International Journal of Pure and Applied Mathematics

Volume 54 No. 4 2009, 521-541

MATLAB APPLICATIONS OF THE OSCILLATIONS OF FORCED NEUTRAL DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

M. Maria Susai Manuel[1 §], A. George Maria Selvam[2], M. Paul Loganathan[3]

[1,2]Department of Mathematics
Sacred Heart College
Vellore Dist., Tirupattur, 635 601, Tamil Nadu, INDIA

[3]Governmental High School, Oddappatti
Krishnagiri, 635 203, Tamil Nadu, INDIA

e-mail: manuelmsm_03@yahoo.co.in

Abstract: In this paper, the authors investigate the oscillation of forced neutral difference equation with positive and negative coefficients of the form

$$\Delta(x_n - c_n x_{n-r}) + p_n x_{n-k} - q_n x_{n-\ell} = e_n, \quad n \geq n_0$$

where p_n, q_n, c_n, e_n $(n = 0, 1, 2 \ldots)$ are real numbers with $p_n \geq 0$, $q_n \geq 0$, $c_n \geq 0$, $e_n \in \mathbb{R}, k, \ell$ and r are integers with $0 \leq \ell \leq k-1, r > 0, h_n = p_n - q_{n-k+\ell} \geq 0$ and not identically zero. Suitable examples are provided to illustrate the results and the trajectories of solutions are obtained using *MATLAB*.

AMS Subject Classification: 39A10, 39A11

Key Words: difference equations, forced, neutral, oscillation, positive and negative coefficients

1. Introduction

Consider the forced neutral difference equations with positive and negative coefficients

$$\Delta(x_n - c_n x_{n-r}) + p_n x_{n-k} - q_n x_{n-\ell} = e_n, \quad n \geq n_0, \tag{1}$$

where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ and p_n, q_n, c_n, e_n $(n = 0, 1, 2 \ldots)$ are real numbers with $p_n \geq 0, q_n \geq 0, c_n \geq 0, e_n \in \mathbb{R}$, k, ℓ

Received: June 10, 2009

§Correspondence author

and r are integers with $0 \le \ell \le k-1, r > 0$.

The problem of determining oscillation and non-oscillation of solutions of difference equations has been a very active area of research in the last two decades and for a survey of recent results, we refer the reader to the monographs [1, 2, 5]. A number of dynamical behaviors of solutions of difference equations is possible; here we will only be concerned with conditions which are sufficient for all solutions of (1) to be oscillatory.

In the absence of the forcing term e_n, equation (1) becomes

$$\Delta(x_n - c_n x_{n-r}) + p_n x_{n-k} - q_n x_{n-\ell} = 0. \tag{2}$$

The oscillatory behavior of equation (2) has been investigated by several authors, see for example [10]-[12], [8], [9]. In [12], the authors established several sufficient conditions for the oscillation of all solutions of (2), assuming

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s \le 1. \tag{3}$$

In [8], the authors studied the oscillatory behavior of equation (2) without the assumption (3) and obtained some general results.

Several authors [3, 11] considered the case where $c_n \equiv 0$ and $c_n = c$ with $0 \le c < 1$. For more general case where c_n is not identically constant, considerably less is known about the oscillatory behavior of the solutions of equation (2).

Our objective in this paper is to study the oscillation of equation (1) without the usual hypothesis (3).

In Section 2, we establish some new sufficient conditions for the oscillation of all solutions of equation (1). Our results improve the known results in the literature, which also can be applied to yield a sufficient and necessary condition for the oscillation of the neutral difference equation

$$\Delta(x_n - x_{n-r}) + cn^{-\beta} x_{n-k} = 0, \quad n > 0. \tag{4}$$

In Section 3, we obtain the existence results for positive solution of equation (1). We establish the sufficient condition for the existence of bounded positive solution under the following hypothesis

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s \equiv 1. \tag{5}$$

For the case $q_n \equiv 0$, our results improve the results in [12].

In Section 4, sufficient examples are provided to illustrate the results obtained in Section 3 and verification of the conditions and the behavior of the solutions were tested using *MATLAB* and trajectories of the solutions are obtained using *MATLAB*.

Let $m^* = \max\{k, r\}$. By a solution of equation (1), we mean a sequence $\{x_n\}$ of real numbers which is defined for $n \geq -m^*$ and satisfies equation (1) for $n = 0, 1, 2 \dots$. It is easy to see under the initial conditions

$$x_n = b_n, \quad n = -m^*, -m^* + 1, \dots, -1\,,$$

equation (1) has a unique solution satisfying the above equation.

As is customary, a nontrivial solution $\{x_n\}$ of (1) is said to be non-oscillatory if the terms x_n are either eventually positive or eventually negative. Otherwise it is said to be oscillatory. Throughout this paper, we assume

$$m^* = \begin{cases} r & \text{when } q_n \equiv 0, \\ \max\{r, k\} & \text{otherwise,} \end{cases}$$

$$m = \begin{cases} r & \text{when } q_n \equiv 0, \\ \min\{\ell + 1, r\} & \text{otherwise,} \end{cases}$$

and the following conditions hold:

$$\begin{cases} h_n = p_n - q_{n-k+\ell} \geq 0, & \text{and not identically zero,} \\ \left| \sum\limits_{s=n}^{\infty} e_s \right| < \infty. \end{cases} \tag{6}$$

2. Some Basic Lemmas

In this section, first we shall prove some basic lemmas which we will use in the proofs of the theorems given in Section 3. For $n \geq n_0$, we consider the following difference inequalities

$$\Delta\left(x_n - c_n x_{n-r} + \sum_{s=n}^{\infty} e_s\right) + p_n x_{n-k} - q_n x_{n-\ell} \leq 0, \tag{7}$$

$$\Delta\left(x_n - c_n x_{n-r} + \sum_{s=n}^{\infty} e_s\right) + p_n x_{n-k} - q_n x_{n-\ell} \geq 0. \tag{8}$$

Lemma 2.1. *Assume that condition (6) holds and*

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s \leq 1 \tag{9}$$

for large n. Let

$$z_n = x_n - c_n x_{n-r} - \sum_{s=n-k+\ell}^{n-1} q_s x_{s-\ell} + \sum_{s=n}^{\infty} e_s. \tag{10}$$

Then, the following statements are true.

(i) If $\{x_n\}$ is an eventually positive solution of (7), then we have $\Delta z_n \le 0$ and $z_n > 0$ for large n.

(ii) If $\{x_n\}$ is an eventually negative solution of (8), then we have $\Delta z_n \ge 0$ and $z_n < 0$ for large n.

Proof. (i) Let n_1 be a positive integer such that $x_{n-m^*} \ge 0$, for $n \ge n_1$. Using (10), (7) becomes

$$\Delta\left(z_n + \sum_{s=n-k+\ell}^{n-1} q_s x_{s-\ell}\right) + p_n x_{n-k} - q_n x_{n-\ell} \le 0\,,$$
$$\Delta z_n + (p_n - q_{n-k+\ell}) x_{n-k} \le 0\,,$$
$$\Delta z_n + h_n x_{n-k} \le 0\,,$$

which implies

$$\Delta z_n \le 0, n \ge n_1.$$

Hence, z_n is non increasing for $n \ge n_1$. We assume the contrary. Let $z_n < 0$. So there exist an integer $n_2 \ge n_1$ and $\alpha > 0$ such that $z_n \le -\alpha$ for $n \ge n_2$, that is

$$x_n \le -\alpha + c_n x_{n-r} + \sum_{s=n-k+\ell}^{n-1} q_s x_{s-\ell} - \sum_{s=n}^{\infty} e_s, \quad n \ge n_2. \tag{11}$$

We consider the following two possible cases.

Case (i) Assume that $\{x_n\}$ is unbounded, i.e., $\limsup_{n\to\infty} x_n = \infty$. Thus, there exists a sequence of points $\{s_i\}_{i=1}^{\infty}$ such that $s_i \ge n_2 + m^*$, $i = 1, 2, 3, \ldots$, $s_i \to \infty$, $x_{s_i} \to \infty$ as $i \to \infty$ and $x_{s_i} = \max\{x_n : n_2 \le n \le s_i\}, i = 1, 2, \ldots$.

From (9) and (11), we find that

$$x_{s_i} \le -\alpha + c_{s_i} x_{s_i - r} + \sum_{s=s_i-k+\ell}^{s_i-1} q_s x_{s-\ell} - \sum_{s=s_i}^{\infty} e_s \le -\alpha + x_{s_i} - \sum_{s=s_i}^{\infty} e_s\,,$$
$$x_{s_i} \le -\alpha + x_{s_i}\,,$$
$$\alpha \le 0,$$

which is a contradiction.

Case (ii) Assume that $\{x_n\}$ is bounded. Let $\limsup_{n\to\infty} x_n = a < \infty$. Choose a sequence $\{\overline{s}_i\}_{i=1}^{\infty}$ such that $\overline{s}_i \to \infty$ and $x_{\overline{s}_i} \to a$ as $i \to \infty$. Let ξ_i be such that $x_{\xi_i} = \max\{x_s : \overline{s}_i - m^* \le s \le \overline{s}_i - m\}$, $\overline{s}_i - m^* \le \xi_i \le \overline{s}_i - m$, $i = 1, 2, \ldots$. Then $\xi_i \to \infty$ as $i \to \infty$ and $\limsup_{i\to\infty} x_{\xi_i} \le a$.

From (11), we get

$$x_{\overline{s}_i} \le -\alpha + c_{\overline{s}_i} x_{\overline{s}_i - r} + \sum_{s=\overline{s}_i-k+\ell}^{\overline{s}_i-1} q_s x_{s-\ell} - \sum_{s=\overline{s}_i}^{\infty} e_s,$$

which yields on applying (9),

$$\begin{aligned} &\le -\alpha + c_{\overline{s}_i} x_{\xi_i} + \sum_{s=\overline{s}_i-k+\ell}^{\overline{s}_i-1} q_s x_{\xi_i} - \sum_{s=\overline{s}_i}^{\infty} e_s \\ &\le -\alpha + x_{\xi_i} - \sum_{s=\overline{s}_i}^{\infty} e_s, \\ x_{\overline{s}_i} &\le -\alpha + x_{\xi_i}. \end{aligned}$$

Taking the limit superior as $i \to \infty$, we obtain

$$\begin{aligned} a &\le -\alpha + \limsup_{i\to\infty} x_{\xi_i}, \\ a &\le -\alpha + a, \\ \alpha &\le 0, \end{aligned}$$

which is a contradiction. Thus the proof of (i) is complete.

(ii) Let n_1 be a positive integer such that $x_{n-m^*} < 0$, for $n \ge n_1$. Using (10), (8) becomes

$$\Delta z_n \ge -h_n x_{n-k} \ge 0$$

which implies

$$\Delta z_n \ge 0, \quad n \ge n_1.$$

Hence, z_n is non decreasing for $n \ge n_1$. We assume the contrary. Let $z_n > 0$. So there exist an integer $n_2 \ge n_1$ and $\mu > 0$ such that $z_n \ge \mu$ for $n \ge n_2$, that is

$$x_n \ge \mu + c_n x_{n-r} + \sum_{s=n-k+\ell}^{n-1} q_s x_{s-\ell} - \sum_{s=n}^{\infty} e_s, n \ge n_2. \tag{12}$$

We consider the following two possible cases.

Case (i) Assume that $\{x_n\}$ is unbounded, i.e., $\liminf_{n\to\infty} x_n = -\infty$. Thus, there exists a sequence of points $\{s_i\}_{i=1}^{\infty}$ such that $s_i \ge n_2 + m^*$, $i = 1, 2, 3, \ldots$,

$s_i \to \infty$, $x_{s_i} \to -\infty$ as $i \to \infty$ and $x_{s_i} = \min\{x_n : n_2 \le n \le s_i\}$, $i = 1, 2, \ldots$.

From (9) and (12), we find that

$$\begin{aligned} x_{s_i} &\ge \mu + c_{s_i} x_{s_i - r} + \sum_{s=s_i-k+\ell}^{s_i-1} q_s x_{s-\ell} - \sum_{s=s_i}^{\infty} e_s, \\ &\ge \mu + x_{s_i} - \sum_{s=s_i}^{\infty} e_s, \\ x_{s_i} &\ge \mu + x_{s_i}, \\ \mu &\ge 0, \end{aligned}$$

which is a contradiction.

Case (ii) Assume that $\{x_n\}$ is bounded. Let $\lim\inf_{n\to\infty} x_n = L \in (-\infty, 0]$. Choose a sequence $\{\overline{s}_i\}_{i=1}^{\infty}$ such that $\overline{s}_i \to \infty$ and $x_{\overline{s}_i} \to L$ as $i \to \infty$. Let ξ_i be such that $x_{\xi_i} = \min\{x_s : \overline{s}_i - m^* \le s \le \overline{s}_i - m\}$, $\overline{s}_i - m^* \le \xi_i \le \overline{s}_i - m$, $i = 1, 2, \ldots$. Then $\xi_i \to \infty$ as $i \to \infty$ and $\lim\inf_{i\to\infty} x_{\xi_i} \ge L$.

From (12), we get

$$x_{\overline{s}_i} \ge \mu + c_{\overline{s}_i} x_{\overline{s}_i - r} + \sum_{s=\overline{s}_i-k+\ell}^{\overline{s}_i-1} q_s x_{s-\ell} - \sum_{s=\overline{s}_i}^{\infty} e_s,$$

which yields on applying (9),

$$\begin{aligned} &\ge \mu + c_{\overline{s}_i} x_{\xi_i} + \sum_{s=\overline{s}_i-k+\ell}^{\overline{s}_i-1} q_s x_{\xi_i} - \sum_{s=\overline{s}_i}^{\infty} e_s \\ &\ge \mu + x_{\xi_i} - \sum_{s=\overline{s}_i}^{\infty} e_s, \\ x_{\overline{s}_i} &\le \mu + x_{\xi_i}. \end{aligned}$$

Taking the limit inferior as $i \to \infty$, we obtain

$$\begin{aligned} L &\ge \mu + \lim_{i\to\infty} \inf x_{\xi_i}, \\ L &\ge \mu + L, \\ \mu &\ge 0, \end{aligned}$$

which is a contradiction. Thus the proof of (ii) is complete.

This completes the proof. □

Lemma 2.2. *Assume that (6) holds, and*

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s \geq 1. \tag{13}$$

Let $\{z_n\}$ be defined by (10). Then we have:

(i) If $\{x_n\}$ is an eventually positive solution of inequality (7) and the following second order difference inequality

$$\Delta^2 y_n + \frac{1}{m^*} h_n y_n \leq 0 \tag{14}$$

has no eventually positive solution, then eventually $\Delta z_n \leq 0$ and $z_n < 0$ for large n.

(ii) If $\{x_n\}$ is an eventually negative solution of inequality (8) and the following second order difference inequality

$$\Delta^2 y_n + \frac{1}{m^*} h_n y_n \geq 0 \tag{15}$$

has no eventually negative solution, then eventually $\Delta z_n \geq 0$ and $z_n > 0$ for large n.

Proof. We give the proof only in the case when $q_n \not\equiv 0$. From (7) and (10), we have eventually

$$\Delta z_n \leq -(p_n - q_{n-k+\ell}) x_{n-k} = -h_n x_{n-k} \leq 0. \tag{16}$$

Assume on the contrary, $z_n \geq 0$. Let n_1 be a positive integer such that

$$x_{n-m^*} > 0, \quad z_n > 0, \quad \Delta z_n \leq 0, \quad n \geq n_1.$$

Set $M = \min\{x_{n_1-m^*},\, x_{n_1-m^*+1}, \ldots, x_{n_1}\}$. Then for $n_1 \leq n \leq n_1 + m$, from (10) and (13) we obtain

$$\begin{aligned} x_n \geq & c_n M + \sum_{s=n-k+\ell}^{n-1} q_s M - \sum_{s=n}^{\infty} e_s, \\ x_n \geq & \left(c_n + \sum_{s=n-k+\ell}^{n-1} q_s \right) M - \sum_{s=n}^{\infty} e_s, \\ x_n \geq & M - \sum_{s=n}^{\infty} e_s. \end{aligned}$$

Suppose this is true for it

$$x_n \geq M - i \sum_{s=n}^{\infty} e_s, \; n_1 + (i-1)m \leq n \leq n_1 + im.$$

By induction

$$\begin{aligned} x_n \geq & c_n x_{n-r} + \sum_{s=n-k+\ell}^{n-1} q_s x_{s-\ell} - \sum_{s=n}^{\infty} e_s \\ \geq & c_n \left(M - i \sum_{s=n-r}^{\infty} e_s \right) + \sum_{s=n-k+\ell}^{n-1} q_s \left(M - i \sum_{t=s-\ell}^{\infty} e_t \right) - \sum_{s=n}^{\infty} e_s. \end{aligned}$$

Replace t by s, s by n, ℓ by r. We get

$$\begin{aligned} \geq & \left(c_n + \sum_{s=n-k+\ell}^{n-1} q_s \right) \left(M - i \sum_{s=n-r}^{\infty} e_s \right) - \sum_{s=n}^{\infty} e_s \\ \geq & M - i \sum_{s=n}^{\infty} e_s - \sum_{s=n}^{\infty} e_s \\ \geq & M - (i+1) \sum_{s=n}^{\infty} e_s. \end{aligned}$$

In general

$$x_n \geq M - i \sum_{s=n}^{\infty} e_s, \ n_1 + (i-1)m \leq n \leq n_1 + im, \quad i = 1, 2, \dots.$$

It follows that

$$x_n \geq M, \quad n \geq n_1 - m^*. \tag{17}$$

Let $\lim_{n\to\infty} z_n = \beta$. Then there exist two possible cases.

Case (i) Assume that $\beta = 0$. Let $n_2 > n_1$ be an integer such that $z_n < \frac{M}{2}$ for $n > n_2$. Then for any integer $n_2' > n_2$, we have

$$x_n \geq \frac{1}{m^*} \sum_{s=n_2'}^{n+m^*-1} z_s, n_2' \leq n \leq n_2' + m^*.$$

Case (ii) Assume that $\beta > 0$. Noting that $\Delta z_n \leq 0$ for $n \geq n_1$, we have $z_n \geq \alpha$ for $n \geq n_1$. From (10), (13) and (17), we get

$$\begin{aligned} x_n \geq \beta + c_n x_{n-r} + \sum_{s=n-k+\ell}^{n-1} q_s x_{s-l} - \sum_{s=n}^{\infty} e_s, n \geq n_1 \geq \beta + c_n M + \sum_{s=n-k+\ell}^{n-1} q_s M \\ - \sum_{s=n}^{\infty} e_s \geq \beta + \left(c_n + \sum_{s=n-k+\ell}^{n-1} q_s \right) M - \sum_{s=n}^{\infty} e_s, \end{aligned}$$

$$x_n \geq \beta + M - \sum_{s=n}^{\infty} e_s.$$

By induction, we have

$$x_n \geq i\beta + M - i\sum_{s=n}^{\infty} e_s, \qquad n \geq n_1 + (i-1)m^*, \qquad i = 1, 2, \dots,$$

and so $\lim_{n\to\infty} x_n = \infty$, which implies that there exist an integer $n_3 \geq n_2$ such that

$$x_n \geq \frac{1}{m^*} \sum_{s=n_3}^{n+m^*-1} z_s, \quad n_3 \leq n \leq n_3 + m^*.$$

Combining Cases (i) and (ii), we know that there is an integer $N > n_2$ such that

$$x_n \geq \frac{1}{m^*} \sum_{s=N}^{n+m^*-1} z_s, \quad N \leq n \leq N + m^*. \tag{18}$$

For $N + m^* \leq n \leq N + m^* + m$, by (10), (13) and (18)

$$\begin{aligned} x_n \geq& z_n + c_n \left(\frac{1}{m^*} \sum_{s=N}^{n-1} z_s \right) + \sum_{s=n-k+\ell}^{n-1} q_s \left(\frac{1}{m^*} \sum_{s=N}^{n-1} z_s \right) - \sum_{s=n}^{\infty} e_s \\ \geq& \frac{1}{m^*} \sum_{s=n}^{n+m^*-1} z_s + \left(c_n + \sum_{s=n-k+\ell}^{n-1} q_s \right) \frac{1}{m^*} \sum_{s=N}^{n-1} z_s - \sum_{s=n}^{\infty} e_s \\ \geq& \frac{1}{m^*} \sum_{s=N}^{n+m^*-1} z_s - \sum_{s=n}^{\infty} e_s. \end{aligned}$$

By induction, we can show in general that

$$x_n \geq \frac{1}{m^*} \sum_{s=N}^{n+m^*-1} z_s - i\sum_{s=n}^{\infty} e_s, \; N+m^*+(i-1)m \leq n \leq n_1+m^*+im, \quad i = 1, 2, \dots$$

Hence, $x_n \geq \frac{1}{m^*} \sum_{s=N}^{n+m^*-1} z_s$, for $n \geq N$, which implies

$$x_{n-k} \geq \frac{1}{m^*} \sum_{s=N}^{n-1} z_s, \quad n \geq N + k. \tag{19}$$

Let $y_n = \sum_{s=N}^{n-1} z_s$, then $\Delta y_n = z_n$, $\Delta^2 y_n = \Delta z_n$, $n \geq N$. From (16) and (19), we obtain

$$\begin{aligned} \Delta z_n + h_n x_{n-k} \leq& 0, \\ \Delta^2 y_n + h_n \left(\frac{1}{m^*} \sum_{s=N}^{n+m^*-1} z_s \right) \leq& 0, \end{aligned}$$

$$\Delta^2 y_n + \frac{1}{m^*} h_n y_n \leq 0, \quad n \geq N + k. \tag{20}$$

We know that every solution of (14) is oscillatory if and only if the inequality (20) has no eventually positive solution. Thus (20) contradicts the assumption that every solution of (14) oscillates. Therefore, the proof of (i) is complete.

In Case (ii) suppose that $\{x_n\}$ is an eventually negative solution of (8). Similarly we can prove that $\Delta z_n \geq 0$ and $z_n > 0$, for large n. □

Now, we are in a position to give new sufficient condition for the oscillation of all solution of equation (1).

3. Main Results

Theorem 3.1. *Assume that conditions (6), (9) and (13) hold. Further assume that every solution of (14) and (15)oscillates. Then every solution of (1) oscillates.*

Proof. In fact, if $\{x_n\}$ is a positive solution of (1), then the conditions of Theorem 3.1 and Lemma 2.1 imply eventually that $z_n > 0$, while Lemma 2.2 imply that $z_n < 0$.

This contradiction shows that $\{x_n\}$ cannot eventually be a positive solution of (1).

On the other hand, if $\{x_n\}$ is a negative solution of (1), then Lemma 2.1 implies that eventually $z_n < 0$, while Lemma 2.2 implies that eventually $z_n > 0$.

This contradiction shows that $\{x_n\}$ cannot be a negative solution of (1). Therefore, every solution of equation (1) oscillates. □

The following result is an immediate consequence of Lemmas 2.1 and 2.2.

Assume that (5) and (6) hold, and

$$\liminf_{n\to\infty} n \sum_{s=n}^{\infty} (p_s - q_{s-k+\ell}) > \frac{m^*}{4}. \tag{21}$$

Then every solution of (1) oscillates.

Theorem 3.2. *Assume that (6), (13) and (21) hold, and suppose that* $\left\{\frac{q_n}{(p_{n+k-\ell} - q_n)}\right\}$ *is non decreasing and*

$$c_{n-k}(p_n - q_{n-k+\ell}) \leq \lambda_1 (p_{n-r} - q_{n-k+\ell-r}), \tag{22}$$

$$(p_n - q_{n-k+\ell}) q_{n-k} \leq \lambda_2 (p_{n-\ell} - q_{n-k}), \tag{23}$$

where λ_1, λ_2 *are nonnegative constants with* $\lambda_1 + \lambda_2(k - \ell) = 1$, *then every* *solution of equation (1) oscillates.*

Proof. If the above conclusion does not hold, equation (1) has an eventually positive solution $\{x_n\}$ and let z_n be defined by (10). From Lemma 2.2, we have $\Delta z_n \leq 0$ and $z_n < 0$ eventually. From Lemma 2.1, we have

$$\Delta z_n = -h_n x_{n-k}.$$

In view of (22) and (23), we obtain

$$\Delta z_n \geq -h_n z_{n-k} + \lambda_1 \Delta z_{n-r} + \lambda_2 (z_{n-\ell} - z_{n-k}) + h_n \sum_{s=n-k}^{\infty} e_s.$$

That is,

$$\Delta(z_n - \lambda_1 z_{n-r}) + (h_n + \lambda_2) z_{n-k} - \lambda_2 z_{n-\ell} - h_n \sum_{s=n-k}^{\infty} e_s \geq 0,$$

$$\Delta(z_n - \lambda_1 z_{n-r}) + (h_n + \lambda_2) z_{n-k} - \lambda_2 z_{n-\ell} - \Delta \left(\sum_{j=n_0}^{n-1} h_j \sum_{s=n-k}^{\infty} e_s \right) \geq 0, \ n \geq n_0,$$

$$\Delta \left(z_n - \lambda_1 z_{n-r} - \sum_{j=n_0}^{n-1} h_j \sum_{s=n-k}^{\infty} e_s \right) + (h_n + \lambda_2) z_{n-k} - \lambda_2 z_{n-\ell} \geq 0,$$

which implies $\{-z_n\}$ is a positive solution of the inequality

$$\Delta \left(u_n - \lambda_1 u_{n-r} - \sum_{j=n_0}^{n-1} h_j \sum_{s=n-k}^{\infty} e_s \right) + (h_n + \lambda_2) u_{n-k} - \lambda_2 u_{n-\ell} \leq 0$$

which yields a contradiction by Lemmas 2.1 and 2.2. The proof is complete. □

Theorem 3.3. *Suppose there exists a number $n \in \{0, 1, \cdots, k-\ell\}$ such that (5) holds eventually. Suppose further that (21) holds and that*

$$c_{n-k}(p_n - q_{n-k+\ell}) \geq (p_{n-r} - q_{n-k+\ell-r}) \tag{24}$$

for large n. Then every solution of (1) oscillates.

Proof. Suppose to contrary that $\{x_n\}$ is an eventually positive solution of (1). Then, in view of Lemma 2.1, we have $z_n > 0$ and $\Delta z_n \leq 0$ for large n. By Lemma 2.1, we have

$$\begin{aligned}\Delta z_n &= -h_n x_{n-k} \\ &= -h_n \left(z_{n-k} + c_{n-k} x_{n-r-k} + \sum_{s=n-k+\ell}^{n-1} q_{s-k} x_{s-k-\ell} - \sum_{s=n-k}^{\infty} e_s \right)\end{aligned}$$

$$= -h_n\left(z_{n-k} + c_{n-k}x_{n-r-k}\right) - h_n\left(\sum_{s=n-k+\ell}^{n-1} q_{s-k}x_{s-k-\ell} - \sum_{s=n-k}^{\infty} e_s\right),$$

$$\Delta z_n + h_n\left(z_{n-k} + c_{n-k}x_{n-r-k}\right) = -h_n\left(\sum_{s=n-k+\ell}^{n-1} q_{s-k}x_{s-k-\ell} - \sum_{s=n-k}^{\infty} e_s\right).$$

Hence

$$\Delta z_n + h_n\left(z_{n-k} + c_{n-k}x_{n-r-k}\right) \le 0 \tag{25}$$

for large n. Using (24), we obtain

$$\begin{aligned} \Delta z_n + h_n z_{n-k} &\le - h_n c_{n-k}x_{n-r-k}, \\ \Delta z_n - \Delta z_{n-r} + h_n z_{n-k} &\le \left(h_{n-r} - h_n c_{n-k}\right)x_{n-r-k}, \\ \Delta\left(z_n - z_{n-r}\right) + h_n z_{n-k} &\le \left(h_{n-r} - h_n c_{n-k}\right)x_{n-r-k}, \\ \Delta\left(z_n - z_{n-r}\right) + h_n z_{n-k} &\le 0, \end{aligned}$$

which gives $\{z_n\}$ is an eventually positive solution of the inequality

$$\Delta\left(z_n - z_{n-r}\right) + h_n z_{n-k} \le 0,$$

which yields a contradiction by Lemmas 2.1 and 2.2.

The proof is now complete. □

Theorem 3.4. *Suppose there exists a number $n \in \{0, 1, \cdots, k-\ell\}$ such that (5) holds eventually. Further, assume that there exists a constant $c \in [0, 1)$ such that*

$$c_{n-k}(p_n - q_{n-k+\ell}) \ge c(p_{n-r} - q_{n-k+\ell-r}) \tag{26}$$

for large n. Then every solution of (1) is oscillatory provided that the following inequality

$$\Delta u_n + \frac{c\left(1 - c^{i+1}\right)}{1-c} h_n u_{n-k-r} \le 0, \quad n = 0, 1, 2, \cdots, \tag{27}$$

does not have an eventually positive solution.

Proof. Suppose to contrary that $\{x_n\}$ is an eventually positive solution of (1). Then, in view of Lemma 2.1, we have $z_n > 0$ and $\Delta z_n \le 0$ for large n. By Lemma 2.1, we have

$$\begin{aligned} \Delta z_n &= -h_n x_{n-k} \\ &= -h_n\left(z_{n-k} + c_{n-k}x_{n-r-k} + \sum_{s=n-k+\ell}^{n-1} q_{s-k}x_{s-k-\ell} - \sum_{s=n-k}^{\infty} e_s\right) \end{aligned}$$

$$= -h_n\left(z_{n-k} + c_{n-k}x_{n-r-k}\right) - h_n\left(\sum_{s=n-k+\ell}^{n-1} q_{s-k}x_{s-k-\ell} - \sum_{s=n-k}^{\infty} e_s\right),$$

$$\Delta z_n + h_n\left(z_{n-k} + c_{n-k}x_{n-r-k}\right) = -h_n\left(\sum_{s=n-k+\ell}^{n-1} q_{s-k}x_{s-k-\ell} - \sum_{s=n-k}^{\infty} e_s\right).$$

Hence

$$\Delta z_n + h_n\left(z_{n-k} + c_{n-k}x_{n-r-k}\right) \le 0 \tag{28}$$

for large n.

That is

$$\Delta z_n + h_n z_{n-k} \le -h_n c_{n-k} x_{n-r-k}.$$

From the definition of h_n and (26), we get $h_n c_{n-k} \ge c h_{n-r}$. Now (28) yields

$$\Delta z_n + h_n z_{n-k} \le c\Delta z_{n-r}.$$

Which yields

$$\Delta\left(z_n - c z_{n-r}\right) + h_n z_{n-k} \le 0, \tag{29}$$

This is true for large n.

Let $u_n = z_n - c z_{n-m}$. Similar to the proof of Lemma 2.1, we have $u_n > 0$ and$\Delta u_n \le 0$ for all large n. Hence, there exists an integer $N > 0$ such that $z_n > 0$ and $\Delta z_n \le 0$ and $u_n > 0$ and $\Delta u_n \le 0$ for $n \ge N$. Thus

$$\begin{aligned} z_n =& u_n + c z_{n-r} \\ =& u_n + c\left(u_{n-r} + c z_{n-2r}\right) \\ =& u_n + c u_{n-r} + c^2 z_{n-2r}. \end{aligned}$$

Proceeding in this fashion, we obtain

$$\begin{aligned} z_n = u_n + c u_{n-r} + c^2 u_{n-2r} + \ldots + c^i u_{n-ir} + c^{i+1} z_{n-(i+1)r} \\ \ge \left(c + c^2 + \ldots + c^{i+1}\right) u_{n-r}, \end{aligned}$$

$$z_n \ge \frac{c\left(1 - c^{i+1}\right)}{1 - c} u_{n-r},$$

for $n \ge (i+1)m + N$. From (29), we obtain

$$\Delta u_n + \frac{c\left(1 - c^{i+1}\right)}{1 - c} h_n u_{n-k-r} \le 0, \quad n = 0, 1, 2, \cdots$$

holds for large n, which is contrary to the hypothesis that (27) has no eventually positive solution. The proof is now complete. □

4. Examples

In this section we provide examples to illustrate the results obtained in the previous section. The examples are accompanied by graphs using *MATLAB*.

Example 4.1. Consider the difference equation

$$\Delta\left(x_n - \frac{1}{2}x_{n-1}\right) + \left(\frac{1}{2} + n^{-\beta}\right)x_{n-2} - \frac{1}{2}x_{n-1} = -ne^{-n}, \quad n > 0. \tag{30}$$

Comparing with equation (1), we get $r = 1,\ k = 2,\ \ell = 1,\ c_n = \frac{1}{2},\ p_n = \frac{1}{2} + n^{-\beta}$, $q_n = \frac{1}{2}$ and $e_n = -ne^{-n}$. Now

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s = \frac{1}{2} + \sum_{s=n-2+1}^{n-1} \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

and

$$h_n = p_n - q_{n-k+\ell} = \frac{1}{2} + n^{-\beta} - \frac{1}{2} = n^{-\beta}.$$

Here $m^* = \max(2,1) = 2$. It is easy to see that all conditions of Theorem 3.1 are satisfied when $-\infty < \beta < 2$ and $m^* < 4$. Thus every solution of equation (30) oscillates.

When $\beta \geq 2$, a solution of equation (30) is non-oscillatory and positive.

Example 4.2. Consider the difference equation

$$\Delta\left(x_n - 0.8x_{n-1}\right) + \left(0.2 + \frac{1}{(n+1)^\beta}x_{n-2}\right) - 0.2x_{n-1} = \frac{1}{n^2}, \quad n > 0. \tag{31}$$

Comparing with equation (1), we get $r = 1,\ k = 2,\ \ell = 1,\ c_n = 0.8,\ p_n = 0.2 + \frac{1}{(n+1)^\beta}$, $q_n = 0.2$ and $e_n = \frac{1}{n^2}$. Now

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s = 0.8 + \sum_{s=n-2+1}^{n-1} 0.2 = 1$$

and

$$h_n = p_n - q_{n-k+\ell} = 0.2 + \frac{1}{(n+1)^\beta} - 0.2 = \frac{1}{(n+1)^\beta}.$$

Here $m^* = \max(2,1) = 2$. It is easy to see that all conditions of Theorem 3.1 are satisfied when $-\infty < \beta < 2$ and $m^* < 4$. Thus every solution of equation (31) oscillates.

When $\beta \geq 2$, a solution of equation (31) is non-oscillatory and negative.

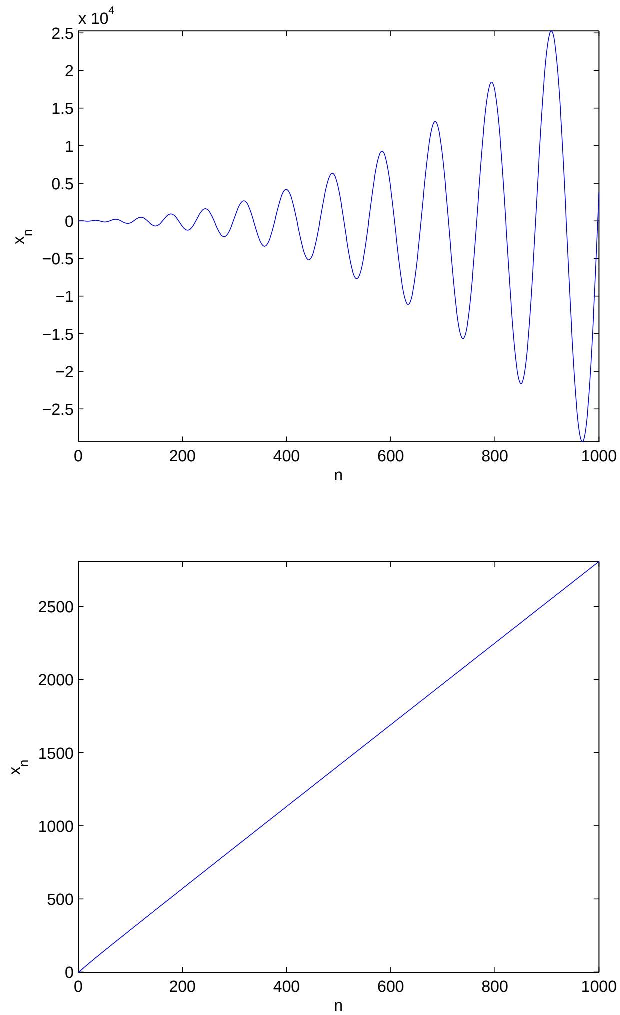

Figure 1: The trajectory solutions x_n of (30) with the initial conditions $x_0 = -2, x_3 = 3$ for $\beta = 0.8$ and $\beta = 2.8$

Example 4.3. Consider the difference equation

$$\Delta\left(x_n - \frac{1}{2}x_{n-1}\right) + \left(\frac{1}{2}x_{n-2}\right) - \left(\frac{1}{2} - n^{-\beta}\right)x_{n-1} = \frac{n!}{n^n}, \quad n > 0. \tag{32}$$

Comparing with equation (1), we get $r = 1,\ k = 2,\ \ell = 1,\ c_n = \frac{1}{2},\ p_n = \frac{1}{2},$

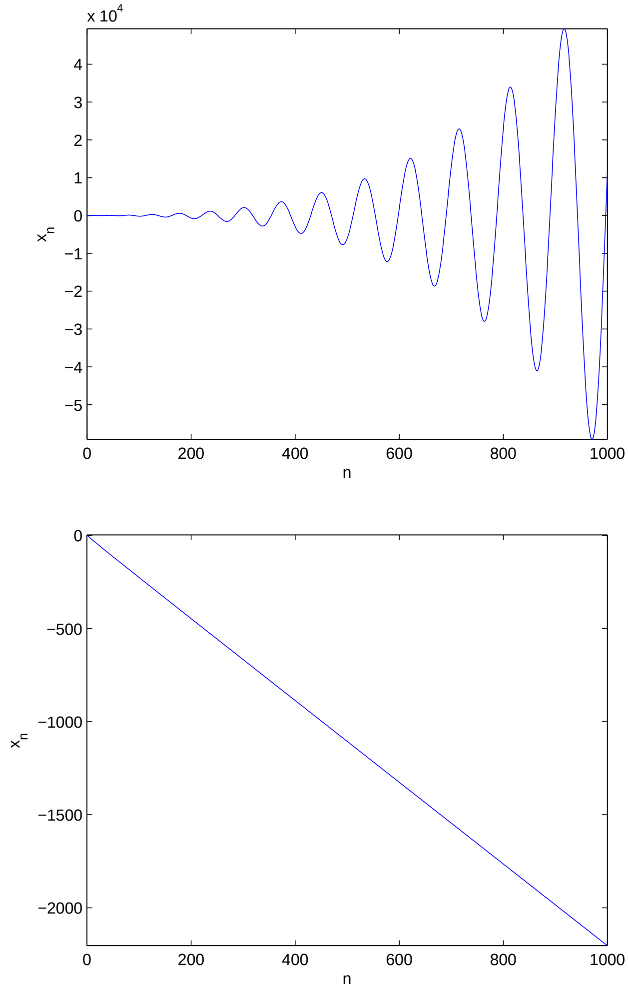

Figure 2: The trajectory solutions x_n of (31) with the initial conditions $x_0 = -10, x_1 = 3, x_2 = -3$ for $\beta = 0.8$ and $\beta = 3$

$q_n = \frac{1}{2} - n^{-\beta}$ and $e_n = \frac{n!}{n^n}$. Now

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s = \frac{1}{2} + \sum_{s=n-2+1}^{n-1} \frac{1}{2} - s^{-\beta} = 1 - (n-1)^{-\beta} \leq 1, \text{ when } n \geq 1$$

and

$$h_n = p_n - q_{n-k+\ell} = \frac{1}{2} - q_{n-1} = \frac{1}{2} - \frac{1}{2} + (n-1)^{-\beta} = (n-1)^{-\beta} \geq 0, \quad n \geq 1.$$

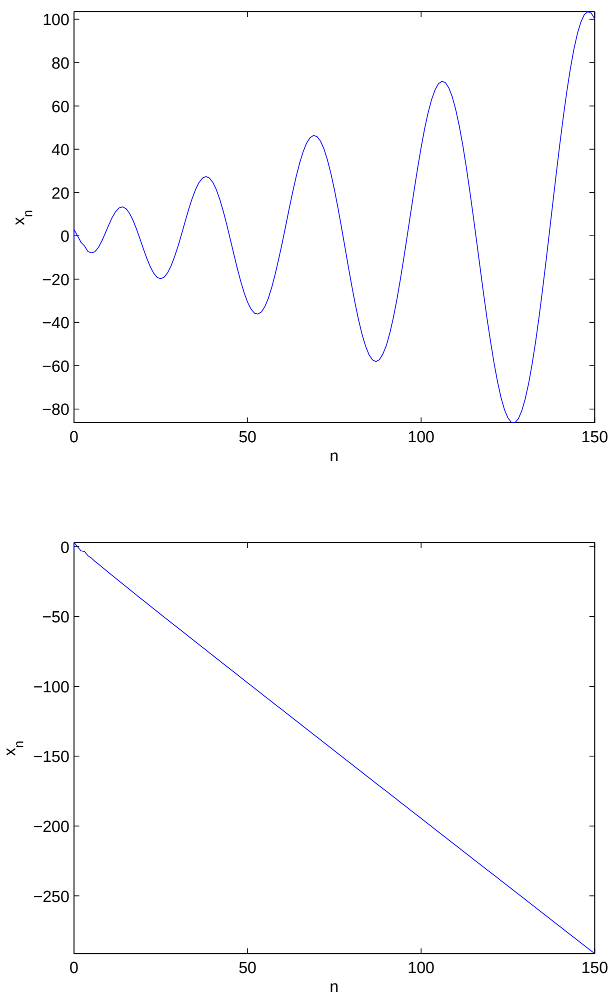

Figure 3: The trajectory solutions x_n of (32) with the initial conditions $x_0 = -2, x_1 = 3, x_2 = -3$ for $\beta = 0.7$ and $\beta = 3$

Here $m^* = \max(2, 1) = 2$. It is easy to see that all conditions of Theorem 3.1 are satisfied when $-\infty < \beta < 2$ and $m^* < 4$. Thus every solution of equation (32) oscillates.

When $\beta \geq 2$, a solution of equation (32) is non-oscillatory and negative.

Example 4.4. Consider the difference equation

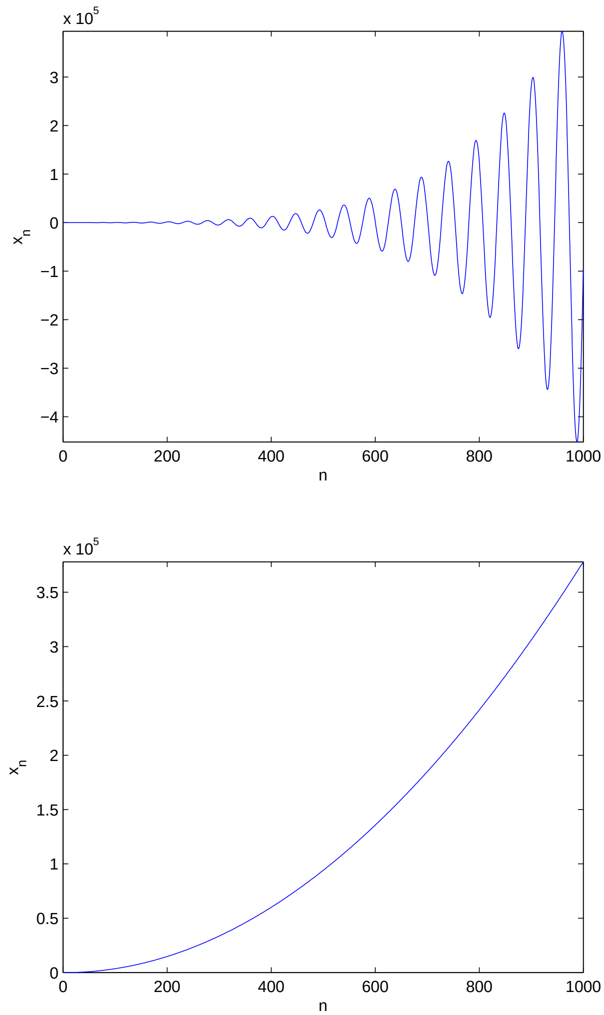

Figure 4: The trajectory solutions x_n of (33) with the initial conditions $x_0 = -2, x_1 = 3, x_2 = -3$ for $\beta = 0.7$ and $\beta = 3$

$$\Delta\left(x_n - \left(1 - \frac{1}{\pi}\right)x_{n-1}\right) + \frac{1}{\pi}x_{n-2} - \left(\frac{1}{\pi} - n^{-\beta}\right)x_{n-1} = \frac{1}{n(n+1)}, \quad n > 0. \tag{33}$$

Comparing with equation (1), we get $r = 1,\ k = 2,\ \ell = 1,\ c_n = 1 - \frac{1}{\pi},\ p_n = \frac{1}{\pi},$

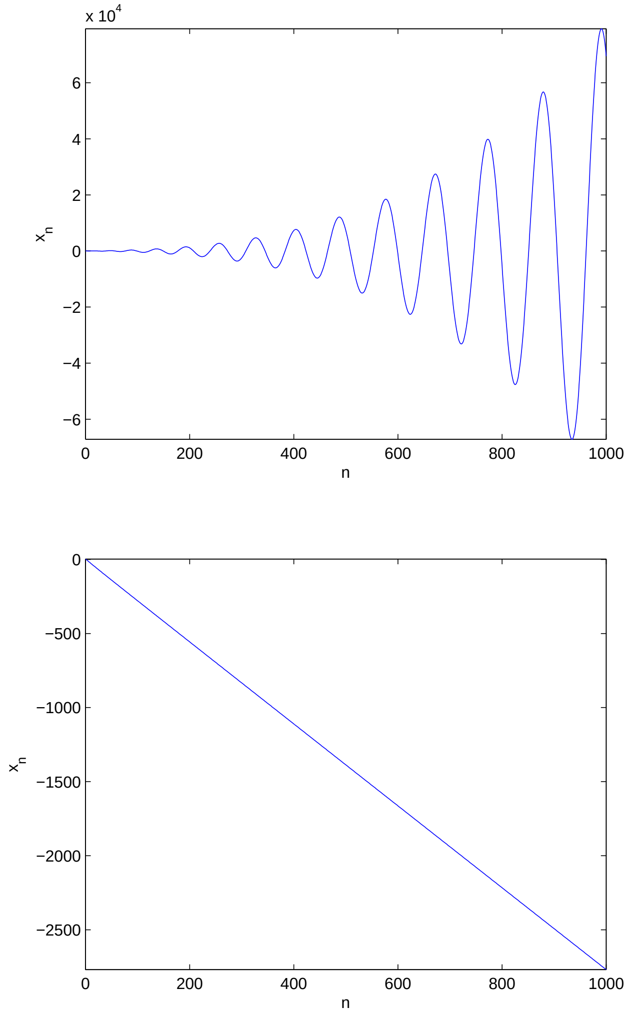

Figure 5: The trajectory solutions x_n of (34) with the initial conditions $x_0 = -2, x_1 = 3, x_2 = -3$ for $\beta = 0.8$ and $\beta = 3$

$q_n = \frac{1}{\pi} - n^{-\beta}$ and $e_n = \frac{1}{n(n+1)}$. Now

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s = 1 - \frac{1}{\pi} + \sum_{s=n-2+1}^{n-1} \frac{1}{\pi} - s^{-\beta} = 1 - (n-1)^{-\beta} \leq 1, \text{ when } n \geq 1$$

and

$$h_n = p_n - q_{n-k+\ell} = \frac{1}{\pi} - q_{n-1} = \frac{1}{\pi} - \left(\frac{1}{\pi} - (n-1)^{-\beta}\right) = (n-1)^{-\beta} \geq 0, \quad n \geq 1.$$

Here $m^* = \max(2,1) = 2$. It is easy to see that all conditions of Theorem 3.1 are satisfied when $-\infty < \beta < 2$ and $m^* < 4$. Thus every solution of equation (33) oscillates.

When $\beta \geq 2$, a solution of equation (33) is non-oscillatory and positive.

Example 4.5. Consider the difference equation

$$\Delta\left(x_n - \left(1 - \frac{1}{\pi}\right)x_{n-1}\right) + \left(\frac{1}{\pi} + n^{-\beta}\right)x_{n-2} - \frac{1}{\pi}x_{n-1} = \frac{(-1)^{(n+1)}}{\log(n+1)}, \quad n > 0. \tag{34}$$

Comparing with equation (1), we get $r = 1,\ k = 2,\ \ell = 1,\ c_n = 1 - \frac{1}{\pi},\ p_n = \frac{1}{\pi} + n^{-\beta}$, $q_n = \frac{1}{\pi}$ and $e_n = \frac{(-1)^{(n+1)}}{\log(n+1)}$. Now

$$c_n + \sum_{s=n-k+\ell}^{n-1} q_s = 1 - \frac{1}{\pi} + \sum_{s=n-2+1}^{n-1} \frac{1}{\pi} = 1$$

and

$$h_n = p_n - q_{n-k+\ell} = \frac{1}{\pi} - q_{n-1} = \frac{1}{\pi} + n^{-\beta} - \frac{1}{\pi} = n^{-\beta} \geq 0, n \geq 0.$$

Here $m^* = \max(2,1) = 2$. It is easy to see that all conditions of Theorem 3.1 are satisfied when $-\infty < \beta < 2$ and $m^* < 4$. Thus every solution of equation (34) oscillates.

When $\beta \geq 2$, a solution of equation (34) is non-oscillatory and negative.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications*, Second Edition: Revised and Expanded, Marcel Dekker, New York (2000).

[2] R.P. Agarwal, S.R. Grace, Donald O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands (2000).

[3] M. Chen, B.G. Zhang, Oscillation and comparison theorems of difference equations with positive and negative coefficients, *Bull. Institute Math. Acad. Sinica*, **22**, No. 4 (1994), 295-306.

[4] Ying Gao, Guang Zhang, Oscillation of nonlinear first order neutral difference equations, *Applied Mathematics E-Letters*, **1** (2001), 5-10.

[5] Walter G. Kelley, Allan C. Peterson, *Difference Equations, An Introduction with Applications*, Second Edition, Academic Press, Indian Reprint (2006).

[6] G. Ladas, Oscillation of difference equations with positive and negative coefficients, *Rockey Mountain J. of Math.*, **20**, No. 4 (1990), 1951-1061.

[7] Qiaoluan Li, Haiyan Liang, Wenlei Dong, Zhenguo Zhang, Existence of nonoscillatory solutions of higher order difference equations with positive and negative coefficients, *Bull. Korean Math. Soc.*, **45**, No. 1 (2008), 23-31.

[8] Wenrui Shan, Weigao Ge, Oscillation of neutral difference equations with positive and negative coefficients, *Computers and Mathematics with Applications*, **47** (2004), 1647-1657.

[9] X.H. Tang, J.S. Yu, D.H. Peng, Oscillation and nonoscillation of neutral difference equations with positive and negative coefficients, *Comput. Math. Applic.*, **39** (2000), 169-181.

[10] Chuan-Jun Tian, Sui Sun Cheng, Oscillation criteria for delay neutral difference equations with positive and negative coefficients, *Bol. Soc. Paran. Math.*, **21**, No-s: 1, 2 (2003), 1-12.

[11] B.G. Zhang, H. Wang, The existence of oscillatory and nonoscillatory solutions of neutral difference equations, *Chinese J. Math.*, **24**, No. 4 (1996), 377-393.

[12] Yong Zhou, Oscillations of neutral difference equations, *Kyungpook Math. J.*, **39** (1999), 283-291.