)$. Since $X \cup Y$ is a set of size $6j+1$, it follows from Theorem 2.2(c), we have $\mathrm{BIBD}(X \cup Y, 3, 4) \neq \emptyset$. We now choose $B_2 \in \mathrm{BIBD}(X \cup Y, 3, 4)$ and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then $(X; Y, \mathcal{B})$ forms a $\mathrm{GDD}(6k+4, 6h+6, 6, 5)$ and $(6k+4, 6h+6) \in S(6,5)$. □

Lemma 3.8. *Let h and k be non-negative integers. Then:*

(a) $(6k+1, 6h+1) \in S(7,6)$ *with* $6k+1+6h+1 \geq 3$,

(b) $(6k+1, 6h+3), (6k+3, 6h+1) \in S(7,6)$, *and*

(c) $(6k+3, 6h+3) \in S(7,6)$.

Proof. We only give a proof for part(a) as those of the other two parts are quite similar.

Let X and Y be two sets of size $6k+1$ and $6h+1$, respectively and $6k+1+6h+1 \geq 3$. Since $X \cup Y$ is a set of size $6j+2 \geq 3$, it follows from Theorem 2.2(a), we have $\mathrm{BIBD}(X \cup Y, 3, 6) \neq \emptyset$. We now choose $B_1 \in \mathrm{BIBD}(X \cup Y, 3, 6), B_2 \in \mathrm{BIBD}(X, 3, 1), B_3 \in \mathrm{BIBD}(Y, 3, 1)$ and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $(X; Y, \mathcal{B})$ forms a $\mathrm{GDD}(6k+1, 6h+1, 7, 6)$ and $(6k+1, 6h+1) \in S(7,6)$. □

Combining the results of Lemmas 3.1, 3.2, 3.4-3.8 and Theorem 2.2(a), we now have the main result:

Theorem 3.9. *Let m and n be positive integers with $m+n \geq 3, (m \neq 2$ and $n \neq 2)$. There exists a* $\mathrm{GDD}(m, n, \lambda, \lambda-1)$ *if and only if:*

1. $3 \mid \lambda[m(m-1) + n(n-1)] + 2(\lambda-1)mn$, *and*

2. $2 \mid \lambda(m-1) + (\lambda-1)n$ *and* $2 \mid \lambda(n-1) + (\lambda-1)m$.

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