International Journal of Pure and Applied Mathematics
Volume 54 No. 4 2009, 477-487

GENERALIZED SOLUTION TO A VISCOUS DIFFUSION EQUATION

Zhengqiu Ling
Department of Mathematics and Computer Science
Yulin Normal University
Yulin, Guangxi, 537000, P.R. CHINA
e-mail: lingzq00@tom.com

Abstract: This paper is devoted to a viscous diffusion equation, some results on the existence and the uniqueness are established based on the time discrete method and Holmgren's approach.

AMS Subject Classification: 35G25, 35K30, 35K35
Key Words: existence of solutions, uniqueness of a solution, viscous, diffusion equation

1. Introduction

This paper is concerned with the generalized solution to the following initial boundary value problem

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial \Delta u}{\partial t} + \Delta A(\Delta u) = 0, \qquad (x,t) \in Q_T, \tag{1}$$

$$u(x,t) = \nabla u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{3}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$, $A(s) = |s|^{p-2}s$ $(p > 1)$, $Q_T = \Omega \times (0,T)$, $\lambda \geq 0$ is the viscosity coefficient, $u_0(x) \in L^\infty(Q_T)$.

Equation (1) arises from mathematical models describing some phenomena which exist in nature extensively. When $\lambda = 0$, the discussion of existence and

Received: March 5, 2009

uniqueness of solutions has been studied [5]. And when $\lambda > 0, A(\Delta s) = B(s)$, the second order viscous diffusion equation

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial \Delta u}{\partial t} + \Delta B(u) = 0 \tag{4}$$

was considered by Choen et al [2], where $B(s)$ has no monotonicity, but their interests center on the steady-state solution. The uniqueness of solutions of the Neumann initial-boundary value problem and Dirichlet initial-boundary value of the linear case of equation (4)

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial \Delta u}{\partial t} = \alpha \Delta u, \tag{5}$$

has been established by Chen et al [1] and Ting et al [3]. Another relevant word by Wang et al [4] is concerned with the uniqueness of solutions to the following equation

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial \Delta u}{\partial t} = \Delta A(u) + \mathrm{div}\vec{B}(u), \tag{6}$$

they prove the uniqueness of solutions to the initial boundary value problem of equation (6) by means of a regularizing technique based on elliptic operators.

In this paper, we prove the existence of generalized solution of the problem (1)–(3) based on the time discrete method and uniqueness by means of Holmgren's approach.

Definition 1. A function $u(x,t)$ is called to be a generalized solution of the initial boundary value problem (1)–(3), if the following conditions are fulfilled:

(1) $u \in L^\infty(0,T; W_0^{2,p}(\Omega)) \cap C(0,T; L^2(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T; W^{-2,p'}(\Omega))$; where p' is the conjugate exponent of p.

(2) For any $\varphi \in C_0^\infty(Q_T)$, the following integral equality holds

$$\iint_{Q_T} u(I - \lambda\Delta)\frac{\partial \varphi}{\partial t}\mathrm{d}x\mathrm{d}t - \iint_{Q_T} A(\Delta u)\Delta\varphi \mathrm{d}x\mathrm{d}t = 0.$$

2. Existence of Generalized Solution

Theorem 1. *Let $u_0 \in W_0^{2,p}(\Omega), p > 2$. Then the boundary value problems* (1)-(3) *admits at least one generalized solution.*

We adopt the time discrete method to construct an approximate solution. Divide the interval $(0,T)$ into N equal segments and denote by $h = \frac{T}{N}$. Consider the problem

$$\frac{1}{h}(u_{k+1}-u_k)-\frac{\lambda}{h}(\Delta u_{k+1}-\Delta u_k)+\Delta A(\Delta u_{k+1})=0, \tag{7}$$

$$u_{k+1}\Big|_{\partial\Omega}=\frac{\partial u_{k+1}}{\partial x_i}\Big|_{\partial\Omega}=0, i=1,2,\cdots,N, k=0,1,\cdots,N-1, \tag{8}$$

where u_0 is the initial datum.

Lemma 1. *For fixed k, if $u_k\in L^2(\Omega)$, then the problem (7)-(8) admits a generalized solution u_{k+1} in the space $W_0^{2,p}(\Omega)$, namely, there exist a function $u_{k+1}\in W_0^{2,p}(\Omega)$, such that for any $\varphi\in C_0^\infty(\Omega)$,*

$$\frac{1}{h}\int_\Omega(u_{k+1}-u_k)\varphi\mathrm{d}x+\frac{\lambda}{h}\int_\Omega(\nabla u_{k+1}-\nabla u_k)\nabla\varphi\mathrm{d}x+\int_\Omega A(\Delta u_{k+1})\Delta\varphi\mathrm{d}x=0. \tag{9}$$

Proof. Consider the functionals

$$\Phi_1[u]=\frac{1}{2}\int_\Omega|u|^2\mathrm{d}x,\quad \Phi_2[u]=\frac{1}{2}\int_\Omega|\nabla u|^2\mathrm{d}x,\quad \Phi_3[u]=\frac{1}{p}\int_\Omega|\Delta u|^p\mathrm{d}x,$$

$$\Psi\ [u]=\frac{1}{h}\Phi_1[u]+\frac{\lambda}{h}\Phi_2[u]+\Phi_3[u]-\int_\Omega fu\mathrm{d}x,$$

where $f\in L^2(\Omega)$ is a given function. By virtue of Young's inequality, there exist the constants $C_1,C_2,C_3>0$,

$$\Psi[u]\ge C_1\int_\Omega|\Delta u|^p\mathrm{d}x+C_2\int_\Omega|\nabla u|^2\mathrm{d}x-C_3\int_\Omega|f|^2\mathrm{d}x.$$

Therefore,

$$\Psi[u]\to+\infty,\qquad \text{if } \|u\|_{2,p}\to+\infty.$$

Here we use the notation $\|u\|_{2,p}$ to denote the norm of u in $W_0^{2,p}(\Omega)$. Noticing that $\Psi[u]$ is obviously a weakly semi-lower continuous functional, we conclude that there exists $u_*\in W_0^{2,p}(\Omega)$, such that

$$\Psi[u_*]=\inf\Psi[u],$$

and u_* is the solution of the Euler equation corresponding to $\Psi[u]$

$$\frac{1}{h}u-\frac{\lambda}{h}\Delta u+\Delta A(\Delta u)=f.$$

Choosing $f=\frac{1}{h}u_k-\frac{\lambda}{h}\Delta u_k$, we then obtain a generalized solution u_{k+1} of the problem (7)-(8). The proof is completed. □

Now, we construct an approximate solution u^h of the problem (1)-(3) by defining

$$u^h(x,t)=u_k(x,t),\quad kh<x\le(k+1)h,\quad k=0,1,\cdots,N-1,$$

$$u^h(x,0)=u_0(x).$$

The desired solution of the problem (1)-(3) will be obtained as the limit of some subsequence of $\{u^h\}$. For this purpose, we need some uniform estimate on u^h.

Lemma 2. *For the generalized solution u_k of the problem (7)-(8), the following estimates hold*

$$h\sum_{k=1}^{N}\int_\Omega |\Delta u_k|^p \mathrm{d}x \le C, \tag{10}$$

$$\sup_{0<t<T}\int_\Omega |\Delta u^h(x,t)|^p \mathrm{d}x \le C, \tag{11}$$

where C is a constant independent of h, k.

Proof. We first proof (10). Notice that, we may choose $\varphi \in W_0^{2,p}(\Omega)$ as the test function in (9). In particular, by the choice $\varphi = u_{k+1}$ and use Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u_{k+1}|^2 \mathrm{d}x + \frac{\lambda}{2}\int_\Omega |\nabla u_{k+1}|^2 \mathrm{d}x + h\int_\Omega |\Delta u_{k+1}|^p \mathrm{d}x \\ \le\, & \frac{1}{2}\int_\Omega |u_k|^2 \mathrm{d}x + \frac{\lambda}{2}\int_\Omega |\nabla u_k|^2 \mathrm{d}x. \end{aligned} \tag{12}$$

Summing up these inequalities for k from 0 up to $N-1$, we have

$$h\sum_{k=0}^{N-1}\int_\Omega |\Delta u_{k+1}|^p \mathrm{d}x \le \frac{1}{2}\int_\Omega |u_0|^2 \mathrm{d}x + \frac{\lambda}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x.$$

So, (10) holds.

To prove (11), we choose $\varphi = u_{k+1} - u_k$ in (9) and obtain

$$\begin{aligned} &\frac{1}{h}\int_\Omega (u_{k+1}-u_k)^2 \mathrm{d}x + \frac{\lambda}{h}\int_\Omega (\nabla u_{k+1} - \nabla u_k)^2 \mathrm{d}x \\ &+ \int_\Omega A(\Delta u_{k+1})\Delta(u_{k+1}-u_k)\mathrm{d}x = 0. \end{aligned}$$

Since the first term and second term of the left-hand side of the above equality are nonnegative, it follows that

$$\int_\Omega A(\Delta u_{k+1})\Delta(u_{k+1}-u_k)\mathrm{d}x \le 0.$$

Young's inequality then yields

$$\begin{aligned} \int_\Omega |\Delta u_{k+1}|^p \mathrm{d}x &\le \int_\Omega |\Delta u_{k+1}|^{p-2}\Delta u_{k+1}\Delta u_k \mathrm{d}x \\ &\le \frac{p-1}{p}\int_\Omega |\Delta u_{k+1}|^p \mathrm{d}x + \frac{1}{p}\int_\Omega |\Delta u_k|^p \mathrm{d}x, \end{aligned}$$

that is,

$$\int_\Omega |\Delta u_{k+1}|^p \mathrm{d}x \le \int_\Omega |\Delta u_k|^p \mathrm{d}x.$$

For any m with $1 \le m \le N-1$, summing up the above inequality for k from 0 up to $m-1$, we have

$$\int_\Omega |\Delta u_m|^p \mathrm{d}x \le \int_\Omega |\Delta u_0|^p \mathrm{d}x.$$

Therefore (11) holds and the proof is complete. □

Lemma 3. *For the generalized solution u_{k+1} of the problem (7)-(8), the following estimates hold*

$$-Ch \le \Big(\int_\Omega |u_{k+1}|^2 \mathrm{d}x - \int_\Omega |u_k|^2 \mathrm{d}x \Big) + \lambda \Big(\int_\Omega |\nabla u_{k+1}|^2 \mathrm{d}x - \int_\Omega |\nabla u_k|^2 \mathrm{d}x \Big) \le 0, \quad (13)$$

where C is a constant independent of h, k.

Proof. The second inequality of (13) is an immediate consequence of (12). To prove the first inequality, we choose $\varphi = u_k$ in (9), applying Hölder inequality and the estimate (11) yields

$$\int_\Omega |u_k|^2 \mathrm{d}x + \lambda \int_\Omega |\nabla u_k|^2 \mathrm{d}x - \int_\Omega u_{k+1} u_k \mathrm{d}x - \lambda \int_\Omega \nabla u_{k+1} \cdot \nabla u_k \mathrm{d}x \le Ch.$$

Therefore

$$\int_\Omega |u_k|^2 \mathrm{d}x + \lambda \int_\Omega |\nabla u_k|^2 \mathrm{d}x \le \int_\Omega u_{k+1} u_k \mathrm{d}x + \lambda \int_\Omega \nabla u_{k+1} \cdot \nabla u_k \mathrm{d}x + Ch$$
$$\le \frac{1}{2} \int_\Omega |u_k|^2 \mathrm{d}x + \frac{1}{2} \int_\Omega |u_{k+1}|^2 \mathrm{d}x + \frac{\lambda}{2} \int_\Omega |\nabla u_{k+1}|^2 \mathrm{d}x + \frac{\lambda}{2} \int_\Omega |\nabla u_k|^2 \mathrm{d}x + Ch.$$

Thus,

$$\Big(\int_\Omega |u_{k+1}|^2 \mathrm{d}x - \int_\Omega |u_k|^2 \mathrm{d}x \Big) + \lambda \Big(\int_\Omega |\nabla u_{k+1}|^2 \mathrm{d}x - \int_\Omega |\nabla u_k|^2 \mathrm{d}x \Big) \ge -Ch.$$

The proof is complete. □

Corollary 1.

$$\sup_{0<t<T} \Big(\int_\Omega |u^h(x,t)|^2 \mathrm{d}x + \lambda \int_\Omega |\nabla u^h(x,t)|^2 \mathrm{d}x \Big) \le \int_\Omega |u_0|^2 \mathrm{d}x + \lambda \int_\Omega |\nabla u_0|^2 \mathrm{d}x. \quad (14)$$

Proof. We need only, for any fixed m with $1 \le m \le N-1$, to sum the second inequality in (13) for k from 0 up to $m-1$. $\square$

Now, we define operator A^t by

$$A^t(\Delta u^h) = A(\Delta u_k),$$
$$\Delta^h u^h = u_{k+1} - u_k,$$

where $kh < t \leq (k+1)h, k = 0, 1, \cdots, N-1$. From the discrete equations (7) and (10), we see that

$$\frac{1}{h}\Delta^h u^h \qquad \text{is bounded in } L^\infty(0,T;W^{-2,p'}(\Omega)). \tag{15}$$

Using the results obtained above, we can complete the proof of Theorem 1.

Proof of Theorem 1. By (11), (14), (15) and (9), we may extract a subsequence from $\{u^h\}$, denoted still by $\{u^h\}$, such that

$$u^h \overset{*}{\rightharpoonup} u, \quad \text{in } L^\infty(0,T;W^{2,p}(\Omega)),$$
$$\frac{1}{h}\Delta^h u^h \overset{*}{\rightharpoonup} \frac{\partial u}{\partial t}, \quad \text{in } L^\infty(0,T;W^{-2,p'}(\Omega)),$$
$$A^t(\Delta u^h) \overset{*}{\rightharpoonup} \omega, \quad \text{in } L^\infty(0,T;L^{p'}(\Omega)),$$

hold for some u, ω. Then in the sense of distributions

$$\frac{\partial u}{\partial t} - \lambda\frac{\partial}{\partial t}(\Delta u) + \Delta\omega = 0. \tag{16}$$

In fact, from the relation (9), we see that for any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T}\Big(\frac{1}{h}\Delta^h u^h\varphi + \lambda\nabla(\frac{1}{h}\Delta^h u^h)\cdot\nabla\varphi + A^t(\Delta u^h)\Delta\varphi\Big)\mathrm{d}x\mathrm{d}t = 0,$$

and (16) follows by letting $h \to 0$.

Now, we turn to proof of $\omega = A(\Delta u)$, a.e. in Q_T. Define

$$\begin{aligned} f_h(t) = \frac{t-kh}{2h}\Big(&(\int_\Omega |u_{k+1}|^2\mathrm{d}x - \int_\Omega |u_k|^2\mathrm{d}x) \\ &+ \lambda(\int_\Omega |\nabla u_{k+1}|^2\mathrm{d}x - \int_\Omega |\nabla u_k|^2\mathrm{d}x)\Big) + \frac{1}{2}\Big(\int_\Omega |u_k|^2\mathrm{d}x + \lambda\int_\Omega |\nabla u_k|^2\mathrm{d}x\Big), \end{aligned}$$

$$kh < t \leq (k+1)h, \quad k = 0, 1, \cdots, N-1.$$

From (13),

$$\frac{1}{2}\Big(\int_\Omega |u_k|^2\mathrm{d}x + \lambda\int_\Omega |\nabla u_k|^2\mathrm{d}x\Big) - Ch \leq f_h(t) \leq \frac{1}{2}\Big(\int_\Omega |u_k|^2\mathrm{d}x + \lambda\int_\Omega |\nabla u_k|^2\mathrm{d}x\Big),$$

$$-C \leq f_h'(t) \leq 0.$$

According to Ascoli-Arzela Theorem, there exists a function $f(t) \in C([0,T])$, such that

$$\lim_{h\to 0} f_h(t) = f(t) \quad \text{uniformly for } t \in [0,T].$$

Using (13) again, we have

$$\lim_{h\to 0} \frac{1}{2}\Big(\int_\Omega |u^h(x,t)|^2 \mathrm{d}x + \lambda \int_\Omega |\nabla u^h(x,t)|^2 \mathrm{d}x\Big) = f(t), \text{ uniformly for } t \in [0,T]. \tag{17}$$

From (12), we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u_N|^2 \mathrm{d}x + \frac{\lambda}{2}\int_\Omega |\nabla u_N|^2 \mathrm{d}x + \iint_{Q_T} |\Delta u^h(x,t)|^p \mathrm{d}x\mathrm{d}t \\ \le &\frac{1}{2}\int_\Omega |u_0|^2 \mathrm{d}x + \frac{\lambda}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x, \end{aligned}$$

that is

$$\begin{aligned} &\iint_{Q_T} |\Delta u^h(x,t)|^p \mathrm{d}x\mathrm{d}t \\ \le &\frac{1}{2}\int_\Omega |u_0|^2 \mathrm{d}x + \frac{\lambda}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega |u_N|^2 \mathrm{d}x - \frac{\lambda}{2}\int_\Omega |\nabla u_N|^2 \mathrm{d}x, \end{aligned}$$

Letting $h \to 0$ and using (17), we have

$$\begin{aligned} &\underline{\lim}_{h\to 0} \iint_{Q_T} |\Delta u^h(x,t)|^p \mathrm{d}x \le f(0) - f(T) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^{T-\varepsilon} (f(t) - f(t+\varepsilon))\mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\lim_{h\to 0} \frac{1}{2\varepsilon}\int_0^{T-\varepsilon} \Big(\big(\int_\Omega |u^h(x,t)|^2 \mathrm{d}x + \lambda\int_\Omega |\nabla u^h(x,t)|^2 \mathrm{d}x\big) \\ &\qquad - \big(\int_\Omega |u^h(x,t+\varepsilon)|^2 \mathrm{d}x + \lambda \int_\Omega |\nabla u^h(x,t+\varepsilon)|^2 \mathrm{d}x\big)\Big) \\ &= \lim_{\varepsilon\to 0}\lim_{h\to 0} \frac{1}{2\varepsilon}\int_0^{T-\varepsilon} \Big(\big(\int_\Omega |u^h(x,t)|^2 \mathrm{d}x - \int_\Omega |u^h(x,t+\varepsilon)|^2 \mathrm{d}x\big) \\ &\qquad + \lambda\big(\int_\Omega |\nabla u^h(x,t)|^2 \mathrm{d}x - \int_\Omega |\nabla u^h(x,t+\varepsilon)|^2 \mathrm{d}x\big)\Big). \end{aligned}$$

Noticing the convexity of $\Phi_1[u]$ and $\Phi_2[u]$, we see that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |u^h(x,t)|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega |u^h(x,t+\varepsilon)|^2 \mathrm{d}x \\ \le &\int_\Omega \Big(u^h(x,t) - u^h(x,t+\varepsilon)\Big) u^h(x,t)\mathrm{d}x, \end{aligned}$$

$$\frac{1}{2}\int_\Omega |\nabla u^h(x,t)|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla u^h(x,t+\varepsilon)|^2 \mathrm{d}x \\ \le \int_\Omega \Big(\nabla u^h(x,t) - \nabla u^h(x,t+\varepsilon)\Big)\nabla u^h(x,t)\mathrm{d}x.$$

Thus

$$\lim_{h\to 0}\frac{1}{2\varepsilon}\int_0^{T-\varepsilon}\Big(\big(\int_\Omega |u^h(x,t)|^2\mathrm{d}x - \int_\Omega |u^h(x,t+\varepsilon)|^2\mathrm{d}x\big) \\ +\lambda\big(\int_\Omega |\nabla u^h(x,t)|^2\mathrm{d}x - \int_\Omega |\nabla u^h(x,t+\varepsilon)|^2\mathrm{d}x\big)\Big)\mathrm{d}t \\ \le \frac{1}{\varepsilon}\int_0^{T-\varepsilon}\Big(\int_\Omega (u(x,t)-u(x,t+\varepsilon))u(x,t)\mathrm{d}x \\ +\lambda\int_\Omega (\nabla u(x,t) - \nabla u(x,t+\varepsilon))\nabla u(x,t)\mathrm{d}x\Big)\mathrm{d}t,$$

and hence

$$\varlimsup_{h\to 0}\iint_{Q_T} |\Delta u^h(x,t)|^p \mathrm{d}x\mathrm{d}t \le -\int_0^T \langle \frac{\partial u}{\partial t} - \lambda\frac{\partial(\Delta u)}{\partial t}, u\rangle \mathrm{d}t,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual produce of the function in $W^{-2,p'}(\Omega)$ and $W^{2,p}(\Omega)$. It follows from (16) that

$$\varlimsup_{h\to 0}\iint_{Q_T} |\Delta u^h(x,t)|^p \mathrm{d}x\mathrm{d}t \le \iint_{Q_T} \omega\Delta u \mathrm{d}x\mathrm{d}t. \tag{18}$$

On the other hand, for any $g\in L^\infty(0,T;W^{2,p}(\Omega))$, from $\frac{\delta\Phi_3(u)}{\delta u} = \Delta A(\Delta u)$, and the convexity of $\Phi_3(u)$, we have

$$\frac{1}{p}\iint_{Q_T}|\Delta g|^p\mathrm{d}x\mathrm{d}t - \frac{1}{p}\iint_{Q_T}|\Delta u^h|^p\mathrm{d}x\mathrm{d}t \ge \iint_{Q_T} A(\Delta u^h)\Delta(g-u^h)\mathrm{d}x\mathrm{d}t.$$

Using (18), the weak lower semi-continuity of $\Phi_3(u)$, and letting $h\to 0$ yield

$$\frac{1}{p}\iint_{Q_T}|\Delta g|^p\mathrm{d}x\mathrm{d}t - \frac{1}{p}\iint_{Q_T}|\Delta u|^p\mathrm{d}x\mathrm{d}t \ge \iint_{Q_T} \omega\Delta(g-u)\mathrm{d}x\mathrm{d}t.$$

Replacing g with $\varepsilon g + u$ leads to

$$\frac{1}{\varepsilon}(\Phi_3(\varepsilon g+u) - \Phi_3(u)) \ge \iint_{Q_T}\omega\Delta g\mathrm{d}x\mathrm{d}t,$$

Therefore

$$\iint_{Q_T}\frac{\delta\Phi_3(u)}{\delta u} g\mathrm{d}x\mathrm{d}t = \iint_{Q_T} A(\Delta u)\mathrm{d}x\mathrm{d}t \ge \iint_{Q_T}\omega\Delta g\mathrm{d}x\mathrm{d}t.$$

Due to the arbitrariness of g, we see that $\omega = A(\Delta u)$ and the proof is complete. □

3. Uniqueness

Theorem 2. *If* $\frac{\partial^2 u}{\partial x_i^2} \in L^\infty(Q_T)$ $(i = 1, 2, \cdots, N)$, *then the initial boundary value problem* (1)-(3) *has at most one generalized solution* $u(x,t)$.

Let u_1, u_2 be two generalized solutions of the problem (1)-(3) such that $\frac{\partial^2 u_1}{\partial x_i^2}, \frac{\partial^2 u_2}{\partial x_i^2} \in L^\infty(Q_T)$ $(i = 1, 2, \cdots, N)$, then for any test function φ and from the definition of generalized solution, we have

$$\iint_{Q_T} (u_1 - u_2)(I - \lambda\Delta)\frac{\partial \varphi}{\partial t}\mathrm{d}x\mathrm{d}t = \iint_{Q_T} \hat{a}(\Delta u_1 - \Delta u_2)\Delta\varphi \mathrm{d}x\mathrm{d}t, \tag{19}$$

where

$$\hat{a} = \int_0^1 a\Big(\theta\Delta u_1 + (1-\theta)\Delta u_2\Big)\mathrm{d}\theta, \quad a(s) = (p-1)|s|^{p-2}.$$

Let $\{\hat{a}_\varepsilon(x,t)\} \subset C_0^\infty(Q_T)$ and be a approximation of $\hat{a}$, such that

$$0 \le \hat{a}_\varepsilon(x,t) \le M, \qquad \iint_{Q_T} \Big|\hat{a}_\varepsilon - \hat{a}\Big|^2 \mathrm{d}x\mathrm{d}t \le \varepsilon^2, \qquad \varepsilon > 0. \tag{20}$$

Consider the boundary value problem for the conjugate equation

$$\frac{\partial \varphi_\varepsilon}{\partial t} - \lambda\frac{\partial}{\partial t}\Big(\Delta\varphi_\varepsilon\Big) = \Delta\Big((\hat{a}_\varepsilon + \varepsilon)\Delta\varphi_\varepsilon\Big) + f, \qquad (x,t) \in Q_T, \tag{21}$$

$$\varphi_\varepsilon(x,t) = \varphi_{\varepsilon x_i}(x,t) = 0 \quad (i = 1, 2, \cdots, N), \qquad (x,t) \in \partial\Omega \times (0,T), \tag{22}$$

$$\varphi_\varepsilon(x,T) = 0, \qquad x \in \Omega, \tag{23}$$

where $f \in C_0^\infty(Q_T)$ is an arbitrarily given function. The classical theory for linear equations implies that the above problem (21)–(23) admits classical solution φ_ε and have the following result.

Lemma 4. *The solution* φ_ε *of problem* (21)-(23) *is such that*

$$\iint_{Q_T} (\hat{a}_\varepsilon + \varepsilon)(\Delta\varphi_\varepsilon)^2 \mathrm{d}x\mathrm{d}t \le \frac{1}{2}e^T \iint_{Q_T} f^2 \mathrm{d}x\mathrm{d}t. \tag{24}$$

Proof. Multiplying (21) by φ_ε and integrate over $Q_t = \Omega \times (t,T)$ and using the integrating by parts, we obtain

$$-\frac{1}{2}\int_\Omega \varphi_\varepsilon^2(x,t)\mathrm{d}x - \frac{\lambda}{2}\int_\Omega (\nabla\varphi_\varepsilon(x,t))^2\mathrm{d}x$$
$$= \iint_{Q_t} (\hat{a}_\varepsilon + \varepsilon)(\Delta\varphi_\varepsilon)^2 \mathrm{d}x\mathrm{d}s + \iint_{Q_t} f\varphi_\varepsilon \mathrm{d}x\mathrm{d}s,$$

that is

$$\frac{1}{2}\int_\Omega \varphi_\varepsilon^2(x,t)\mathrm{d}x + \frac{\lambda}{2}\int_\Omega (\nabla\varphi_\varepsilon(x,t))^2\mathrm{d}x + \iint_{Q_t}(\hat{a}_\varepsilon+\varepsilon)(\Delta\varphi_\varepsilon)^2\mathrm{d}x\mathrm{d}s$$
$$= -\iint_{Q_t} f\varphi_\varepsilon\mathrm{d}x\mathrm{d}s \le \frac{1}{2}\iint_{Q_t} f^2\mathrm{d}x\mathrm{d}s + \frac{1}{2}\iint_{Q_t}\varphi_\varepsilon^2\mathrm{d}x\mathrm{d}s,$$

therefore,

$$\frac{1}{2}\int_\Omega \varphi_\varepsilon^2(x,t)\mathrm{d}x + \int_t^T\int_\Omega(\hat{a}_\varepsilon+\varepsilon)(\Delta\varphi_\varepsilon)^2\mathrm{d}x\mathrm{d}s$$
$$\le \frac{1}{2}\iint_{Q_T} f^2\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_t^T\int_\Omega \varphi_\varepsilon^2\mathrm{d}x\mathrm{d}s.$$

Hence, from the Gronwall inequality we obtain (24). The proof is complete. □

Proof of Theorem 2. Let $\varphi = \varphi_\varepsilon$ in (19), where φ_ε is a solution of the problem (21)–(23), we have

$$\iint_{Q_T}(u_1-u_2)\cdot\Delta\Big((\hat{a}_\varepsilon+\varepsilon)\Delta\varphi_\varepsilon\Big)\mathrm{d}x\mathrm{d}t + \iint_{Q_T}(u_1-u_2)f\mathrm{d}x\mathrm{d}t$$
$$= \iint_{Q_T}\hat{a}(\Delta u_1-\Delta u_2)\Delta\varphi_\varepsilon\mathrm{d}x\mathrm{d}t,$$

using (20) and (24), we obtain

$$\Big|\iint_{Q_T}(u_1-u_2)f\mathrm{d}x\mathrm{d}t\Big| \le \Big|\iint_{Q_T}(\Delta u_1-\Delta u_2)(\hat{a}_\varepsilon-\hat{a})\Delta\varphi_\varepsilon\mathrm{d}x\mathrm{d}t\Big|$$
$$+\Big|\iint_{Q_T}\varepsilon(\Delta u_1-\Delta u_2)\Delta\varphi_\varepsilon\mathrm{d}x\mathrm{d}t\Big|$$
$$\le C_1\Big(\iint_{Q_T}(\hat{a}_\varepsilon-\hat{a})^2\mathrm{d}x\mathrm{d}t\Big)^{1/2}\Big(\iint_{Q_T}(\Delta\varphi_\varepsilon)^2\mathrm{d}x\mathrm{d}t\Big)^{1/2}$$
$$+ C_2\varepsilon\Big(\iint_{Q_T}(\Delta\varphi_\varepsilon)^2\mathrm{d}x\mathrm{d}t\Big)^{1/2}$$
$$\le C_3\varepsilon\Big(\iint_{Q_T}(\Delta\varphi_\varepsilon)^2\mathrm{d}x\mathrm{d}t\Big)^{1/2} = C_3\sqrt{\varepsilon}\Big(\iint_{Q_T}\varepsilon(\Delta\varphi_\varepsilon)^2\mathrm{d}x\mathrm{d}t\Big)^{1/2}$$
$$\le C_3\sqrt{\varepsilon}\Big(\iint_{Q_T}(\hat{a}_\varepsilon+\varepsilon)(\Delta\varphi_\varepsilon)^2\mathrm{d}x\mathrm{d}t\Big)^{1/2}$$
$$\le C_3\sqrt{\varepsilon}\Big(\frac{1}{2}e^T\iint_{Q_T} f^2\mathrm{d}x\mathrm{d}t\Big)^{1/2} \to 0 \quad (\varepsilon\to 0).$$

Owing the arbitrariness of f, we have $u_1 = u_2$ a.e. in Q_T, the proof of Theorem 2 is completed. □

Acknowledgements

This work is supported by the NNSF of China grant 10626024.

References

[1] P.J. Chen, M.E. Gurtin, On a theory of heat conduction involving two temperatures, *Z. Angew. Math. Phys.*, **19** (1968), 614-627.

[2] A. Novick-Cohen, R.L. Pego, Stable patterns in a viscous diffusion equation, *Comm. PDE*, **15**, No. 12 (1991), 331-351.

[3] R.E. Showalter, T.W. Ting, Pseudo-parabolic partial differential equation, *SIAM J. Math. Anal.*, **1** (1970), 1-26.

[4] Z. Wang, J. Yin, Uniqueness of solutions to a viscous diffusion equation, *Appl. Math. Letters*, **17** (2004), 1317-1322.

[5] Z. Wu, J. Zhao, J. Yin, H. Li, *Nonlinear Diffusion Equations*, World Scientific Publishing, Singpore (2001).