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PLANE STRESS POLYCRYSTAL PLASTICITY AS A LIMITING CASE OF THE POWER-LAW MODEL VIA Γ-CONVERGENCE

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Abstract: A model problem in polycrystal plasticity involving plane stress is considered. A variational principle which characterizes the yield set of the polycrystal is obtained as a limiting case of variational principles associated to a class of power-law functionals, via Γ-convergence.

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1. Introduction

Yield in a crystalline solid is determined by a finite number of slip systems, each being defined by a pair of orthogonal vectors (n_k, m_k), where n_k is the normal to the slip plane, and m_k is the direction of slip. The yield set of the crystal has the form

$$K = \left\{ A \in \mathbb{M}^{3\times 3}_{\text{sym}} \ : \ \langle A, \mu_k \rangle \leq \tau_k^{\text{c}}, k = 1, \cdots, s \right\},$$

which is a closed convex subset of the space of symmetric 3×3 real matrices $\mathbb{M}^{3\times 3}_{\text{sym}}$. Here s stands for the number of slip systems, τ_k^{c} is the critical shear stress for the k-th slip system, and

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$$\mu_k := \frac{1}{2}\left(m_k \otimes n_k + n_k \otimes m_k\right)$$

is the k-th slip tensor.

A polycrystal is a collection of grains (single crystals) which are bonded together in different orientations. The texture of the polycrystal, consisting of the shapes and orientations of the grains, is described by means of a rotation-valued function $R : \Omega \to \mathrm{SO}(3)$ which is constant in each grain, with $R(x)$ indicating the orientation of the grain which contains the point $x \in \Omega$. The stress in the polycrystal occupying the region $\Omega \subset \mathbb{R}^3$ must satisfy the constraint

$$\sigma(x) \in R(x) K R^T(x),\ \ x \in \Omega, \tag{1.1}$$

where K is the yield set of the basic crystal. The set of all average stresses $\overline{\sigma} := \fint_\Omega \sigma(x)dx$ where σ satisfies the pointwise constraint (1.1) and the equilibrium equation

$$\mathrm{Div}\ \sigma = 0 \text{ in } \Omega, \tag{1.2}$$

is called the effective yield set of the polycrystal. We write

$$K_{\mathrm{eff}} := \left\{ \overline{\sigma} := \fint_\Omega \sigma(x)dx\ \ :\ \ (1.1) \text{ and } (1.2) \text{ hold} \right\}.$$

The central problem in polycrystal plasticity is to describe the yield set K_{eff}, given K and some information on the texture of the polycrystal. To overcome some of the difficulties of the problem alternative schemes have been introduced in order to estimate the macroscopic response of polycrystals without solving the equilibrium equation (1.2) directly. Natural inner and outer estimates (the so-called Sachs and Bishop-Hill-Taylor bounds; see [1], [15], [16]) for the effective yield set and their optimality have been studied by many authors. We refer to Garroni et al [9], Garroni et al [10], Goldsztein [11], [12], Kohn et al [14] for recent work on the subject. In [10] the authors introduce natural variational principles that can be used to characterize the yield set of a polycrystal in the simpler context of (first failure) dielectric breakdown, which corresponds to replacing the equilibrium equation (1.2) with the differential constraint that the underlying fields are gradients (curl-free instead of divergence-free), and where (1.1) consists of a single pointwise constraint. This is achieved by means of an efficient mathematical derivation of the (first-failure) dielectric breakdown model as a limiting case of the power-law model via De Giorgi's Γ-convergence. The advantage of this new approach is two-fold: first, it provides a new rigorous justification, via Γ-convergence, of the dielectric breakdown model (the traditional one uses convex duality; see, e.g., Jikov et al [13]); on the other hand, it leads to a new variational principle for the effective yield set in the conductivity setting which is less degenerate than the traditional one. For recent generalizations of these results to the framework of electrical resistivity we refer to Bocea et al [2].

In this paper we extend the ideas in [10] concerning the derivation of the traditional dielectric breakdown model to the case where the underlying fields take values in stress space, are divergence-free (as opposed to being gradient fields) and where, depending on the number of slip systems present in the basic crystal, (1.1) consists of several distinct pointwise constraints that need to be simultaneously satisfied. This more general setting is appropriate for handling realistic models of polycrystal plasticity previously considered in the literature. To simplify the exposition we will restrict our attention to the two-dimensional model of plane stress, but our results can be easily adapted to other classical models, such as, for example, antiplane shear (see, e.g., Kohn et al [14]). In the plane stress model the stresses are given by

$$\sigma(x) = \begin{pmatrix} \sigma_{11}(x) & \sigma_{12}(x) & 0 \\ \sigma_{12}(x) & \sigma_{22}(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ x \in \Omega \subset \mathbb{R}^2. \tag{1.3}$$

The slip tensor is $\mu = \frac{1}{2}(m \otimes n + n \otimes m)$, where $m \perp n, \mu_{13} = \mu_{23} = \mu_{33} = 0$, and it belongs to the two-dimensional space spanned by $\mu^{(1)} = \frac{1}{2}\left(m^{(1)} \otimes n^{(1)} + n^{(1)} \otimes m^{(1)}\right)$ and $\mu^{(2)} = \frac{1}{2}\left(m^{(2)} \otimes n^{(2)} + n^{(2)} \otimes m^{(2)}\right)$, with $m^{(1)} = (1,0,0)$, $n^{(1)} = (0,1,0), m^{(2)} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, and $m^{(2)} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$. In what follows we identify the stress at $x \in \Omega$ with the 2×2 upper-left corner of the 3×3 matrix in (1.3). The yield set of the basic crystal has the form

$$K = K_{MN} = \left\{\sigma = (\sigma_{ij}) \in \mathbb{M}^{2\times 2}_{\text{sym}} \ : \ |\sigma_{12}| \le M, |\sigma_{11} - \sigma_{22}| \le 2, \ |\sigma_{11} + \sigma_{22}| \le N\right\}.$$

Let

$$R(x) = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \tag{1.4}$$

be the rotation describing the orientations of the grain containing the point $x \in \Omega$. The pointwise constraint (1.1) then becomes

$$\begin{pmatrix} \sigma_{11}\cos^2\theta + \sigma_{12}\sin 2\theta + \sigma_{22}\sin^2\theta & \sigma_{12}\cos 2\theta - \frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta \\ \sigma_{12}\cos 2\theta - \frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta & \sigma_{11}\sin^2\theta(x) - \sigma_{12}\sin 2\theta + \sigma_{22}\cos^2\theta \end{pmatrix}(x) \in K_{MN}$$

and thus it can be written in the form

$$\sigma(x) \in \left\{\eta = (\eta_{ij}) \in \mathbb{M}^{2\times 2}_{\text{sym}} \ : \left|\eta_{12}\cos(2\theta(x)) - \frac{1}{2}(\eta_{11} - \eta_{22})\sin(2\theta(x))\right| \tag{1.5}$$

$$\leq M,\ |(\eta_{11}-\eta_{22})\cos(2\theta(x))+2\eta_{12}\sin(2\theta(x))|\leq 2, |\eta_{11}+\eta_{22}|\leq N\Big\}.$$

The plan of the paper is as follows. In Section 2 we give a very brief account on Γ-convergence and $\mathcal{A}$-quasiconvexity, and we state and prove a Γ-convergence result for a class of power-law functionals acting on divergence-free fields which are adapted to the plane stress setting. The last section of the paper is devoted to the characterization of the effective yield set of a polycrystal in terms of a variational principle associated to the Γ-limit of the sequence of power-law functionals considered in the previous section.

2. A Γ-Convergence Result

We start this section by recalling the definition of De Giorgi's Γ-convergence (see [6], [7]) in metric spaces. For a comprehensive introduction to the subject we refer to [5]. See also [3], and [4].

Definition 2.1. Let X be a metric space. A sequence $\{I_p\}$ of functionals $I_p : X \to \overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$ is said to $\Gamma(X)$-converge to $I : X \to \overline{\mathbb{R}}$ (we write $\Gamma(X) - \lim_{p\to\infty} I_p = I$) if:

(i) for every $u \in X$ and $\{u_p\} \subset X$ such that $u_p \to u$ in X, we have

$$I(u) \leq \liminf_{p\to\infty} I_p(u_p);$$

(ii) for every $u \in X$ there exists a sequence $\{u_p\} \subset X$ such that $u_p \to u$ in X, and

$$I(u) = \lim_{p\to\infty} I_p(u_p).$$

Let Ω be an open, bounded domain in $\mathbb{R}^2$, and for $i = 1, 2, 3$, let $f_i : \Omega \times \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to [0, +\infty)$ be defined by

$$f_1(x,\eta) := \frac{1}{M}\left|\eta_{12}\cos(2\theta(x)) - \frac{1}{2}(\eta_{11}-\eta_{22})\sin(2\theta(x))\right|, \tag{2.1}$$

$$f_2(x,\eta) := \frac{1}{2}\left|(\eta_{11}-\eta_{22})\cos(2\theta(x)) + 2\eta_{12}\sin(2\theta(x))\right|, \tag{2.2}$$

and

$$f_3(x,\eta) := \frac{1}{N}\left|\eta_{11}+\eta_{22}\right|. \tag{2.3}$$

It is not difficult to check that there exists a constant $C > 0$ such that for every $i \in \{1, 2, 3\}$ we have

$$f_i(x,\eta) \leq C\,(1+|\eta|) \ \text{ for } \mathcal{L}^2\text{−a.e. } x \in \Omega, \text{ and all } \eta \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \tag{2.4}$$

and, in addition,

$$\sum_{i=1}^{3} f_i(x,\eta) \geq c|\eta| \text{ for } \mathcal{L}^2\text{–a.e. } x \in \Omega, \text{ and all } \eta \in \mathbb{M}^{2\times 2}_{\text{sym}}, \tag{2.5}$$

where $c > 0$ is a constant. In fact, one can take

$$c = \left(\max\{N, 2+\sqrt{2}, (2+\sqrt{2})M\}\right)^{-1}.$$

Define $I_p : L^1(\Omega; \mathbb{M}^{2\times 2}_{\text{sym}}) \to [0, +\infty]$ by

$$I_p(\sigma) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega \left(\sum_{i=1}^{3} f_i(x,\sigma(x))^p\right) dx & \text{if } \sigma \in L^p(\Omega; \mathbb{M}^{2\times 2}_{\text{sym}}) \text{ and Div } \sigma = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Before we can state the Γ-convergence result regarding the functionals I_p, we need to give a brief overview of $\mathcal{A}$-quasiconvexity, and to state a lower semicontinuity result due to Fonseca et al [8] which will play an important role in the sequel. We refer the reader to [8] for more details on these issues.

Let $N, d, l \in \mathbb{N}$ be given, $\Omega \subset \mathbb{R}^N$ open and bounded, and let $1 < p < \infty$. Consider a family of linear operators $A^{(1)}, A^{(2)}, \cdots, A^{(N)} \in \text{Lin}(\mathbb{R}^d; \mathbb{R}^l)$, and define the differential operator $\mathcal{A} : L^p(\Omega; \mathbb{R}^d) \to W^{-1,p}(\Omega; \mathbb{R}^l)$ by

$$\mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i}. \tag{2.6}$$

The basic assumption regarding the operator $\mathcal{A}$ is that it satisfies the following constant rank property: there exists $r \in \mathbb{N}$ such that

$$\text{rank}\,(\mathbb{A}(w)) = r \text{ for all } w = (w_1, \cdots, w_N) \in S^{N-1}, \tag{2.7}$$

where

$$\mathbb{A}(w) := \sum_{i=1}^{N} w_i A^{(i)} \in \text{Lin}(\mathbb{R}^d; \mathbb{R}^l).$$

A function $g : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-quasiconvex if

$$g(A) \leq \int_Q g(A + w(x))dx$$

for all $A \in \mathbb{R}^d$, and all Q-periodic $w \in C^\infty(Q; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_Q w(x)dx = 0$, where $Q = (0,1)^N$ is the unit cube in $\mathbb{R}^N$. In view of Jensen's inequality, it is immediate that convex functions are $\mathcal{A}$-quasiconvex. Fonseca and Müller have shown in [8] that if $\mathcal{A}$ satisfies constant rank property (2.7) then, under natural additional assumptions, $\mathcal{A}$-quasiconvexity of $g(x,u,\cdot)$ is a necessary and sufficient condition for the sequential lower semicontinuity of integral functionals of the form

$$(u,v) \mapsto \int_\Omega g(x,u(x),v(x))dx$$

along sequences such that $u_n \to u$ in measure, $v_n \to v$ in L^p, and $\mathcal{A}v_n \to 0$ in $W^{-1,p}$. In particular, the following holds.

Proposition 2.2. (see [8, Theorem 3.7]) *Let $1 \le p \le +\infty$ and suppose that $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,+\infty)$ is a normal integrand such that $z \mapsto g(x,u,z)$ is $\mathcal{A}$-quasiconvex and continuous for $\mathcal{L}^N$-a.e. $x \in \Omega$, and all $u \in \mathbb{R}^d$. If $1 \le p < +\infty$, assume further that there exists $a \in L^\infty_{\mathrm{loc}}\left(\Omega \times \mathbb{R}^d; [0,+\infty)\right)$ such that*

$$0 \le g(x,u,v) \le a(x,u)(1+|v|^p), \text{ for } \mathcal{L}^N-\text{a.e } x \in \Omega, \text{ and all } (u,v) \in \mathbb{R}^m \times \mathbb{R}^d.$$

If

$$u_n \to u \text{ in measure},$$
$$v_n \rightharpoonup v \text{ in } L^p(\Omega;\mathbb{R}^d), \tag{2.8}$$

and

$$\mathcal{A}v_n \to 0 \text{ in } W^{-1,p}(\Omega;\mathbb{R}^l) \tag{2.9}$$

then

$$\int_\Omega g(x,u(x),v(x))dx \le \liminf_{n\to\infty} \int_\Omega g(x,u_n(x),v_n(x))dx. \tag{2.10}$$

If $p = +\infty$, then (2.10) still holds provided that in (2.8) the weak convergence of v_n to v in $L^p(\Omega;\mathbb{R}^d)$ is replaced by the weak convergence in $L^\infty(\Omega;\mathbb{R}^d)$, and in (2.9) $\mathcal{A}v_n \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^l)$ is replaced by $\mathcal{A}v_n = 0$.*

We are now ready to state our Γ-convergence result.

Theorem 2.3. *Define $I_\infty : L^1(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}) \to [0,+\infty]$ by*

$$I_\infty(\sigma) := \begin{cases} 0 & \text{if } \operatorname{Div}\sigma = 0, \text{ and (1.5) holds for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Then:

(i) for every $\sigma \in L^1(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$, and $\{\sigma_p\} \subset L^1(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ such that $\sigma_p \rightharpoonup \sigma$ weakly in $L^1(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$, we have

$$I_\infty(\sigma) \le \liminf_{p\to\infty} I_p(\sigma_p). \tag{2.11}$$

(ii) for every $\sigma \in L^1(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$, there exists a sequence $\{\sigma_p\} \subset L^1(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$

such that $\sigma_p \to \sigma$ *strongly in* $L^1(\Omega;\mathbb{M}^{2\times 2}_{\rm sym})$*, and*

$$\limsup_{p\to\infty} I_p(\sigma_p) \le I_\infty(\sigma). \tag{2.12}$$

In particular,

$$\Gamma(L^1(\Omega;\mathbb{M}^{2\times 2}_{\rm sym})) - \lim_{p\to\infty} I_p = I_\infty.$$

Proof. Let $\{\sigma_p\} \subset L^1(\Omega;\mathbb{M}^{2\times 2}_{\rm sym})$ be such that $\sigma_p \rightharpoonup \sigma$ weakly in $L^1(\Omega;\mathbb{M}^{2\times 2}_{\rm sym})$. After passing to a subsequence (not relabelled), we may assume without loss of generality that we have

$$\sigma_p \in L^p(\Omega;\mathbb{M}^{2\times 2}_{\rm sym}),\ \mathrm{Div}\ \sigma_p = 0, \tag{2.13}$$

and

$$\liminf_{p\to\infty} I_p(\sigma_p) = \lim_{p\to\infty} I_p(\sigma_p) < +\infty. \tag{2.14}$$

Let $i \in \{1,2,3\}$ be fixed, and let $x \in \Omega$ be a Lebesgue point for $f_i(\cdot,\sigma(\cdot)) \in L^1(\Omega)$. For any ball $B(x,r) \subset \Omega$ and $p > 1$ sufficiently large, we have

$$\begin{aligned}\int_{B(x,r)} f_i(y,\sigma_p(y))dy &\le \left(\int_\Omega (f_i(y,\sigma_p(y)))^p\,dy\right)^{1/p} \left(\mathcal{L}^2(B(x,r))\right)^{(p-1)/p}\\ &\le (I_p(\sigma_p))^{1/p}\,p^{1/p}\left(\mathcal{L}^2(B(x,r))\right)^{(p-1)/p},\end{aligned} \tag{2.15}$$

where we have used Hölder's inequality. Letting $p \to \infty$, we obtain

$$\limsup_{p\to\infty} \int_{B(x,r)} f_i(y,\sigma_p(y))dy \le \mathcal{L}^2(B(x,r)). \tag{2.16}$$

Taking into account (2.4), and the fact that the functions $f_i(x,\cdot)$ defined by (2.1), (2.2), and (2.3) are convex for all $x \in \Omega$, we are in position to apply Proposition 2.2, and we deduce that

$$\int_{B(x,r)} f_i(y,\sigma(y))dy \le \liminf_{p\to\infty} \int_{B(x,r)} f_i(y,\sigma_p(y))dy. \tag{2.17}$$

We remark that Proposition 2.2 applies in the particular case where $N = l = 2, d = 4$, and for the differential operator $\mathcal{A}$ given by

$$\mathcal{A}\sigma := \mathrm{Div}\ \sigma = \begin{pmatrix} \mathrm{div}\sigma^{(1)} \\ \mathrm{div}\sigma^{(2)} \end{pmatrix},\ \sigma \in L^p\left(\Omega;\mathbb{M}^{2\times 2}\right),$$

where $\sigma^{(i)}(x) := (\sigma_{i1}(x),\sigma_{i2}(x))$ stands for the i-th row of the matrix $\sigma(x),\ x \in \Omega$ $(i = 1,2)$. It is not difficult to see that the differential constraint $\mathcal{A}\sigma = 0$ (see

(2.6)) may be written in the form

$$\sum_{k=1}^{2} A^{(k)} \frac{\partial \sigma}{\partial x_k} = 0$$

provided that we define, for $i, k = 1, 2$ and $j = 1, \cdots, 4$,

$$A_{ij}^{(k)} = \begin{cases} \delta_{i1} & \text{if } j = 2k-1 \\ \delta_{i2} & \text{if } j = 2k \\ 0 & \text{else,} \end{cases}$$

where the symbol δ_{ij} stands for the Kronecker's delta. We note that the constant rank condition (2.7) is satisfied since for every $w \in S^1$ we have

$$\ker(\mathbb{A}(w)) = \left\{ V \in \mathbb{M}^{2\times 2} \ : \ wV = 0 \right\},$$

and thus $\dim(\ker \mathbb{A}(w)) = 2$.

Combining (2.17) with (2.16), we have that

$$\frac{1}{\mathcal{L}^2(B(x,r))} \int_{B(x,r)} f_i(y, \sigma(y)) dy \le 1.$$

Since $\mathcal{L}^2$-almost every $x \in \Omega$ is a Lebesgue point for $f_i(\cdot, \sigma(\cdot))$, passing to the limit $r \to 0^+$ in the above inequality yields

$$f_i(x, \sigma(x)) \le 1, \ \mathcal{L}^2 - \text{a.e. } x \in \Omega.$$

Since $i \in \{1, 2, 3\}$ was arbitrary, it follows that $I_\infty(\sigma) = 0$, and this implies that (2.11) holds. Next, we need to show that for any $\sigma \in L^1(\Omega; \mathbb{M}^{2\times 2}_{\text{sym}})$, there exists a recovery sequence for the Γ-limit, that is, $\{\sigma_p\} \subset L^1(\Omega; \mathbb{M}^{2\times 2}_{\text{sym}})$ with $\sigma_p \to \sigma$ strongly in $L^1(\Omega; \mathbb{M}^{2\times 2}_{\text{sym}})$, and such that (2.12) holds. To this aim, we may assume, without loss of generality, that $I_\infty(\sigma) = 0$. This implies that for every $i \in \{1, 2, 3\}$ we have $f_i(x, \sigma(x)) \le 1$ for $\mathcal{L}^2$-a.e. $x \in \Omega$, and that Div $\sigma = 0$. By (2.5), $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2}_{\text{sym}})$. Define $\sigma_p = \sigma, \ p \in \mathbb{N}$. We have

$$0 \le I_p(\sigma_p) = \frac{1}{p} \int_\Omega \sum_{i=1}^{3} f_i(x, \sigma(x))^p dx \le \frac{3}{p} \mathcal{L}^N(\Omega),$$

which implies that $\lim_{p\to\infty} I_p(\sigma_p) = 0 = I_\infty(\sigma)$. We conclude that (2.12) holds. □

3. Variational Characterization of the Yield Set

We have seen in Introduction that the pointwise constraint (1.1) on the stress may be written in the form (see (1.5))

$$\sigma(x) \in \left\{ \eta \in \mathbb{M}^{2\times 2}_{\text{sym}} \ : \ f_i(x, \eta) \le 1 \text{ for all } i = 1, 2, 3 \right\}, \tag{3.1}$$

where $f_i : Q \times \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}$ ($i \in \{1,2,3\}$) are the Carathéodory integrands introduced in the previous section (see (2.1)-(2.3)). In this case the yield set of the polycrystal becomes

$$K_{\mathrm{eff}} = \Bigg\{ \eta \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \ : \ \exists\, \sigma \in L^\infty\left(Q; \mathbb{M}^{2\times 2}_{\mathrm{sym}}\right) \text{ such that } \eta = \int_Q \sigma(x)dx,$$
$$\mathrm{Div}\ \sigma = 0,\ \ f_i(x,\sigma(x)) \le 1\ \mathcal{L}^2\text{-a.e. } x \in Q,\ \ i = 1,2,3 \Bigg\}.$$

Equivalently,

$$K_{\mathrm{eff}} = \Bigg\{ \eta \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \ : \ \exists\, \sigma \in L^\infty\left(Q; \mathbb{M}^{2\times 2}_{\mathrm{sym}}\right) \text{ such that } \int_Q \sigma(x)dx = 0,$$
$$\mathrm{Div}\ \sigma = 0,\ \ f_i(x,\sigma(x)+\eta) \le 1\ \mathcal{L}^2\text{-a.e. } x \in Q,\ i = 1,2,3 \Bigg\}. \quad (3.2)$$

For $\eta \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ we consider the variational principle

$$i_p^{\mathrm{eff}}(\eta) := \inf \Bigg\{ \frac{1}{p} \int_Q \left(\sum_{i=1}^{3} f_i(x,\sigma(x)+\eta)^p \right) dx : \sigma \in L^p\left(Q; \mathbb{M}^{2\times 2}_{\mathrm{sym}}\right), \int_Q \sigma\ dx = 0$$
$$\mathrm{Div}\ \sigma = 0 \Bigg\}. \quad (3.3)$$

In view of our Theorem 2.3, standard arguments in the theory of Γ-convergence (see, e.g., [4] and [5]) imply, in particular, that for any $\eta \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$, $i_p^{\mathrm{eff}}(\eta)$ converges, as $p \to \infty$, to $i_\infty^{\mathrm{eff}}(\eta)$ given by

$$i_\infty^{\mathrm{eff}}(\eta) :=$$
$$\inf \Bigg\{ I_\infty(\sigma+\eta) \ : \ \sigma \in L^\infty\left(Q; \mathbb{M}^{2\times 2}_{\mathrm{sym}}\right), \int_Q \sigma(x)\ dx = 0, \mathrm{Div}\ \sigma = 0 \Bigg\}. \quad (3.4)$$

The limiting variational principle i_∞^{eff} allows us to provide the following simple description of the yield set K_{eff}.

Theorem 3.1.

$$K_{\mathrm{eff}} = \left\{ \eta \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \ : \ i_\infty^{\mathrm{eff}}(\eta) = 0 \right\}.$$

Proof. Let $\eta \in K_{\mathrm{eff}}$. By (3.2), there exists $\sigma \in L^\infty\left(Q; \mathbb{M}^{2\times 2}_{\mathrm{sym}}\right)$ such that

$\int_Q \sigma(x)dx = 0$, Div $\sigma = 0$, and $f_i(x, \sigma(x) + \eta) \leq 1$ for $\mathcal{L}^N-$a.e. $x \in Q$ ($i \in \{1, 2, 3\}$). We have $I_\infty(\sigma+\eta) = 0$, and thus $i_\infty^{\text{eff}}(\eta) = 0$. Conversely, let $\eta \in \mathbb{M}^{2\times 2}_{\text{sym}}$ be such that

$$i_\infty^{\text{eff}}(\eta) = 0. \tag{3.5}$$

Consider a sequence $\{\sigma_n\} \subseteq L^\infty\left(Q; \mathbb{M}^{2\times 2}_{\text{sym}}\right)$ such that Div $\sigma_n = 0$, $\int_Q \sigma_n(x)dx = 0$ for any $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} I_\infty(\sigma_n + \eta) = 0. \tag{3.6}$$

In particular, $f_i(x, \sigma_n(x) + \eta) \leq 1$ for $\mathcal{L}^2-$a.e. $x \in Q$ ($i \in \{1, 2, 3\}$), and taking into account the coercivity condition (2.5) we deduce that the sequence $\{\sigma_n\}$ is bounded in $L^\infty\left(Q; \mathbb{M}^{2\times 2}_{\text{sym}}\right)$. Thus, we may extract a subsequence (not relabelled) of $\{\sigma_n\}$ such that $\sigma_n \rightharpoonup \sigma$ weakly* in $L^\infty\left(Q; \mathbb{M}^{2\times 2}_{\text{sym}}\right)$, with Div $\sigma = 0$, and $\int_Q \sigma(x)dx = 0$. If $x \in Q$ is a Lebesgue point for each of the $f_i(\cdot, \sigma(\cdot) + \eta)$, we have, applying again Proposition 2.2,

$$\int_{B(x,r)} f_i(y, \sigma(y) + \eta)dy \leq \liminf_{n\to\infty} \int_{B(x,r)} f_i(y, \sigma_n(y) + \eta)dy \leq \mathcal{L}^2(B(x, r)),$$

for sufficiently small $r > 0$ and all $i \in \{1, 2, 3\}$. Thus, letting $r \to 0^+$, since almost every point $x \in Q$ is a Lebesgue point for all $f_i(\cdot, \sigma(\cdot)+\eta)$, we have that $f_i(x, \sigma(x) + \eta) \leq 1$ for $\mathcal{L}^2$-a.e. $x \in Q$, ($i \in \{1, 2, 3\}$). Taking (3.5) into account, we deduce that $\eta \in K_{\text{eff}}$. This concludes the proof. □

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