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MINIMAL LIGHTLIKE MONGE HYPERSURFACES IN SEMI-EUCLIDEAN SPACES

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Abstract: We give the condition for a lightlike Monge hypersurface in the semi-Euclidean space to be minimal. Using the condition we discuss some examples of minimal lightlike Monge hypersurfaces in semi-Euclidean spaces.

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1. Introduction

A submanifold M in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called lightlike if the induced metric $g = \bar{g}|_M$ is degenerate (cf. [3]). The study of lightlike submanifolds is much different from that of semi-Riemannian submanifolds. For a lightlike submanifold M, the tangent bundle TM and the normal bundle $TM^\perp$ have a non-trivial intersection, which is called the radical distribution and denoted by $\mathrm{Rad}(TM)$. Then we may choose a (non-unique) semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, which is called the screen distribution and denoted by $S(TM)$.

In the case where M is a lightlike hypersurface, the normal bundle $TM^\perp$ coincides with the radical distribution $\mathrm{Rad}(TM)$, and there exists a canonical transversal vector bundle $\mathrm{tr}(TM)$ corresponding to the screen distribution $S(TM)$, which is called the lightlike transversal vector bundle.

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In [3, Section 4.7] Duggal and Bejancu introduced the notion of minimal lightlike hypersurfaces in the 4-dimensional Minkowski space R_1^4. In [2] Burdujan defined minimal lightlike hypersurfaces in Lorentzian manifolds, and showed that a minimal lightlike Monge hypersurface in the Minkowski space is a part of a lightlike hyperplane. Recently, Bejan and Duggal [1] defined minimal lightlike submanifolds in semi-Riemannian manifolds.

In a previous paper [5], we give the necessary and sufficient condition for a lightlike hypersurface with integrable screen distribution in the 4-dimensional semi-Euclidean space R_2^4 of index 2 to be minimal. Using the condition, we obtain a class of minimal lightlike hypersurfaces in R_2^4 which are not totally geodesic. On the other hand, in [4], we show that a lightlike hypersurface with integrable screen distribution in a Lorentzian space form is minimal if and only if it is totally geodesic. As a corollary, we find that a lightlike hypersurface in the Minkowski space is minimal if and only if it is totally geodesic.

In this paper, we will give the condition for a lightlike Monge hypersurface in the semi-Euclidean space to be minimal, which is a generalization of [3, Chapter 4, Theorem 7.1] and [2].

Let R_q^{m+2} be the $(m+2)$-dimensional semi-Euclidean space of index q $(1 \leq q \leq m+1)$ with standard coordinate system $x = (x_0, x_1, \cdots, x_{m+1})$ and semi-Euclidean metric

$$\bar{g}(x,y) = -\sum_{i=0}^{q-1} x_i y_i + \sum_{j=q}^{m+1} x_j y_j.$$

Let M be a Monge hypersurface in R_q^{m+2} defined by

$$x_0 = F(x_1, \cdots, x_{m+1}), \qquad (x_1, \cdots, x_{m+1}) \in D \subset R_{q-1}^{m+1}, \tag{1}$$

where $F : D \to R$ is a smooth function. It is known that M is lightlike if and only if

$$1 + \sum_{i=1}^{q-1} F_{x_i}^2 = \sum_{j=q}^{m+1} F_{x_j}^2 \tag{2}$$

(cf. [3, Chapter 4, Theorem 6.3]). Set $\varepsilon_i = -1$ if $1 \leq i \leq q-1$, and $\varepsilon_i = 1$ if $q \leq i \leq m+1$. The Laplacian Δ on R_{q-1}^{m+1} is defined by

$$\Delta = \sum_{i=1}^{m+1} \varepsilon_i \frac{\partial^2}{\partial x_i^2}.$$

Now we state the result as follows:

Theorem. *A lightlike Monge hypersurface M in R_q^{m+2} given by (1) with*

(2) is minimal if and only if F is a harmonic function on $D \subset R^{m+1}_{q-1}$, that is, $\Delta F = 0$.

Using the theorem we discuss some examples of minimal lightlike Monge hypersurfaces in semi-Euclidean spaces.

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2. Preliminaries

In this section, following [3, Chapter 4] and [1], we recall some basic facts on lightlike hypersurfaces.

Let $\bar{M}$ be a semi-Riemannian manifold with metric $\bar{g}$ and Levi-Civita connection $\bar{\nabla}$. Let M be a lightlike hypersurface in $\bar{M}$, so that the induced metric $g = \bar{g}|_M$ is degenerate. Then the normal bundle $TM^{\perp}$ coincides with the radical distribution $\mathrm{Rad}(TM)$, defined by

$$\mathrm{Rad}(T_xM) = \{\xi \in T_xM | g(\xi, X) = 0, \quad X \in T_xM\},$$

where $\dim(\mathrm{Rad}(T_xM)) = 1$. We may choose a (non-unique) semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, which is called the screen distribution and denoted by $S(TM)$. So we have

$$TM = S(TM) \perp \mathrm{Rad}(TM) = S(TM) \perp TM^{\perp}.$$

By [3, Chapter 4, Theorem 1.1], for a screen distribution $S(TM)$, there exists a unique vector bundle $\mathrm{tr}(TM)$ of rank 1 such that, for any non-zero local section ξ of $TM^{\perp}$ on U there is a unique section N of $\mathrm{tr}(TM)|_U$ satisfying

$$\bar{g}(\xi, N) = 1, \qquad \bar{g}(N, N) = \bar{g}(N, W) = 0$$

for any $W \in \Gamma(S(TM)|_U)$. This vector bundle $\mathrm{tr}(TM)$ is called the lightlike transversal vector bundle with respect to $S(TM)$. Then we have the following decomposition

$$T\bar{M}|_M = TM \oplus \mathrm{tr}(TM).$$

We choose a non-zero local section ξ of $\mathrm{Rad}(TM)$. From the above decomposition, we have the Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where $X, Y \in \Gamma(TM)$. Then ∇ is a torsion-free linear connnection on M, and h is a symmetric $C^{\infty}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(\mathrm{tr}(TM))$. The form h is called the second fundamental form of M. Locally on U, we may write

$$h(X,Y) = B(X,Y)N.$$

The form B is called the local second fundamental form of M, which is independent of the choice of $S(TM)$. When $B = 0$, M is called totally geodesic.

By Definition 2 of [1] in the case of lightlike hypersurfaces, we say that M is minimal if $\mathrm{trace}(B) = 0$, where the trace is written with respect to g restricted to $S(TM)$. Noting that $B(\xi, X) = 0$ for any $X \in \Gamma(TM|_U)$ (cf. [3, Chapter 4, Corollary 2.1]), we can see that the above minimality condition is independent of the choice of $S(TM)$ and ξ.

3. Proof of Theorem and Examples

First, following [3, Section 4.6], we recall some basic facts on lightlike Monge hypersurfaces in semi-Euclidean spaces.

Let M be a Monge hypersurface in R_q^{m+2} given by (1). We consider the parametrization of M by

$$p = (F(u_1, \cdots, u_{m+1}), u_1, \cdots, u_{m+1}),$$

and the frame field on M by

$$p_{u_i} = F_{u_i}\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_i}, \qquad 1 \leq i \leq m+1.$$

Then

$$\xi = \frac{\partial}{\partial x_0} - \sum_{i=1}^{q-1} F_{u_i}\frac{\partial}{\partial x_i} + \sum_{j=q}^{m+1} F_{u_j}\frac{\partial}{\partial x_j}$$

spans $TM^\perp$. So we can see that M is lightlike if and only if

$$1 + \sum_{i=1}^{q-1} F_{u_i}^2 = \sum_{j=q}^{m+1} F_{u_j}^2.$$

If $F_{u_{m+1}} \neq 0$, the natural lightlike transversal vector bundle $\mathrm{tr}^*(TM)$ and the natural screen distribution $S^*(TM)$ are spanned by

$$N = -\frac{\partial}{\partial x_0} + \frac{1}{2}\xi = \frac{1}{2}\left(-\frac{\partial}{\partial x_0} - \sum_{i=1}^{q-1} F_{u_i}\frac{\partial}{\partial x_i} + \sum_{j=q}^{m+1} F_{u_j}\frac{\partial}{\partial x_j}\right),$$

and

$$W_i = F_{u_{m+1}}\frac{\partial}{\partial x_i} - F_{u_i}\frac{\partial}{\partial x_{m+1}} = F_{u_{m+1}}p_{u_i} - F_{u_i}p_{u_{m+1}}$$

$$= p_* \left(F_{u_{m+1}} \frac{\partial}{\partial u_i} - F_{u_i} \frac{\partial}{\partial u_{m+1}} \right), \quad 1 \le i \le m,$$

respectively.

Proof of Theorem. We may assume that $F_{u_{m+1}} \neq 0$. Restricted to the natural screen distribution $S^*(TM)$, we have

$$\hat{g}_{ij} := \bar{g}(W_i, W_j) = F^2_{u_{m+1}} \varepsilon_i \delta_{ij} + F_{u_i} F_{u_j}, \quad 1 \le i, j \le m.$$

Noting that

$$\sum_{j=1}^{m} \varepsilon_j F^2_{u_j} = 1 - F^2_{u_{m+1}},$$

we find that

$$\hat{g}^{jk} = \frac{1}{F^2_{u_{m+1}}} \varepsilon_j (\delta_{jk} - \varepsilon_k F_{u_j} F_{u_k}), \quad 1 \le j, k \le m$$

are the components of the inverse matrix of $(\hat{g}_{ij})_{1 \le i,j \le m}$.

With respect to the local second fundamental form B of M, we have on $S^*(TM)$,

$$\hat{B}_{ij} := B(W_i, W_j) = \bar{g}(\bar{\nabla}_{W_i} W_j, \xi)$$
$$= -F^2_{u_{m+1}} F_{u_i u_j} + F_{u_j} F_{u_{m+1}} F_{u_i u_{m+1}} + F_{u_i} F_{u_{m+1}} F_{u_j u_{m+1}} - F_{u_i} F_{u_j} F_{u_{m+1} u_{m+1}},$$

where $1 \le i, j \le m$ and $\bar{\nabla}$ is the flat connection on R^{m+2}_q.

By the definition, M is minimal if and only if

$$\sum_{i,j=1}^{m} \hat{B}_{ij} \hat{g}^{ji} = 0.$$

We may compute that

$$F^2_{u_{m+1}} \sum_{i,j=1}^{m} \hat{B}_{ij} \hat{g}^{ji} = -F^2_{u_{m+1}} \sum_{i=1}^{m} \varepsilon_i F_{u_i u_i} + F^2_{u_{m+1}} \sum_{i,j=1}^{m} \varepsilon_i \varepsilon_j F_{u_i} F_{u_j} F_{u_i u_j}$$
$$+ 2F_{u_{m+1}} \sum_{i=1}^{m} \varepsilon_i F_{u_i} F_{u_i u_{m+1}} - 2F_{u_{m+1}} \sum_{i=1}^{m} \varepsilon_i F_{u_i} F_{u_i u_{m+1}} \sum_{j=1}^{m} \varepsilon_j F^2_{u_j}$$
$$- F_{u_{m+1} u_{m+1}} \sum_{i=1}^{m} \varepsilon_i F^2_{u_i} + F_{u_{m+1} u_{m+1}} \sum_{i=1}^{m} \varepsilon_i F^2_{u_i} \sum_{j=1}^{m} \varepsilon_j F^2_{u_j}.$$

Noting that

$$\sum_{i=1}^{m} \varepsilon_i F_{u_i u_i} = \Delta F - F_{u_{m+1} u_{m+1}}, \quad \sum_{i=1}^{m} \varepsilon_i F^2_{u_i} = 1 - F^2_{u_{m+1}},$$

and

$$\sum_{i=1}^{m} \varepsilon_i F_{u_i} F_{u_i u_a} = -F_{u_{m+1}} F_{u_{m+1} u_a}, \qquad 1 \leq a \leq m+1,$$

we have

$$\begin{aligned} F_{u_{m+1}}^2 \sum_{i,j=1}^{m} \hat{B}_{ij} \hat{g}^{ji} &= -F_{u_{m+1}}^2 (\Delta F - F_{u_{m+1} u_{m+1}}) + F_{u_{m+1}}^4 F_{u_{m+1} u_{m+1}} \\ &\quad -2F_{u_{m+1}}^2 F_{u_{m+1} u_{m+1}} + 2F_{u_{m+1}}^2 (1 - F_{u_{m+1}}^2) F_{u_{m+1} u_{m+1}} \\ &\quad -(1 - F_{u_{m+1}}^2) F_{u_{m+1} u_{m+1}} + (1 - F_{u_{m+1}}^2)^2 F_{u_{m+1} u_{m+1}} \\ &= -F_{u_{m+1}}^2 \Delta F. \end{aligned}$$

From this equation, we find that M is minimal if and only if F is a harmonic function on $D \subset R_{q-1}^{m+1}$. Thus we get the conclusion. □

Using the theorem, let us consider some examples. When the ambient space is the Minkowski space R_1^{m+2}, we have the following corollary.

Corollary. (cf. [2]) *A lightlike Monge hypersurface M in R_1^{m+2} given by (1) with (2) is minimal if and only if F is a linear function.*

It was shown by Burdujan [2] using spherical harmonics. Here we give a simpler proof.

Proof of Corollary. By the theorem, M is minimal if and only if $\Delta F = 0$ and $||\mathrm{grad}(F)||^2 = 1$ on $D \subset R_0^{m+1} = R^{m+1}$. Noting the formula

$$\frac{1}{2} \Delta ||\mathrm{grad}(F)||^2 = ||\mathrm{Hess}(F)||^2 + \bar{g}(\mathrm{grad}(F), \mathrm{grad}(\Delta F)),$$

we can see that M is minimal if and only if F is a linear function. □

Remark. The corollary is also a spacial case of the result in [4].

Example 1. Let $Q_1(t), \cdots, Q_r(t)$ be smooth functions, and set

$$F(x_1, \cdots, x_{2r+1}) = \sum_{i=1}^{r} Q_i(x_i + x_{r+i}) + x_{2r+1}.$$

Then F satisfies

$$1 + \sum_{i=1}^{r} F_{x_i}^2 = \sum_{j=r+1}^{2r+1} F_{x_j}^2, \qquad \sum_{i=1}^{r} F_{x_i x_i} = \sum_{j=r+1}^{2r+1} F_{x_j x_j}.$$

By the theorem, the Monge hypersurface M in R_{r+1}^{2r+2} given by

$$x_0 = \sum_{i=1}^{r} Q_i(x_i + x_{r+i}) + x_{2r+1}$$

is lightlike and minimal. By the computation in the proof of the theorem, we have

$$\hat{B}_{ii} = -Q_i''(x_i + x_{r+i}), \quad 1 \leq i \leq r.$$

So, choosing $Q_i(t)$ nonlinear, we get minimal lightlike Monge hypersurfaces which are not totally geodesic.

Example 2. Let M be a Monge hypersurface in R_2^4 given by

$$x_0 = F(x_1, x_2, x_3),$$

where $F(x_1, x_2, x_3)$ is a quadratic function on R_1^3. Then the lightlike and minimal conditions are

$$1 + F_{x_1}^2 = F_{x_2}^2 + F_{x_3}^2, \quad F_{x_1x_1} = F_{x_2x_2} + F_{x_3x_3}.$$

By the second condition and a suitable choice of the coordinate system, we may assume that

$$F(x_1, x_2, x_3) = \frac{1}{2}(a_1 + a_2)x_1^2 + \frac{1}{2}a_1x_2^2 + \frac{1}{2}a_2x_3^2 + a_3x_1x_2 + a_4x_1x_3 + a_5x_1 + a_6x_2 + a_7x_3,$$

and $a_1 > 0,\ a_3 \geq 0,\ a_7 \geq 0$. By the first condition, we can get

$$a_2 = a_4 = 0, \quad a_3 = a_1, \quad a_5 = a_6, \quad a_7 = 1.$$

Thus we have

$$F(x_1, x_2, x_3) = \frac{1}{2}a_1(x_1 + x_2)^2 + a_5(x_1 + x_2) + x_3,$$

which is a spacial case of Example 1.

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