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THE LIMIT THEOREMS FOR EXPECTATION OF FUZZY RANDOM VARIABLES

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Abstract: In this paper, the limit theorems for expectation of random variables which extend previous results by M.L. Puri, D.A. Ralescu (Fuzzy random variables, *J. Math. Anal. Appl.*, **114** (1986), 409-422), and Mila Stojakovic (Fuzzy random variable, expectation, and martingales, *J. Math. Anal. Appl.*, **134** (1994), 594-606) are presented.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [8] in 1965. After that, many applications of fuzzy sets have been developed. One of them is the fuzzy stochastic optimization in operation research, which utilizes the fuzziness and randomness in the optimization problems. The concept of expectation of fuzzy random variables is very important in stochastic optimization problems, thus it is natural to optimize the expectation of fuzzy random variables in this field. The fuzzy random variables and their expectation were defined by Kwakernaak [4] and Puri and Ralescu [6], but the limit theorems they gave are very strong. In 1994, Mila Stojakovic gave a uniformly convergent theorem [7]. The expectation as an integral for the measurable fuzzy-valued functions and its properties have

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been defined and discussed by R. Goetschel and W. Voxman [2], O. Kaleva [3]. In this paper, we shall give a convergent theorem, which is an extension of [6], [7]. At first, we recall the definitions of fuzzy random variables and its expectation. Next, we present and discuss the concept of absolute continuity of fuzzy random variables. Finally, we give several limit theorems for random variables.

2. Preliminaries

In this section, we describe some basic concepts of fuzzy numbers [1], [6], [7]. Let R denote the real line. A fuzzy number is a fuzzy set $\widetilde{u} : R \to [0,1]$ with the following properties:

(1) $\widetilde{u}$ is normal, i.e., there exists $x \in R$ such that $\widetilde{u}(x) = 1$.

(2) $\widetilde{u}$ is upper semicontinuous.

(3) supp $\widetilde{u} = cl\{x \in R : \widetilde{u}(x) > 0\}$ is compact.

(4) $\widetilde{u}$ is a convex fuzzy set, i.e., $\widetilde{u}(\lambda x + (1-\lambda)y) \geq \min(\widetilde{u}(x), \widetilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

Let $F(R)$ be the family of all fuzzy numbers. By the results of [1] and [6], for a fuzzy set $\widetilde{u}$, if we define

$$u_\alpha = \begin{cases} \{x : \widetilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ \text{support } \widetilde{u}, & \alpha = 0, \end{cases}$$

then $\widetilde{u}$ is completely determined by the intervals $u_\alpha = [u_\alpha^-, u_\alpha^+]$. Suppose now that $\widetilde{u}, \widetilde{v} \in F(R)$ are fuzzy numbers represented by $\{[u_\alpha^-, u_\alpha^+] : 0 \leq \alpha \leq 1\}$ and $\{[v_\alpha^-, v_\alpha^+] : 0 \leq \alpha \leq 1\}$, respectively, and define [1, 2]

$$(\widetilde{u} + \widetilde{v})(z) = \sup_{x+y=z} \min(\widetilde{u}(x), \widetilde{v}(x)),$$

$$(\lambda \widetilde{u})(z) = \begin{cases} \widetilde{u}(\frac{z}{\lambda}), & \lambda \neq 0, \\ \widetilde{0}, & \lambda = 0, \end{cases}$$

where $\widetilde{0} = \chi_{(0)}(x)$ is the indicator function of $\{0\}$, then

$$\widetilde{u} + \widetilde{v} = \{[u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] : 0 \leq \alpha \leq 1\},$$

$$\lambda \widetilde{u} = \begin{cases} \{(\lambda u_\alpha^-, \lambda v_\alpha^+) : 0 \leq \alpha \leq 1\}, & \lambda \geq 0, \\ \{(\lambda u_\alpha^+, \lambda v_\alpha^-) : 0 \leq \alpha \leq 1\}, & \lambda < 0. \end{cases}$$

Now, we define a metric on $F(R)$ by

$$D(\widetilde{u}, \widetilde{v}) = \sup_{\alpha \in [0,1]} d(u_\alpha, v_\alpha) = \sup_{\alpha \in [0,1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}.$$

Then, it is well known that $(F(R), D)$ is a metric space, and

(1) $D(\widetilde{u}+\widetilde{w}, \widetilde{v}+\widetilde{w}) = D(\widetilde{u}, \widetilde{v})$;

(2) $D(k\widetilde{u}, k\widetilde{v}) = |k|D(\widetilde{u}, \widetilde{v})$.

3. Fuzzy Random Variables

Throughout this paper, (Ω, A, P) denotes a complete probability space. If $\widetilde{X} : \Omega \to F(R)$ is a fuzzy number-valued function and B is a subset of R, then $\widetilde{X}^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$\widetilde{X}^{-1}(B)(w) = \sup_{x \in B} \widetilde{X}(w)(x)$$

for every $x \in \Omega$. The fuzzy-number-valued function $\widetilde{X} : \Omega \to F(R)$ is said to be a fuzzy random variable if for every closed subset B of R, the fuzzy subset $\widetilde{X}^{-1}(B)$ is measurable when considered as a function from Ω to [0,1] (cf. [6]), i.e., if we denote $\widetilde{X}(w) = \{(X_\alpha^-(w), X_\alpha^+(w)) : 0 \le \alpha \le 1\}$, then $\widetilde{X} : \Omega \to F(R)$ is a fuzzy random variable if and only if for $\alpha \in [0,1], X_\alpha^-, X_\alpha^+$ are random variable in the classical sense (cf. [6]).

Definition 3.1. (see Puri and Ralescu [6]) Let $\widetilde{X}(w)$ be an integrably and bounded fuzzy random variable, i.e., there exists an (L) integrable function h such that $|\dot{x}| \le h(w)$ for all $\dot{x} \in X_\alpha(w)$. $\widetilde{X}(w)$ is said to be integrable on [a, b] if the set

$$(\int_a^b X(w)dw)_\alpha = \{(L)\int_a^b f(w)dw : f(w) \text{ is a measurable selection for } X_a(w)\}$$

for all $\alpha \in [0,1]$, determine a fuzzy number. We define $E(\widetilde{X}) = \int_a^b X(w)dw$ as the expectation of $\widetilde{X}$ on [a, b].

By the result of [6, 7], $\widetilde{X}(w)$ is integrable on [a, b] if and only if for every $\alpha \in [0,1], X_\alpha^-, X_\alpha^+$ are Lebesgue integrable on [a, b], and

$$E\widetilde{X} = \{(\int X_\alpha^- d\mu, \int X_\alpha^+ d\mu) : 0 \le \alpha \le 1\}.$$

Note that this definition is exactly the same as Yun Kyong Kim and Byung Moon Ghil [5] and applied by Kaleva [3].

Lemma 3.1. (see Kaleva [3]) *Let $\widetilde{X}(x), \widetilde{X}(y) : [a,b] \to [a,b]$ be integrable and $\lambda \in R$. Then:*

(1) $\int(\widetilde{X}(w) + \widetilde{Y}(w))dw = \int \widetilde{X}(w)dw + \int \widetilde{Y}(w)dw$;

(2) $\int \lambda\widetilde{X}(w)dw \le \lambda \int X(w)dw$;

(3) $D(\widetilde{X},\widetilde{Y})$ *is (L) integrable;*

(4) $D(\int \widetilde{X}(w)dw, \int \widetilde{Y}(w)dw) \leq (L)\int D(\widetilde{X}(w),\widetilde{Y}(w))dw.$

4. The Limit Theorems of Expectations

In this section, we discuss the problems of convergence for the expectation of fuzzy random variables. $\lim_{n\to\infty}\widetilde{X}_n(w)=\widetilde{X}(w)$ for $\widetilde{X}_n(w), X(w) : [a,b] \to F(R), n = 1,2,3.\cdots$ means that for any $\varepsilon > 0$, there exists a N such that for any $n > N$, we have

$$D(\widetilde{X}_n(w),\widetilde{X}(w)) < \varepsilon.$$

Definition 4.1. A fuzzy random variable $\widetilde{X} : [a,b] \to F(R)$ is said to be absolutely continuous on $[a,b]$ if for any $\varepsilon > 0$, there exists a $\eta > 0$ such that for any finite or infinite sequences of non-overlapping intervals $\{[a_i,b_i]\}$ satisfying $\sum(b_i - a_i) < \eta$, we have

$$\sum D(\widetilde{X}(b_i),\widetilde{X}(a_i)) < \varepsilon.$$

Theorem 4.1. *A fuzzy random variable $\widetilde{X} : [a,b] \to F(R)$ is absolutely continuous on [a, b] if and only if for any X^-_α, X^+_α are absolutely continuous on $[a,b]$ uniformly for $\alpha \in [0,1]$, i.e., for any $\varepsilon > 0$, there exists a $\eta > 0$ such that for any finite or infinite sequences of non-overlapping intervals $\{[a_i,b_i]\}$ satisfying $\sum(b_i - a_i) < \eta$, for any $\alpha \in [0,1]$, we have*

$$\sum |X^-_\alpha(b_i) - X^-_\alpha(a_i)| < \varepsilon, \qquad \sum |X^+_\alpha(b_i) - X^+_\alpha(a_i)| < \varepsilon.$$

This theorem is completely proved by the following inequality:

$$\begin{aligned}\max\{\sum |X^-_\alpha(b_i) - X^-_\alpha(a_i)|, \sum |X^+_\alpha(b_i) - X^+_\alpha(a_i)|\} &\leq \sum D(\widetilde{X}(b_i),\widetilde{X}(a_i))\\ &= \sup_{\alpha\in[0,1]} \max\{|X^-_\alpha(b_i) - X^-_\alpha(a_i)|, |X^+_\alpha(b_i) - X^+_\alpha(a_i)|\}\\ &\leq \sup_{\alpha\in[0,1]} \max\{|X^-_\alpha(b_i) - X^-_\alpha(a_i)|, |X^+_\alpha(b_i) - X^+_\alpha(a_i)|\}.\end{aligned}$$

Theorem 4.2. *Let $\widetilde{X}_n : [a,b] \to F(R), n = 1,2,3,\cdots$ be a sequence of fuzzy random variables and $E(\widetilde{X}_n)$ the expectation of $E(\widetilde{X}_n)$. If the following conditions are satisfied:*

(1) $\lim_{n\to\infty}\widetilde{X}_n(w)=\widetilde{X}(w)$ *a.e on* $[a,b]$,

(2) $E(\widetilde{X}_n)$ *is absolutely continuous uniformly on* $[a,b]$, *i.e., there* $\eta > 0$ *is*

independently for $n = 1, 2, 3, \cdots$.

Then, $\widetilde{X}$ *is integrable, and furthermore*

$$E(\widetilde{X}) = \lim_{n\to\infty} E(\widetilde{X}_n).$$

Proof. By using Vitali uniformly integrable convergence theorem, according to Definition 3.1, $\widetilde{X}(w)$ is integrable on $[a, b]$. We can show that $E(\widetilde{X})$ is absolutely continuous on $[a, b]$. In fact, let $\varepsilon > 0$ be fixed, by Lemma 3.1, we know that $D(\widetilde{X}(w), 0)$ is Lebesgue integrable on $[a, b]$, and

$$D(\int_{[a,b]} \widetilde{X}(w)dw, \int_{[a,b]} \widetilde{0}dw) \leq (L) \int_{[a,b]} D(\widetilde{X}(w), 0)dw.$$

Because $H(w) = (L)\int_a^w D(\widetilde{X}(t), 0)dt$, as a primitive of $D(\widetilde{X}(w), 0)$ is absolutely continuous on $[a, b]$, then there exists a $\eta > 0$ such that for any finite or infinite sequences of non-overlapping intervals $\{[a_i, b_i]\}$ satisfying $\sum(b_i - a_i) < \eta$, we have

$$\sum_i |H(b_i) - H(a_i)| < \varepsilon.$$

On the other hand,

$$\begin{aligned}\sum D(E(\widetilde{X}(b_i)), E(\widetilde{X}(a_i))) &= \sum D(\int_{a_i}^{b_i} \widetilde{X}(w)dw, \widetilde{0}) \\ &\leq \sum (L) \int_{a_i}^{b_i} D(\widetilde{X}(w), \widetilde{0})dw = \sum |H(b_i) - H(a_i)| < \varepsilon.\end{aligned}$$

It follows that $E(\widetilde{X})$ is absolutely continuous on $[a, b]$. Note now that Lemma 3.1, then $D(\widetilde{X}_n(w), \widetilde{X}(w))$ is measurable on $[a, b]$. Also,

$$\lim_{n\to\infty} \widetilde{X}_n(w) = \widetilde{X}(w) \text{ a.e on } [a, b],$$

by using Egorof Theorem, for above $\eta > 0$, there exists an open set $G = \bigcup_i (a_i, b_i) \subset [a, b]$ satisfying $\sum(b_i - a_i) < \eta, D(\widetilde{X}_n(w), \widetilde{X}(w))$ is convergent to 0 uniformly for $w \in [a, b] \setminus G$. That is, there exists a N such that for any $n > N$, and any $w \in [a, b] \setminus G$, we have

$$D(\widetilde{X}_n(w), \widetilde{X}(w)) < \varepsilon.$$

It follows that

$$\begin{aligned}D(E(\widetilde{X}_n(w)), E(\widetilde{X}(w))) &= D(\int_a^b \widetilde{X}_n(w)dw, \int_a^b \widetilde{X}(w)dw) \\ &= D(\int_{[a,b]\setminus G} \widetilde{X}_n(w)dw, \int_{[a,b]\setminus G} \widetilde{X}(w)dw) + D(\int_G \widetilde{X}_n(w)dw, \int_G \widetilde{X}(w)dw)\end{aligned}$$

$$
\begin{aligned}
&\leq (L)\int_a^b D(\widetilde{X}_n(w),\widetilde{X}(w)) + D(\int_G \widetilde{X}_n(w)dw,\widetilde{0}) + D(\int_G \widetilde{X}(w)dw,\widetilde{0})\\
&\quad \leq (L)\int_{[a,b]\setminus G} D(\widetilde{X}_n(w),\widetilde{X}(w))dw + \sum D(\int_{[a_i,b_i]} \widetilde{X}_n(w)dw,\widetilde{0})\\
&\qquad\qquad + \sum D(\int_{[a_i,b_i]} \widetilde{X}(w)dw,\widetilde{0})\\
&< \varepsilon + \sum D(E(\widetilde{X}_n(b_i)),E(\widetilde{X}_n(a_i))) + \sum D(E(\widetilde{X}(b_i)),E(\widetilde{X}(a_i))) < 3\varepsilon.
\end{aligned}
$$

Hence $E(\widetilde{X}) = \lim_{n\to\infty} E(\widetilde{X}_n)$. The proof is completed. □

Corollary 4.1. (see Puri and Ralescu [6], Yun Kyong Kim and Byung Moon Ghil [5]) *Let*

$$\widetilde{X}_n : [a,b] \to F(R), n = 1,2,3,\cdots,$$

be a sequence of fuzzy random variables and $E(\widetilde{X}_n)$ *the expectation of* $\widetilde{X}_n$. *If the following conditions are satisfied:*

(1) $\lim_{n\to\infty} \widetilde{X}_n(w) = \widetilde{X}(w)$ *a.e on* $[a,b]$,

(2) there exists an (L) integrable function $h(x)$ *such that for any* n, *we have*

$$\|\widetilde{X}(w)\| = D(\widetilde{X}_n(w),\widetilde{0}) \leq h(w).$$

Then, $\widetilde{X}$ *is integrable, and furthermore*

$$E(\widetilde{X}) = \lim_{n\to\infty} E(\widetilde{X}_n).$$

Corollary 4.2. (see Mila Stojakovic [7]) *Let* $\widetilde{X}_n : [a,b] \to F(R), n = 1,2,3,\cdots$, *be a sequence of fuzzy random variables and the expectation of . If*

$$\lim_{n\to\infty} \widetilde{X}_n(w) = \widetilde{X}(w) \text{ uniformly on } [a,b],$$

Then, $\widetilde{X}$ *is integrable, and furthermore*

$$E(\widetilde{X}) = \lim_{n\to\infty} E(\widetilde{X}_n).$$

Example 4.1. Let $\widetilde{u} : R \to [0,1]$ be a triangular fuzzy number, i.e., its membership function is

$$
\widetilde{u}(x) = \begin{cases}
0 & \text{if } x < u_1,\\
\dfrac{x-u_1}{u_2-u_1} & \text{if } u_1 \leq x \leq u_2,\\
\dfrac{u_3-x}{u_3-u_2} & \text{if } u_2 \leq x \leq u_3,\\
0 & \text{if } x > u_3.
\end{cases}
$$

$\widetilde{u}$ is denoted by (u_1,u_2,u_3). If $\widetilde{u},\widetilde{v}$ are two triangular fuzzy numbers, Then:

(1) $u_\alpha = [(u_2 - u_1)\alpha + u_1, (u_2 - u_3)\alpha + u_3]$;

(2) $\widetilde{u} + \widetilde{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$;

(3) $k\widetilde{u} = \begin{cases} (ku_1, ku_2, ku_3) & \text{if } k \geq 0, \\ (ku_3, ku_2, ku_1) & \text{if } k < 0. \end{cases}$

Let $\widetilde{X} : [a, b] \to F(R)$ be fuzzy random variable on $[a, b]$ assuming values on the set of triangular fuzzy numbers. We write $\widetilde{X}(w) = (X_1(w), X_2(w), X_3(w))$. Then

$$X_\alpha^-(w) = (X_2(w) - X_1(w))\alpha + X_1(w),$$
$$X_\alpha^+(w) = (X_2(w) - X_3(w))\alpha + X_3(w),$$

are random variables for all $\alpha \in [0, 1]$. From the definition of the fuzzy expectation of $\widetilde{X}(w)$, we have

$$(E(\widetilde{X}))_\alpha = [(E(X_2) - E(X_1))\alpha + E(X_1), (E(X_2) - E(X_3))\alpha + E(X_3)].$$

In other words, $E(\widetilde{X})$ is also a triangular fuzzy number with

$$E(\widetilde{X}) = (E(X_1), E(X_2), E(X_3)).$$

Let $\widetilde{X}_n : [a, b] \to F(R), n = 1, 2, 3, \cdots$, be a sequence of fuzzy random variables on $[a, b]$ assuming values on the set of triangular fuzzy numbers and satisfy the conditions of Theorem 4.1. By the difference from the convergence theorems of the classical random variables, we can see that Theorem 4.1 is a real extension of the Corollaries 4.1 and 4.2.

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