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THERMAL NOISE IN A MODIFIED DRUDE MODEL

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Abstract: A modified Drude model for the gas of electrons in a conductor is introduced as a dynamical system of charged mass points. The passage process through a cut is studied; based on a computer experiment, a Poisson point process is proposed for the description of the passage process. Formulas for stochastic pendants of electrodynamic quantities are derived; in particular a formula for the variance of time averages of the thermal voltage at the ends of a conductor is motivated from local Ohm's law.

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1. Introduction

The Drude model is an accepted description for the gas of electrons in a conductor (cf. [8]). In particular, the phenomenon of thermal motion of charge carriers can be represented in the picture of this model.

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The aim of the present contribution is the exploration of some electrodynamic consequences that result from a modified Drude model.

In Section 2 we introduce a system of mass points based on [6], Section 2.3; a point process is proposed and evaluated in context of a description of the passage process of mass points (electrons) through a cut through a conductor.

In Section 3 some stochastic pendants of electrodynamic quantities based on the point process are derived. In Section 4 we motivate a spatial covariance function for passage processes at different cuts through the conductor.

In Section 5 a formula for the variance of the voltage between the ends of the conductor is derived based on a local version of Ohm's law.

The authors believe that the search for empirical pendants of the presented formulas would be a challenging task for the experimental exploration of thermal noise phenomena.

2. A Stochastic Model of the Passage Process

Let us consider a 3-dimensional container C which is modelled by a hyperrectangle

$$C := [0, L] \times [-a/2, +a/2]^2 \subset \mathbb{R}^3$$

with edge lengths $L, a > 0$. We inject N mass points of mass $m > 0$ into C according to the uniform distribution over C. The initial velocities $v^{(1)}(0), \ldots,$ $v^{(N)}(0) \in \mathbb{R}^3$ of the points are generated according to the centered normal distribution $N(0, \sigma^2 \cdot I_3)$ with mean $0 \in \mathbb{R}^3$ and covariance matrix $\sigma^2 \cdot I_3$ where I_3 denotes the 3×3-identity matrix; parameter σ can be interpreted thermally according to

$$\sigma = \left(\frac{k_B \cdot T}{m} \right)^{1/2},$$

where $k_B = 1.380662 \cdot 10^{-23}\mathrm{J/K}$ and $T > 0$ denote Boltzmann constant and temperature of the system, respectively.

Let the system (gas) evolve according to the Newtonian dynamics entailing that the micro-constituents do not mutually interact and are reflected at the walls of container C at appropriate time points. The system can be interpreted as a kinetic model of the ideal gas ([6], Section 2.3).

Fixing mass $m = 9.109534 \cdot 10^{-31}\mathrm{kg}$ of the micro-constituents implies that the system can be viewed as gas of electrons confined to an electric conductor in the sense of a modified Drude model (cf. [8]).

Let $x \in (0, L)$ be fixed. Let us consider the cut

$$S_x := \{x\} \times [-a/2, +a/2]^2$$

of area a^2 through conductor C.

We are interested in the process of passages of electrons through S_x from the left. We describe the construction of a stochastic model for this passage process.

Put

$$\lambda := \frac{1}{\sqrt{2\pi}} \cdot \varrho \cdot \sigma \cdot a^2, \tag{2.1}$$

where

$$\varrho := \frac{N}{La^2}$$

denotes the (particle) density of the gas. Kinematic considerations suggest that λ is the intensity of the passage process in the sense of the expected number of passages per a time interval of unit length.

Let P_σ denote the probability distribution (on $\mathbb{R}_+$) with the Lebesgue density

$$f_\sigma(v) = \frac{v}{\sigma^2} \cdot \exp\left(-\frac{v^2}{2\sigma^2}\right) \cdot 1_{\mathbb{R}_+}(v).$$

Plausibility considerations suggest that P_σ is the distribution of horizontal velocity components of electrons passing through cut S_x from the left.

Let us define the (intensity) measure

$$\mu_x := \lambda \cdot P_\sigma \otimes \lambda^1|_{\mathbb{R}_+}, \tag{2.2}$$

where $\lambda^1|_{\mathbb{R}_+}$ denotes the restriction of the 1-dimensional Lebesgue measure to $(\mathbb{R}_+, \mathcal{B} \cap \mathbb{R}_+)$ and $\mathcal{B}$ denotes the Borel σ-field on $\mathbb{R}$.

The proposed stochastic model for the passage process of electrons through cut S_x from the left is the Poisson point process N_x^L with intensity measure μ_x; for the construction of a Poisson point process with a prescribed σ-finite intensity measure, cf. [10].

For an illustration of this model, let $[\tau_1, \tau_2] \subset \mathbb{R}_+$ be a time interval and $[v_1, v_2] \subset \mathbb{R}_+$ an interval of velocities; the $\mathbb{Z}_+$-valued random variable

$$N_x^L([\tau_1, \tau_2] \times [v_1, v_2]) \tag{2.3}$$

models the number of electrons passing through S_x from the left in the time interval $[\tau_1, \tau_2]$ with horizontal velocity components lying in the interval $[v_1, v_2]$; ($\mathbb{Z}_+$ denotes the set of non-negative integers). According to the construction of

N_x^L, the random variable given in (2.3) is distributed according to the (discrete) Poisson distribution with parameter

$$\alpha := \mu_x([\tau_1, \tau_2] \times [v_1, v_2]) = \lambda \cdot (\tau_2 - \tau_1) \cdot P_\sigma([v_1, v_2]).$$

Note that random variable $N_x^L([\tau_1, \tau_2] \times \mathbb{R}_+)$ models the number of electrons passing through cut S_x from the left in the time interval $[\tau_1, \tau_2]$ where the velocities are disregarded in the counting process.

To evaluate the proposed stochastic model, a computer experiment has been implemented and carried out where the passages of electrons through cut S_x in container C have been registered in the sense that the passage times together with the horizontal velocity components have been stored. The described model assumption of Poissonity of point process N_x^L implies that $(N_x^L([0, \tau] \times \mathbb{R}_+))_{\tau \geq 0}$ is a classical Poisson process with intensity λ; this entails that the inter-passage times can be modelled by a sequence of stochastically independent random variables that are distributed according to the exponential distribution with parameter λ given in (2.1).

Figure 1 shows a typical screen shot obtained in the course of the experiment. In the upper window container C filled with $N = 15000$ electrons is visualized together with cut S_x. In the lower left window the graphical comparison between the probability density

$$g_\lambda(t) = \lambda \cdot \exp(-\lambda \cdot t) \cdot 1_{\mathbb{R}_+}(t)$$

of the exponential distribution and the kernel density estimate based on sampled inter-passage times is shown. The lower right window shows the model probability density f_σ of horizontal velocity components of electrons passing through cut S_x from the left and a kernel density estimate based on velocities sampled during the experiment (for an introduction and a practical guide to the concept of kernel density estimator the reader is referred [11] and [12]; for further application of the kernel method to the statistical evaluation of computer experiments cf. [3], [4], [6], [7]).

Since the kernel density estimates in both windows are close to the corresponding probability densities implied by the stochastic model, the experiment confirms the validity of point process N_x^L for the description of the passage process in the modified Drude model. The reported agreement between stochastic model and outcome of the computer experiment has been observed independently of temperature T and particle density ϱ and independently of the positioning cut S_x at $x \in (0, L)$.

For reasons of symmetry, a natural model for the passage process through cut S_x from the right is an independent copy N_x^R of point process N_x^L for

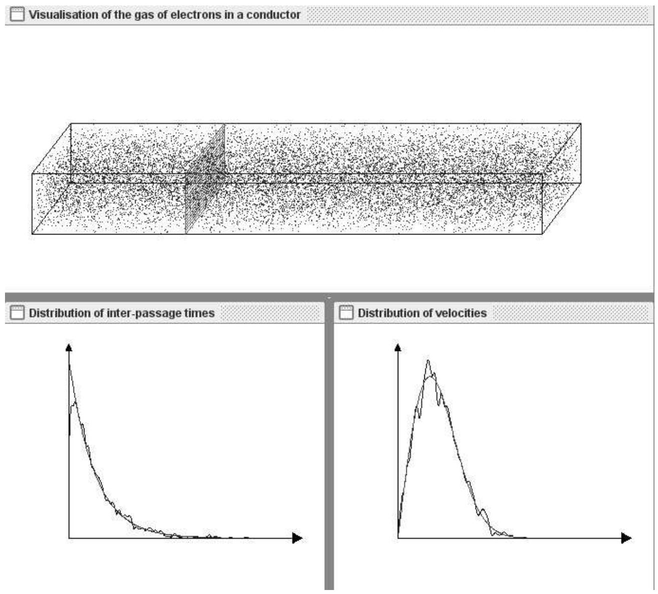

Figure 1: Screen shot of the computer experiment

$x \in (0, L)$.

3. Stochastic Pendants of Some Electrodynamic Quantities

Let us consider container (conductor) C of length L and a cut S_x through C at $x \in (0, L)$ as introduced in Section 2.

If the mass points confined to C are interpreted as electrons of charge $e = -1.60219 \cdot 10^{-19}\text{C}$, then

$$Q(x,\tau) := e \cdot (N_x^L([0,\tau] \times \mathbb{R}_+) - N_x^R([0,\tau] \times \mathbb{R}_+)) \tag{3.1}$$

is the net electric charge transported through cut S_x in the time interval $[0,\tau]; (Q(x,\tau))_{\tau\geq 0}$ can be viewed as a stochastic process. Since the processes $(N_x^L([0,\tau] \times \mathbb{R}_+))_{\tau\geq 0}$ and $(N_x^R([0,\tau] \times \mathbb{R}_+)_{\tau\geq 0}$ are stochastically independent Poisson processes with intensity λ given in (2.1), it follows that

$$\mathbb{E}(Q(x,\tau)) = 0 \qquad (\tau \geq 0, x \in (0, L)), \tag{3.2}$$

where $\mathbb{E}(.)$ denotes the expected value of a real random variable; (3.2) means that the expectation of the net charge transport through cut S_x vanishes, which is typical for thermal noise phenomena. We can, moreover, express the variance of the (centered) random variable $Q(x,\tau)$ by

$$\mathrm{Var}(Q(x,\tau)) = \mathbb{E}(Q(x,\tau)^2) = 2\lambda e^2 \tau \qquad (\tau > 0, x \in (0,L)) \tag{3.3}$$

which follows from the Poissonity of the processes $(N_x^L([0,\tau] \times \mathbb{R}_+))_{\tau \geq 0}$ and $(N_x^R([0,\tau] \times \mathbb{R}_+))_{\tau \geq 0}$ and from the validity of

$$\mathbb{E}((X_1 - X_2)^2) \;=\; 2\alpha$$

for independent random variables X_1, X_2 distributed according to the (discrete) Poisson distribution with parameter $\alpha > 0$. In other words, the expected value of the squared charge transport through cut S_x increases linearly with time, which is typical for diffusion phenomena.

Put

$$I(x,\tau) := \frac{1}{\tau} \cdot Q(x,\tau) \qquad (x \in (0,L), \tau > 0). \tag{3.4}$$

$(I(x,\tau))_{\tau \geq 0}$ can be viewed as a stochastic process which models the averaging of the net charge transport over a time interval $[0,\tau]$ for $\tau > 0$. We call the random variable $I(x,\tau)$ the average net (electric) current through cut S_x based on the time interval $[0,\tau]$.

Obviously,

$$\mathrm{Var}(I(x,\tau)) \;=\; \frac{2\lambda e^2}{\tau} \;=\; \frac{\varrho \sigma e^2 a^2}{\sqrt{2\pi} \cdot \tau} \tag{3.5}$$

which means that the variance of the current decreases with the length of the time interval allowed for the averaging process. Note that the variance of the current is proportional to the cross-area of the conductor.

Example 3.1. Let us consider a copper conductor of cross-area $1\mathrm{m}^2$ at temperature $T = 300$ K. Under the assumption that each copper atom contributes two conducting electrons to the gas, the density of the charge carriers is:

$$\varrho_{\mathrm{Cu}} = 1.6906 \cdot 10^{29} \mathrm{m}^{-3}.$$

According to (3.5), the dispersion of the current averaged over the period of 10^{-9}s is given by

$$(\mathrm{Var}(I(x, 10^{-9}\mathrm{s}))^{1/2} = 3.41681 \cdot 10^2 \mathrm{A}.$$

Let us define the time average of current density through cut S_x:

$$j(x,\tau) := \frac{I(x,\tau)}{a^2} \qquad (x \in (0,L), \tau > 0). \tag{3.6}$$

(3.4) implies

$$\mathrm{Var}(j(x,\tau)) \;=\; \mathbb{E}(j(x,\tau)^2) \;=\; \frac{\varrho\sigma e^2}{\sqrt{2\pi}a^2\tau} \qquad (x \in (0,L), \tau > 0). \tag{3.7}$$

(3.7) entails that the variance of the average current density decreases with increasing cross area a^2 which is in some sense plausible, because noise current density $j(x,\tau)$ as defined in (3.6) is the cross-area average of the current which can be viewed as an estimator of the macroscopic current density whose variance is reduced if a larger cross-area is considered. In context of Example 3.1 the numerical value of the dispersion of the current density based on averaging time of 10^{-9}s for a copper conductor of cross-area $a^2 = 10^{-6}\mathrm{m}^2$ is $\mathrm{Var}(j(x,10^{-9}\mathrm{s}))^{1/2} \;=\; 3.4168 \cdot 10^5\mathrm{A/m}^2$.

Let s denote the specific (electric) resistance of the substance constituting conductor C. Under the assumption of spatial homogeneity of the considered current, the local version of Ohm's law (cf. [5], [9]) implies the formula

$$E(x,y,z,\tau) = E(x,\tau) = s \cdot j(x,\tau) \qquad ((x,y,z) \in C, \tau > 0), \tag{3.8}$$

where $E(x,y,z,\tau)$ denotes the time average of the horizontal component of the electric field at $(x,y,z) \in C$ induced by the thermal motion of charge carriers. The corresponding electric power density $W(\tau)$ of the thermal noise based on the time average $j(x,\tau)$ of the current density in the time interval $[0,\tau]$ is given by

$$W(\tau) \;=\; j(x,\tau) \cdot E(x,y,z,\tau) \;=\; j(x,\tau)^2 \cdot s \qquad (\tau > 0, (x,y,z) \in C). \tag{3.9}$$

Combining (3.9) and (3.7) entails

$$\mathbb{E}(W(\tau)) = \frac{\varrho\sigma e^2 s}{\sqrt{2\pi}a^2\tau} \tag{3.10}$$

for the expectation of the time average of power density. From (3.10) we obtain for the expected time average $P_d(\tau)$ of total electric power of the thermal noise in C w.r.t. the horizontal direction:

$$P_d(\tau) \;:=\; \mathbb{E}(W(\tau)) \cdot La^2 \;=\; \frac{\varrho\sigma e^2 s}{\sqrt{2\pi}\tau} \cdot L \qquad (\tau > 0). \tag{3.11}$$

(3.11) suggests that the time average of the total electric power performed by the thermal motion of charge carriers is an extensive quantity which is proportional to length L of the conductor. As a consequence of (3.7), $P_d(\tau)$ does not depend on cross-area a^2. For reasons that become apparent in Section 5, we speak of dissipated electric power that is expressed by $P_d(\tau)$.

For a numerical illustration, let us consider a copper conductor of length $L = 10^3$m in context of Example 3.1. The time average of the total electric power based on the averaging time 10^{-9}s is given by $P_d(10^{-9}\mathrm{s}) = 2.078$W.

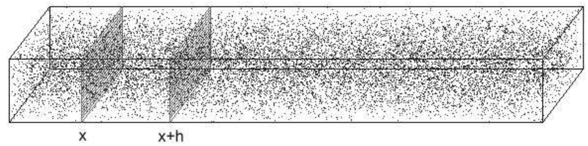

Figure 2: Conductor with two cuts

4. The Spatial Covariance Function between Passages

Let us consider conductor C filled with electron gas as introduced in Section 2.

Let us, moreover, consider two cuts S_x and S_{x+h}, where $0 < x < x+h < L$, cf. Figure 2. According to Section 2, the particle passage processes through S_x and S_{x+h} from the left are modelled by Poisson point processes N_x^L and N_{x+h}^L. Let us suppose that N_x^L and N_{x+h}^L can be defined on a conjoint probability space $(\Omega, \mathcal{A}, P)$. In the present section we are in particular interested in the expected value

$$\mathbb{E}(N_x^L([0,\tau] \times \mathbb{R}_+) \cdot N_{x+h}^L([0,\tau] \times \mathbb{R}_+)) \tag{4.1}$$

whose knowledge will enable us to compute the variance of the voltage between the ends $\{0\} \times [-a/2, a/2]^2$ and $\{L\} \times [-a/2, a/2]^2$ of conductor C (cf. Section 5).

Put

$$M_y^L = N_y^L([0,\tau] \times \mathbb{R}_+) \qquad (0 < y < L)$$

and

$$M_y^R = N_y^R([0,\tau] \times \mathbb{R}_+) \qquad (0 < y < L).$$

Let us assume that N_{x+h}^L can be additively decomposed by a thinning $M^{(1)}$ of N_x^L and by a point process $M^{(2)}$ which is independent of N_x^L:

$$N_x^L = M^{(1)} + M^{(2)}. \tag{4.2}$$

Let us, moreover, assume that the parameter $p(h)$ of the thinning is given by

$$p(h) = \exp\left(-\frac{h^2}{2\sigma^2\tau^2}\right) - \sqrt{2\pi} \cdot \frac{h}{\sigma\tau} \cdot \left(1 - \Phi\left(\frac{h}{\sigma\tau}\right)\right), \tag{4.3}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. We obtain

$$\mathbb{E}(M^{(1)}([0,\tau]\times\mathbb{R}_+)) = p(h)\lambda\tau \tag{4.4}$$

and

$$\mathbb{E}(M^{(2)}([0,\tau]\times\mathbb{R}_+)) = (1-p(h))\cdot\lambda\tau. \tag{4.5}$$

Combining (4.2)-(4.5) yields

$$\mathbb{E}(M_x^L\cdot M_{x+h}^L) = (\lambda\tau)^2 + p(h)\lambda\tau, \tag{4.6}$$

where $p(h)$ is given according to (4.3).

Analogous reasoning for the passage processes from the right yields

$$\mathbb{E}(M_x^R\cdot M_{x+h}^R) = (\lambda\tau)^2 + p(h)\lambda\tau. \tag{4.7}$$

Under the assumption of stochastic independence between the passage processes from the left and from the right we obtain for the covariance between the net passages through S_x and S_{x+h}:

$$\mathbb{E}((M_{x+h}^L - M_{x+h}^R)\cdot(M_x^L - M_x^R)) = 2p(h)\lambda\tau. \tag{4.8}$$

5. Cumulated Properties of the Thermal Noise

Equation (4.8) offers in particular the possibility to determine the spatial covariance between local time averages of the electric field $E(x,\tau)$ because $E(x,\tau)$ is can be obtained from $(M_x^L - M_x^R)$ by multiplication with constants (cf. Section 3) for $x\in(0,L)$. We have

$$E(x,\tau) = \frac{es}{\tau a^2}\cdot(M_x^L - M_x^R) \qquad (x\in(0,L)). \tag{5.1}$$

The time average of the voltage between the ends of conductor C based on the time interval $[0,\tau]$ is given by

$$U(\tau) = \int_0^L E(y,\tau)dy; \tag{5.2}$$

in this sense $(U(\tau))_{\tau>0}$ can be interpreted as a stochastic average process whose expected value vanishes for all $\tau>0$.

Now we would like to compute the variance of $U(\tau)$ for $\tau>0$. First we point out that

$$\operatorname{Var}(U(\tau)) = \mathbb{E}\left(\left(\int_0^L E(y,\tau)dy\right)^2\right); \tag{5.3}$$

Fubini theorem implies that

$$\mathrm{Var}(U(\tau)) = \int_0^L \int_0^L \mathbb{E}(E(x,\tau) \cdot E(y,\tau))\, dxdy. \tag{5.4}$$

Since the integrand in (5.4) depends only on $|x-y|$ (cf. (4.8)), we may write

$$\mathrm{Var}(U(\tau)) = \frac{e^2 s^2}{\tau^2 a^4} \cdot 4\lambda\tau \int_0^L (L-h) \cdot p(h) dh. \tag{5.5}$$

We call

$$P_c(\tau) = \mathbb{E}\left(\frac{U(\tau)^2 \cdot a^2}{sL}\right) \tag{5.6}$$

the cumulated power of the thermal noise.

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