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INTEGRAL SPACE CURVES IN THE RANGE C WITH PRESCRIBED GEOMETRIC AND ARITHMETIC GENERA

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Abstract: Fix integers $s \geq 5$ and $d > s(s-1)$ (range C). For many pairs (g, q) we prove the existence of curves $X \subset \mathbb{P}^3$ with geometric genus g, $p_a(X) = q$ and $h^0(\mathbb{P}^3, \mathcal{I}_X(s-1)) = 0$.

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1. Introduction

Fix positive integers d, s. Let $G(d, s)$ (respectively $G'(d, s)$) denote the maximal genus of a smooth and connected space curve not contained in a surface of degree $< s$ (respectively not contained in a surface of degree $< s$ and contained in a surface of degree s). The computation of $G(d, s)$, the study of the degree d space curves not contained in a surface of degree $< s$ and with "very large genus", and the determination of all triples (d, s, g) such that there is a smooth and connected degree d space curve not contained in a surface of degree $< s$ are usually called "Halphen's problem", after [14]. On this subject there is a huge literature (see [12], [13], [15], [16], [17], [2], [6], [7], [8], [9], [10], [24], [11], [3], [18], [20], [24], [25]). For singular space curves there are the following classical problems.

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♣: Find the set $A(d,s,g,q)$ of all quadruples of integers (d,s,g,q) such that there exists an integral degree d curve $X \subset \mathbb{P}^3$ with geometric genus g, $p_a(X) = q$ and $h^0(\mathbb{P}^3, \mathcal{I}_X(s-1)) = 0$.

♠: Find the set $B(d,s,g,q)$ of all quadruples of integers (d,s,g,q) such that there exists an integral degree d curve $X \subset \mathbb{P}^3$ with geometric genus g, $p_a(X) = q$, $h^0(\mathbb{P}^3, \mathcal{I}_X(s-1)) = 0$ and $h^0(\mathbb{P}^3, \mathcal{I}_X(s)) > 0$.

It was stressed in [12] and [16] that to get good theorems one has first to distinguish the set of all pairs (d,s) in the following way:

Range $\emptyset$: If $d < (s^2+4s+6)/6$, then every degree d smooth space curve is contained in a surface of degree $< s$.

Range A: If $(s^2+4s+6)/6 \le d < (s^2+4s+6)/3$, then $G(d,s) \le G_A(d,s)$, where $G_A(d,s) := 1 + d(s-1) - \binom{s+2}{3}$.

Range B: $(s^2+4s+6)/3 \le d \le s(s-1)$.

Range C: If $d > s(s-1)$, then $G(d,s) = G'(d,s) = G_C(d,s) := 1 + d(s + (d/s) - 4)/2 - r(s-r)(s-1)/2s$, where r is the only integer such that $0 \le r < s$ and $d + r \equiv 0 \pmod{s}$.

If $s^2 - 2s + 2 \le d \le s(s-1)$, then $G(d,s)$ is known (see [13]) and sometimes this interval is put in the range C.

In the range A it is expected that $G(d,s) = G_A(d,s)$ and for many (d,s) this equality holds (see [11], [3] and end of the introduction of [3]). In the range C we have $G(d,s) = G_C(d,s)$ and all space curves computing $G(d,s)$ are known (see [12], Theorem 3.1) (see Remark 5 for the case of singular integral curves). For the closely related computation of the value of $G'(d,s)$, see [12], Theorem 3.1, and [15]. In this paper we only consider the range C. Ch. Walter made a conjecture concerning ♠ for the case $q = g$ (i.e. for smooth space curves) if $d \ge \binom{s-2}{2}$ (a range larger that range C) (see [25], Conjecture 0.3). He proved a related theorem in a smaller range, finding curves in a degree s surface S containing a line with multiplicity $s-3$ (see [25], Theorem 0.4). Here we will just follow his proof, trying to get curves with controlled singularities on the same surface. In Section 2 we prove the following results.

Theorem 1. *Fix integers d, s, q such that $s > 5$ $d > s^2 - 2s + 6$ and*

$$(s-1)d - (s^3 - 5s^2 + 18s - 18)/2 < q \le (d^2 + 3(s-4)d - (\epsilon^2 - \epsilon s + 12s - 36))/2s, \quad (1)$$

where ϵ is the unique integer such that $0 \le \epsilon < s$ and $\epsilon \equiv d - 6 \pmod{s}$. There are integers d', g', m such that $d' \ge 6$, $d' - 2 \le g' \le 2d' - 9$, $s - 2 \le m <$

$(d+s-6)/s$, $d=d'+ms$, $q=g'+m(2d'+ms+3s-12)/2$. Set

$$\beta := (s-2)(m-s+2)(m-s+1)/2 + \lfloor((s-2)(12+(s-2)^2)-1)/3\rfloor - (m-s+3). \quad (2)$$

Fix an integer g such that $q-\beta \le g \le q$. Let $S \subset \mathbb{P}^3$ be a general degree s surface containing a line L with multiplicity $s-3$. Then there exists an integral curve $X \subset S$ with degree d, geometric genus g and arithmetic genus q.

Theorem 2. *Fix integers d, s, q such that $s > 5$ $d > s^2-2s+6$ and*

$$(s-1)d-(s^3-5s^2+18s-18)/2 < q \le (d^2+3(s-4)d-(\epsilon^2-\epsilon s+12s-36))/2s, \quad (3)$$

where ϵ is the unique integer such that $0 \le \epsilon < s$ and $\epsilon \equiv d-6 \pmod{s}$. There are integers d', g', m such that $d' \ge 6$, $d'-2 \le g' \le 2d'-9$, $s-2 \le m < (d+s-6)/s$, $d = d'+ms$, $q = g'+m(2d'+ms+3s-12)/2$. Set $\alpha := (m-s+3)((12-2s+(m+1)m/2-(s-3)(s-4)/2)$. Fix an integer g such that $q-\lfloor(\alpha-1)/3\rfloor+1 \le g \le q$. Let $S \subset \mathbb{P}^3$ be a general degree s surface containing a line L with multiplicity $s-3$. Then there exists an integral and nodal curve $X \subset S$ with degree d, geometric genus g and arithmetic genus q.

Take X as in Theorems 1 or 2. If $d > s(s-1)$, then X is not contained in a surface of degree at most $s-1$ by Bezout Theorem.

Then we go on and consider the case $g = G_C(d,s)$. In Section 3 we prove the following result.

Theorem 3. *Fix integers d, s, g such that $s \ge 5$, $d \ge s^2$ and $G_C(d,s) - \alpha(d,s) \le g \le G_C(d,s)$, where $t := \lceil d/s\rceil$, $r := t-ds$, $t' := t-1$ if $r > 0$, $t' := t$ if $r = 0$, $\epsilon = 0$ if $t'-s+1 \notin \{2,4\}$, $\epsilon = 1$ if $t'-s+1 \in \{2,4\}$, $u_0 := \lfloor(t'-s+3)(t'-s+2)/6-\epsilon\rfloor$ and $\alpha(d,s) = u_0 + \binom{t'-1}{3} - \binom{t'-s+4}{3} - 1$. Then there exists an integral curve $X \subset \mathbb{P}^3$ such that $\deg(X) = d$, $p_a(X) = G_C(d,s)$, X has geometric genus g, $h^0(\mathbb{P}^3, \mathcal{I}_X(s-1)) = 0$ and $h^0(\mathbb{P}^3, \mathcal{I}_X(s)) > 0$, i.e. $B(d,s,g,G_C(d,s)) \neq \emptyset$.*

If $d > s^2$ below $G_C(d,s)$ there are gaps for the possible arithmetic genera of an integral degree d space curve not contained in a surface of degree $< s$ (see [6], [7]). Hence there is no hope to extend Theorem 3 to arithmetic genera very near to $G_C(d,s)$.

If $d \gg s^2$, i.e. if $t := \lceil d/s\rceil \gg s$, then $G_C(d,s) \sim d^2/2s \sim st^2/2$ and $\alpha(d,s) \sim st^2/2$.

If $d > s^2$ below $G_C(d,s)$ there are gaps for the possible arithmetic genera of an integral degree d space curve not contained in a surface of degree $< s$ (see [6], [7]). Thus there is no hope of any extension of Theorem 3 to all arithmetic genera. We prove Theorem 3 by fixing a smooth plane curve $D \subset \mathbb{P}^3$ with degree r (with the convention $D = \emptyset$ if $r = 0$), a general degree Y surface containing D and then proving the existence of $X \in |\mathcal{O}_Y(t)(-D)|$ with certain prescribed singularities. Thus any such curve X is linked to D by Y and a degree t surface. Thus the minimal free resolution of X is known (it is uniquely determined by s, t and the minimal free resolution of D), and it is the same as the one of a connected smooth curve C of degree d and genus $G_C(d, s)$ not contained in a surface of degree $s-1$. Since the general element of the linear system $|\mathcal{O}_Y(t)(-D)|$ is a connected smooth curve C with genus $G_C(d, s)$ not contained in a surface of degree $s-1$, X is a flat limit of a flat family of smooth space curves with constant minimal free resolution and hence with constant postulation. Let X' be any integral curve as in Theorem 3. By Remark 5 below X' is contained in an integral degree s surface Y' containing a degree r plane curve D' and it is linked to D' by Y' and a degree t surface. In a smaller range of integers g we were able prove the existence of X as in Theorem 3 lying on a singular degree s surface Y' containing D such that Y' has certain prescribed singular points. Here $X' \cup D$ is the intersection of Y' with a general degree t surface containing $D \cup \mathrm{Sing}(Y')$. We omit this approach, because it gave a weaker statement.

For integral nodal curves we are only able to prove the following result.

Proposition 1. *Fix integers d, s, g such that $s \geq 4$, $d > s(s-1)$ and*

$$G_C(d,s) - \max\{0, \lfloor (\binom{t'+3}{3} - \binom{t'-s+3}{3}) - 4)/3 \rfloor - 1, 0\} \leq g \leq G_C(d,s),$$

where $t := \lceil d/s \rceil$ and $t' := t$ if $d/s \in \mathbb{Z}$, $t' := t-1$ if $d/s \notin \mathbb{Z}$. Then there exists an integral and nodal curve $X \subset \mathbb{P}^3$ such that $\deg(X) = d$, $p_a(X) = G_C(d,s)$, X has geometric genus g, $h^0(\mathbb{P}^3, \mathcal{I}_X(s-1)) = 0$ and $h^0(\mathbb{P}^3, \mathcal{I}_X(s)) > 0$.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. Proofs of Theorems 1 and 2

As in [25] set $F(d, s, r) := (r+1)d - (r-1)rs/2 - 6s - 3$ (note the misprint in [25], p. 56). Note that $F(d, s, s-2) = (s-1)d - (s^3 - 5s^2 + 18s - 18)/2$, i.e. $F(d, s, d-2) + 1$ is the lower bound of the arithmetic genera q considered in Theorems 1 and 2.

For any scheme A, any $P \in A_{reg}$ and any integer $m > 0$ let $\{mP, A\}$ denote the infinitesimal neighborhood of order $m-1$ of P in A, i.e. the closed subscheme of A with $(\mathcal{I}_{P,A})^m$ as its ideal sheaf. Hence $\{mP, A\}$ is a zero-dimensional scheme, $\{mP, A\}_{reg} = \{P\}$ and $\mathrm{length}(\{mP, A\}) = \binom{m+n-1}{m}$, where n is the dimension of A at P. For any finite subset $B \subset A_{reg}$ set $\{mB, A\} := \cup_{P \in B}\{mP, A\}$. Set $\{0P, A\} := \emptyset$ and $\{0B, A\} := \emptyset$. The scheme $\{2P, A\}$ (respectively $\{mP, A\}$) is often called a double point of Y (respectively a fat point with multiplicity m of A or an m-fat point of A). We usually write mP or mB instead of $\{mP, A\}$ or $\{mB, A\}$ if either A is the smooth surface Y introduced below or (sometimes) if A is a blowing-up of Y.

Remark 1. Let A be a projective scheme, E an effective Cartier divisor of A, $R \in \mathrm{Pic}(A)$ and Z a closed subscheme of A. Let $\mathrm{Res}_E(Z)$ denote the residual scheme of Z with respect to E, i.e. the closed subscheme of A with $\mathcal{I}_{Z,A} : \mathcal{I}_{E,A}$ as its ideal sheaf. We have an exact sequence of sheaves:

$$0 \to \mathcal{I}_{\mathrm{Res}_E(Z),A} \otimes R(-E) \to \mathcal{I}_{Z,A} \otimes R \to \mathcal{I}_{Z\cap E,E} \otimes (R|E) \to 0. \qquad (4)$$

From (4) we get:

$$h^0(A, \mathcal{I}_{Z,A} \otimes R) \le h^0(A, \mathcal{I}_{\mathrm{Res}_E(Z),A} \otimes R(-E)) + h^0(E, \mathcal{I}_{Z\cap E,E} \otimes (R|E)),$$

$$h^1(A, \mathcal{I}_{Z,A} \otimes R) \le h^1(A, \mathcal{I}_{\mathrm{Res}_E(Z),A} \otimes R(-E)) + h^1(E, \mathcal{I}_{Z\cap E,E} \otimes (R|E)).$$

Remark 2. Fix an integer $c > 0$, an integral projective variety A, an integral effective Cartier divisor E of A, a closed subscheme Z of A such that $Z \cap E \neq E$, and $R \in \mathrm{Pic}(A)$. Fix a general $B \subset A$ and a general $B' \subset E$ such that $\sharp(B) = \sharp(B') = c$. The multiplicity two case of [1], Lemma 2.3 and Figure 1, gives

$$h^0(A, \mathcal{I}_{Z\cup 2B,A} \otimes R) \le h^0(A, \mathcal{I}_{\mathrm{Res}_E(Z)\cup\{2B',E\},A} \otimes R(-E)) + h^0(E, \mathcal{I}_{Z\cap E\cup B',E} \otimes (R|E)),$$

$$h^1(A, \mathcal{I}_{Z\cup 2B,A} \otimes R) \le h^1(A, \mathcal{I}_{\mathrm{Res}_E(Z)\cup\{2B',E\},A} \otimes R(-E)) + h^1(E, \mathcal{I}_{Z\cap E\cup B',E} \otimes (R|E)).$$

Remark 3. Let D be a smooth and projective curve. Fix $R \in \mathrm{Pic}(D)$, a linear subspace $V \subseteq H^0(D, R)$ and integers $c \ge 1$, $m_i > 0$, $1 \le i \le c$. Fix c general points $P_1, \dots, P_c \in D$ and set $B := \sum_{i=1}^c \{m_i P_i, D\}$. Since $\mathrm{char}(\mathbb{K}) = 0$, for any integral and non-degenerate curve $M \subset \mathbb{P}^n$ the general point of M is not an osculating point of M (see [22], Theorem 14). Hence by induction on c the generality of the c-ple $(P_1, \dots, P_c) \in D^c$ gives $\dim(V \cap H^0(B, \mathcal{I}_D \otimes R)) = \max\{0, \dim(V) - \sum_{i=1}^c m_i\}$ (if $c \ge 2$ use $c-1$ general linear projections from suitable general osculating spaces).

We use the following set-up. Fix a line $L \subset \mathbb{P}^3$ and let $\beta : \widetilde{\mathbb{P}} \to \mathbb{P}^3$ be the blowing-up of L. Fix an integral surface S with degree $s > 5$ such that S contains L with multiplicity $s-3$ and such that its strict transform Y in $\widetilde{\mathbb{P}}$ is smooth. Set $u := \beta|Y$, $H := u^*(\mathcal{O}_S(1))$ and $E := u^{-1}(L)$. As in [25], Lemma 2.5, we assume that E is integral. All these conditions are satisfied if we take as S a general degree s surface containing L with multiplicity $s-3$. The line bundle H is spanned and $H^2 = s$. We have $h^1(Y, \mathcal{O}_Y(nH - mE)) = 0$ for all integers n, m such that $0 \le m \le s-3$ (see [25], Lemma 2.2). Fix a smooth and irreducible curve $C_1 \subset S$ and set $d' := \deg(C_1)$ and $g' := p_a(C_1)$. Let C' be the strict transform of C_1 in Y. Since C_1 is smooth, $C' \cong C_1$ and in particular $p_a(C') = g'$. As in [25] we assume $\sharp(C_1 \cap L) = C' \cdot E = d' - 6$. Hence $C' \cdot H = 6$. Consider the linear system $|C' + rH|$, $r \in \mathbb{Z}$. Since $d' \ge 6$ and $C' \cdot E = d' - 6$, for every $r \ge s-3$ the linear system $|C' + rH|$ is spanned and its general member is a smooth and connected curve of genus $g' + r(2d' + rs + 3s - 12)/2$ mapped isomorphically onto a smooth space curve of degree $d' + rs$ (see [25], Lemma 2.5). Fix an integer $r' \ge 2$ such that $h^1(Y, \mathcal{O}_Y(C' + r'H)) = 0$. The minimal such integer $r' \ge 2$ depends only from C'. The proof of [25], Lemma 2.5, gives $h^1(C', \mathcal{O}_{C'}(C' + rH)) = 0$ for every $r \ge s-3$. Hence $r' \le s-3$. Let T be any smooth element of $|H|$. T is the normalization of a degree s plane curve with an ordinary multiple point with multiplicity $s-3$ as its only singularity. Hence T is a connected smooth curve with genus $(s-2)(s-1)/2 - (s-3)(s-4)/2 = 2s-5$ and $\deg(H|T) = s$.

Lemma 1. *We have* $h^2(Y, \mathcal{O}_Y(mH)) = h^2(Y, \mathcal{O}_Y(C' + mH)) = 0$ *for all* $m > 0$.

Proof. We have $\omega_Y \cong \mathcal{O}_Y((s-4)H - (s-4)E)$. Hence $h^0(Y, \omega_Y(tH)) = 0$ for all $t < 0$. Serre duality gives $h^2(Y, \mathcal{O}_Y(mH)) = 0$ for all $m > 0$. Since C' is effective and $\dim(Y) = 2$, we have $h^2(Y, \mathcal{O}_Y(C' + mH)) \le h^2(Y, \mathcal{O}_Y(mH))$ for every $m \in \mathbb{Z}$. Hence $h^2(Y, \mathcal{O}_Y(C' + mH)) = 0$ for all $m > 0$. □

Remark 4. Fix any smooth $T \in |H|$. Consider the exact sequence:

$$0 \to \mathcal{O}_Y(C' + (m-1)H) \to \mathcal{O}_Y(C' + mH) \to \mathcal{O}_T(C' + mH) \to 0. \quad (5)$$

Since $r' \ge 2$, Lemma 1, induction on m and the exact sequence (5) give $h^1(T, \mathcal{O}_T(C' + mH)) = 0$ for all $m \ge r' + 1$. Hence Riemann-Roch gives $h^0(T, \mathcal{O}_T(C' + mH)) = 12 + ms - 2s$ for all $m \ge r' + 1$. The definition of the integer r' implies the surjectivity of the restriction map $H^0(Y, \mathcal{O}_Y(C' + mH)) \to H^0(T, \mathcal{O}_T(C' + mH))$ for all $m \ge r' + 1$.

Lemma 2. *We have*

$$h^0(T, \mathcal{O}_T(C' + nH)) + 1 \le 2(h^0(T, \mathcal{O}_T(C' + (n-1)H))) \quad (6)$$

for all integers $n \geq \max\{4, r'+1\}$.

Proof. Recall that $h^0(T, \mathcal{O}_T(C'+mH)) = 12+(m-2)s$ for all integers $m \geq r'$. By assumption $r' \geq 2$. Obviously, $12+(n-2)s+1 \leq 2(12+(n-3)s)$ for all $n \geq 4$. □

Lemma 3. *Assume* $r \geq r'+1$ *and set* $\alpha := (r-r')(12-2s)+(s-2)((r+1)r/2 - r'(r'-1)/2)$. *Fix any integer* a *such that* $0 \leq 3a \leq \alpha - 1$. *Then* $h^1(Y, \mathcal{I}_{2B}(C'+rH)) = 0$ *for a general* $B \subset Y$ *such that* $\sharp(B) = a$.

Proof. Note that $(r+1)r/2 - r'(r'-1)/2 = \sum_{m=r'+1}^{r} m$. Set $a_1 := \lfloor h^0(T, \mathcal{O}_T(C'+rH))/2 \rfloor$ and $e_1 := h^0(T, \mathcal{O}_T(C'+rH)) - 2a_1$. First assume $a \geq a_1+e_1$. We specialize B to a general union B' of $a-a_1-e_1$ double points of Y, a_1 double points of Y with support on T and (if $e_1 = 1$) we apply Remark 2 with respect to the integer $c := e_1$ and the Cartier divisor $E := T$. The scheme $B' \cap T$ is a general union of a_1 double points of T and e_1 points of T. The case $c = a_1+e_1$, $m_i \leq 2$ for $1 \leq i \leq a_1$ and (if $e_1 > 0$) $m_{a_1+1} = 1$ of Remark 3 gives $h^i(T, \mathcal{I}_{B' \cap T}(C'+rH)) = 0$, $i = 0, 1$. By Remark 2 it is sufficient to prove $h^1(Y, \mathcal{I}_{\mathrm{Res}_T(B')}(C'+(r-1)H)) = 0$. The virtual residue $\mathrm{Res}_T(B')$ of B' with respect to T is a general union of $a-a_1-e_1$ double points of Y, a_1 points of T and (if $e_1 > 0$) a double point of T. Set $a_2 := \lfloor (h^0(T, \mathcal{O}_T(C'+(r-1)H)) - a_1 - 2e_1)/2 \rfloor$ and $e_2 := h^0(T, \mathcal{O}_T(C'+(r-1)H)) - a_1 - 2e_1 - 2a_2$. Hence $e_2 \in \{0, 1\}$. Assume $a \geq a_1+e_1+a_2+e_2$. We specialize $\mathrm{Res}_T(B')$ to a general union of $a-a_1-e_1-a_2-e_2$ double points of Y, a_2 double points of Y with support on T, a_1 points of T, e_1 double points of T and (if $e_2 > 0$) a length 3 scheme obtained applying Remark 2 with respect to a unique point of T. To do this step (i.e. to have $a_2 \geq 0$) we need $a_1+2e_1 \leq h^0(T, \mathcal{O}_T(C'+(r-1)H))$. Since $2a_1+e_1 = h^0(T, \mathcal{O}_T(C'+rH))$, we may apply Lemma 2. Then we continue specializing some of the remaing $a-a-1-e_1$ double points of Y. At the end we have finitely many general points of T (at most $a_{r-r'-1}$ of them), and at most a double point of T (which again may be taken general). Call B this scheme. It is sufficient to prove $h^1(Y, \mathcal{I}_B(C'+(r'+1)H)) = 0$. Since $\mathrm{length}(B) \leq 12+(r'-1)(s-2)$, it is sufficient to quote Remark 3.

If $a \leq a_1+e_1-1$, the first step is sufficient to conclude, without quoting Remark 2 and taking as double points only double points of Y with support on T. If $\sum_{i=1}^{j}(a_i+e_i) \leq a < \sum_{i=1}^{j=1}(a_i+e_i)$ for some $j \leq r-r'-1$, then we conclude after $j+1$ steps. □

From now on we use $s-3$ instead of r'. We loose something for certain curves C_1, i.e. for certain integers d', g', but we do not how to use any additional information to get at the end a better result.

Lemma 4. *Fix integers $x \geq s-2 \geq 3$, $\alpha \geq 0$, $\alpha_1 \geq 0$, $0 < \mu \leq x-s+2$, $0 < \mu_1 \leq x-s+2$, $0 < \mu'$, $u \geq 0$, $v \in \{0,1\}$, $u_1 \geq 0$ and $v_1 \in \{0,1\}$ such that*

$$\alpha\mu + \mu' + 2u + v = 12 + x(s-2), \tag{7}$$

$$\alpha\mu_1 + 2u_1 + v_1 = 12 + x(s-2), \tag{8}$$

Then

$$\alpha(\mu-1) + \mu' - 1 + u + 2v \leq 12 + (x-1)(s-2), \tag{9}$$

$$\alpha_1(\mu_1 - 1) + u + 2v \leq 12 + x(s-2). \tag{10}$$

Proof. To prove (9) (respectively (10)) it it sufficient to check that $\alpha+1+u \geq s-2+v$ (respectively $\alpha_1 + u_1 \geq s-2+v_1$). Since $v \in \{0,1\}$ (respectively $v_1 \in \{0,1\}$), it is sufficient to prove the inequality $\alpha + u \geq s-2$ (respectively $\alpha_1 + u_1 \geq s-1$). Since $0 < \mu \leq x-s+2$ (resp. $\mu_1 \leq x-s+2$), (7) (respectively (8)) proves this inequality. □

Notation 1. Fix a smooth surface Y', $P \in Y'$ and an integer $m > 0$. Set $\mathrm{ad}(mP) := (m-1)P$. For any disjoint union $Z \subset Y'$ of fat points with some multiplicities, say $Z = \sqcup Z_i$ with $Z_i = m_i P = i$ and $P_i \neq P_j$ for all $i \neq j$, set $\mathrm{ad}(Z) := \sqcup \mathrm{ad}(Z_i) = \sqcup (m_i - 1)P_i$.

Lemma 5. *Fix integers $s \geq 5$ and $m \geq s-1$. Fix a smooth and irreducible curve $C_1 \subset S$ and set $d' := \deg(C_1)$ and $g' := p_a(C_1)$. Let C' be the strict transform of C_1 in Y. We assume $C' \cdot H = 6$. Fix an integer z such that $0 \leq z \leq (s-2)(m-s+2)(m-s+1)/2 + \lfloor ((s-2)(12+(s-2)^2)-1)/3 \rfloor$. There is a zero-dimensional scheme $Z \subset Y$ such that $h^1(Y, \mathcal{I}_Z(C'+mH)) = 0$, each connected component of Z is a fat point of Y with multiplicity between 2 and $m-s+2$ and length$(\mathrm{ad}(Z)) = z$.*

Proof. Fix any smooth $T \in |H|$. Recall that $h^0(T, \mathcal{O}_T(x)) = 12 + x(s-2)$ for all $x \in \{s-2, \ldots, m\}$.

(a) Fix an integer δ such that $0 \leq \delta \leq \lfloor (m-s+2)(12+(s-2)^2)/3 \rfloor - 1$. Let $W \subset X$ be a general union of $s-2$ fat points with multiplicity $m-s+2$ and δ double points. Here we prove $h^1(Y, \mathcal{I}_W(C'+mH)) = 0$. It is sufficient to prove the vanishing for a virtual specialization of a general union of $s-2$ fat points with multiplicity $m-s+2$ and $b \geq \lfloor (m-s+2)(12+(s-2)^2)/2 \rfloor - 1$ double points. Set $a_1 := \lfloor (12+(s-2)^2)/2 \rfloor$ and $e_1 := 12 - 2a_1 + (s-2)^2$. Hence $e_1 \in \{0,1\}$. Let W_1 be a general union of $s-2$ fat points of Y with multiplicity $m-s+2$ and support on T, $b-a_1-e_1$ double points of Y, a_1 double points of Y with support on T and (if $e_1 > 0$) a scheme obtained applying Remark 2 with $E := T$ and a unique general point of T. We have $\text{length}(W_1 \cap T) = (s-2)(m-s+2)+2a_1+e_1 = h^0(T, \mathcal{O}_T(C'+mH))$. Since the support of $W_1 \cap T$ is general in T, Remark 3 gives $h^i(T, \mathcal{I}_{W_1 \cap T}(C'+mH)) = 0$, $i = 0, 1$. Hence it is sufficient to prove $h^1(Y, \mathcal{I}_{\text{Res}_T(W_1)}(C'+(m-1)H)) = 0$ (Remark 2). The scheme $\text{Res}_T(W_1) \cap T$ is a general union of $s-2$ fat points with multiplicity $m-s+1$, a_1 points of T and e_1 double points of T. Set $a_2 := \lfloor (12+(s-2)^2 - a_1 - 2e_1)/2 \rfloor$ and $e_2 = 12+(s-2)^2 - a_1 - 2e_1 - 2a_2$. We specialize $\text{Res}_T(W_1)$ to a general union of $s-2$ fat points of Y with multiplicity $m-s+1$ and support on T, a_1 points of T, e_1 double points of T, a_2 double points of Y with support on T and (if $e_2 > 0$) a scheme obtained applying Remark 2 with $E := T$ at a unique general point of T. And so on. At each step we need to check that the integer a_i is non-negative. This is true by the case $\mu_1 = x-s+2$ of Lemma 4. At the end (after taking $m-s+1$ times a residual scheme) we obtain a scheme $A \subset T$ union of $s-2$ general points of T (one for each of the $s-2$ fat points with multiplicity $m-s+2$ of Y with support on T, a_{m-s+2} general points of T and e_{m-s+2} double points of T. To get $h^1(Y, \mathcal{I}_W(C'+mH)) = 0$ it is sufficient to prove $h^1(Y, \mathcal{I}_A(C'+(s-2)H)) = 0$. This is true, because $h^1(T, \mathcal{I}_A(C'+(s-2)H)) = 0$ by Lemma 4 and an obvious case of Remark 3.

(b) Part (a) proves the lemma for all integers z such that $(s-2)(m-s+3)(m-s+2)/2 \le z \le (s-2)(m-s+2)(m-s+1)/2 + 3 \cdot \lfloor (m-s+2)(12+(s-2)^2)/3 \rfloor - 1$. We make a similar construction, except that we only use $s-3$ fat points of Y of multiplicity $m-s+2$ with support on T and (if $m-s+1 \ge 2$) a fat point of Y with multiplicy $m-s+1$ and support on Y (plus as much double points as possible). At each step we may apply the case $\mu = x-s+2$ and $\mu' = x-s+1$ of Lemma 4.

Claim. *With the construction made so far in part (b) we get a connected range for the integers length(ad(Z)) whose upper bound is* $\ge (s-2)(m-s+2)(m-s+1)/2$.

Proof of the Claim. We have $\text{length}(\text{ad}((m-s+2)P)) - \text{length}(\text{ad}((m-s+1)P)) = (m-s+2)(m-s+1)/2 - (m-s+1)(m-s)/2 = m-s+1$. Since in the construction just done in part (a) we are able to add at least the same number of double points, it is sufficient to check if $\lfloor (m-s+2)(12+(s-2)^2)/3 \rfloor - 1 \ge m-s+1$. The last equality is obviously true because $12+(s-2)^2 \ge 4$. □

Then we make a similar construction, except that we only use $s-3$ fat points of Y of multiplicity $m-s+2$ with support on T and (if $m-s+1 \ge 2$) a fat point of Y with multiplicy $m-s+1$ and support on Y (plus as much double points as possible). At each step we may apply the case $\mu = x-s+2$

and $\mu' = x - s$ of Lemma 4. At each step we get an interval for the lengths which overlap the previous interval: just use the proof of the claim, i.e. just use that $12 + (s-2)^2 \ge 4$. For low z we may also use only double points and just quote Lemma 3. □

Proposition 2. *Fix integers $s \ge 5$ and $m \ge s-1$. Fix a smooth and irreducible curve $C_1 \subset S$ and set $d' := \deg(C_1)$ and $g' := p_a(C_1)$. Let C' be the strict transform of C_1 in Y. Assume $C' \cdot H = 6$. Fix an integer g such that $g' + md' + m^2s/2 + 3ms/2 - 6m - (s-2)(m-s+2)(m-s+1)/2 + (m-s+3) - \lfloor (s-2)(12+(s-2)^2)/3 \rfloor \le g \le g' + md' + m^2s/2 + 3ms/2 - 6m$. Then there exists $X \in |C' + mH|$ such that X has geometric genus g and the only singular points of X are ordinary multiple points of various multiplicities between 2 and $m-s+2$.*

Proof. Recall that $q := p_a(A) = g' + md' + m^2s/2 + 3ms/2 - 6m$ for any $A \in |C' + mH|$.

(a) Fix integers $c \ge 0$, $m_i \ge 3$, $1 \le i \le c$, $b \ge 0$. We assume $h^1(Y, \mathcal{I}_Z(C' + mH)) = 0$ for a general union Z of c fat points of Y with multiplicity $m_1, \dots, m_c$ and b double points of Y. Lemma 5 shows that the assumption "$h^1(Y, \mathcal{I}_Z(C' + mH)) = 0$" is satisfied in one of the following cases:

(a1) $c \ge 0$, $m_i = m - s + 2$ for all $1 \le i \le c$, and $0 \le a \le \lfloor (h^0(Y, \mathcal{O}_Y(C' + mH)) - h^0(Y, C' + (s-2)H)) - \sum_{i=1}^{c} (m_i+1)m_i/2)/3 \rfloor - 1$.

(a2) $c > 0$, $m_i = m-s+2$ for all $1 \le i \le c-1$, $3 \le m_c \le m-s+1$, and $0 \le a \le \lfloor (h^0(Y, \mathcal{O}_Y(C' + mH)) - h^0(Y, C' + (s-2)H)) - \sum_{i=1}^{c} (m_i+1)m_i/2)/3 \rfloor - 1$.

However, we take c, m_i, a satisfying a stronger condition, i.e. we assume that one of the following conditions holds:

(a3) $c \ge 0$, $m_i = m - s + 2$ for all $1 \le i \le c$, and $0 \le a \le \lfloor (h^0(Y, \mathcal{O}_Y(C' + mH)) - h^0(Y, C' + (s-2)H)) - \sum_{i=1}^{c} (m_i+1)m_i/2)/3 \rfloor - 2$.

(a4) $c > 0$, $m_i = m-s+2$ for all $1 \le i \le c-1$, $3 \le m_c \le m-s+1$, and $0 \le a \le \lfloor (h^0(Y, \mathcal{O}_Y(C' + mH)) - h^0(Y, C' + (s-2)H)) - \sum_{i=1}^{c} (m_i+1)m_i/2)/3 \rfloor - 2$.

At one step we were forced to use the following stronger assumptions:

(a5) $c \ge 0$, $m_i = m - s + 2$ for all $1 \le i \le c$, and $0 \le a \le \lfloor (h^0(Y, \mathcal{O}_Y(C' + mH)) - h^0(Y, C' + (s-2)H)) - \sum_{i=1}^{c} (m_i+1)m_i/2 - (m-s+3))/3 \rfloor - 2$.

(a6) $c > 0$, $m_i = m - s + 2$ for all $1 \le i \le c - 1$, $3 \le m_c \le m - s + 1$, and $0 \le a \le \lfloor (h^0(Y, \mathcal{O}_Y(C' + mH)) - h^0(Y, C' + (s-2)H)) - \sum_{i=1}^{c}(m_i + 1)m_i/2 - (m - s + 3))/3 \rfloor - 2$.

Take a general $W \in |\mathcal{I}_Z(C' + mH)|$. Assume for the moment that W is integral and that its only singularities are c ordinary multiple points of multiplicity $m_1, \dots, m_c$ and a ordinary nodes. In this case W would have geometric genus $g' + md' + m^2 s/2 + 3ms/2 - 6m - b - \sum_{i=1}^{c} m_i(m_i - 1)/2$.

(b) Here and in the next two steps we assume (a5) or (a6) and prove that a general $W \in |\mathcal{I}_Z(C' + mH)|$ is integral and that its only singularities are c ordinary multiple points of multiplicity $m_1, \dots, m_c$ (say at the points $P_1, \dots, P_c$) and a ordinary nodes, say at the points $Q_1, \dots, Q_a$. Since W is general $(P_1, \dots, P_c, Q_1, \dots, Q_a)$ is general in Y^{a+c}. We only use (a5) or (a6) instead of (a3) or (a4) in step (d) to check that W is integral. Here in step (b) we prove that W has an ordinary point with multiplicity m_i at each P_i and an ordinary node at each $Q_1, \dots, Q_a$. Since $|\mathcal{I}_Z(C' + mH)|$ is a non-empty projective space, it is irreducible. Since W is general in $|\mathcal{I}_Z(C' + mH)|$ and $(P_1, \dots, P_a)$ is general in Y^a, to prove that W has an ordinary point with multiplicity m_i at each P_i it is sufficient (at least for general Z) to fix $i \in \{1, \dots, c\}$ and show that W has an ordinary point with multiplicity m_i at P_i. Let $u : A \to Y$ be the blowing-up of Y at P_i. Let $D := u^{-1}(P_i)$ be the exceptional divisor. Set $R := u^*(\mathcal{O}_Y(C' + mH)))$ and $Z' := u^{-1}(Z \backslash m_i P_i)$. Since $P_j \neq P_i$ for all $j \neq i$ and $Q_h \neq P_i$ for all h, u induces an isomorphism between Z' and $Z \backslash m_i P_i$. Since $h^1(Y, \mathcal{I}_Z(C' + mH)) = 0$, we have $h^1(A, \mathcal{I}_{Z'} \otimes R(-m_i D)) = 0$. Let B be a general element of $|\mathcal{I}_{Z'} \otimes R(-m_i D)|$. W has an ordinary point with multiplicity m_i at P_i if and only B is transversal to D. This is an open condition. A curve $B' \subset A$ is transversal to D if and only if it contains no scheme $\{2O, D\}$ with $O \in D$. Since B is general in $|\mathcal{I}_{Z'} \otimes R(-m_i D)|$, to show that it is transversal to D it is sufficient to show that for every $O \in D$ the projective space in $|\mathcal{I}_{Z' \cup \{2O, D\}} \otimes R(-m_i D)|$ has codimension 2 in the projective space $|\mathcal{I}_{Z'} \otimes R(-m_i D)|$. Hence it is sufficient to prove $h^1(A, \mathcal{I}_{Z' \cup \{2O, D\}} \otimes R(-m_i D)) = 0$ for every $O \in D$. Remember that we assumed (a3) or (a4), while Lemma 5 gives $h^1(Y, \mathcal{I}_Z(C' + mH)) = 0$ if (a1) or (a2) are satisfied. It is sufficient to repeat the proof of Lemma 5 on A instead of Y with at the beginning the additional scheme $\{2O, A\}$. In the same way we prove that W has an ordinary node at each Q_j.

(c) Here we prove that W is smooth ouside $\{P_1, \dots, P_c, Q_1, \dots, Q_a\}$. Let $v : A' \to Y$ be the blowing-up of $\{P_1, \dots, P_c\}$. Set $D_i := v^{-1}(P_i)$, $1 \le i \le c$. Let J be the strict transform of W in A. If (a3) (respectively (a4)) is satisfied, then (a1) (respectively (a2)) is satisfied for the integer $a' := a + 1$.

Hence Lemma 5 gives $h^1(Y, \mathcal{I}_{Z\cup 2Q}(C' + mH)) = 0$ for a general $Q \in A$, i.e. $h^1(A', \mathcal{I}_{2B\cup 2Q'} \otimes v^*(\mathcal{O}_Y(C' + mH))(-\sum_{i=1}^{c} m_i D_i)) = 0$ for a general $B \subset A'$ such that $\sharp(B) = a$ and a general $Q' \in A'$. Take a general

$$J \in |\mathcal{I}_{2B} \otimes v^*(\mathcal{O}_Y(C' + mH))(-\sum_{i=1}^{c} m_i D_i))|.$$

By [5], Theorem 1.4, $\mathrm{Sing}(J) = S'$, i.e. W is smooth outside $\{P_1, \dots, P_c, Q_1, \dots, P_c\}$.

(d) Here we check that W is integral. Our assumption on a gives

$$\dim(|\mathcal{I}_{Z\cup 2O}(C' + mH)|) = \dim(|\mathcal{I}_Z(C' + mH)|) - 3,$$

for a general $O \in Y$. Hence at a general $O \in Y$ the differential of the rational map induced by $|\mathcal{I}_Z(C'+mH)|$ has rank 2. Hence the image of the rational map induced by $|\mathcal{I}_Z(C' + mH)|$. Hence if $|\mathcal{I}_Z(C' + mH)|$ has no base component, then a theorem of Bertini gives that a general member of $|\mathcal{I}_Z(C' + mH)|$ is irreducible (part (4) of [19], Theorem 6.3). Now we check that $|\mathcal{I}_Z(C' + mH)|$ has no base component. Assume the contrary and call J the 1-dimensional part of its base locus. Write $W = W' + J$ (as Cartier divisors of Y) with W' effective and non-empty. Let e_i, $1 \le i \le c$, be the multiplicity of J at P_i and f_j, $1 \le j \le a$, the multiplicity of J at Q_j. For any $j \in \{0, 1, 2\}$ set $F_i := \{j \in \{1, \dots, a\} : f_j = i\}$. Hence $\{1, \dots, a\}$ is the disjoint union of the sets F_0, F_1, and F_2. Since Y^a is irreducible, for general $(Q_1, \dots, Q_a) \in Y^a$ only one of the sets F_0, F_1, and F_2 is non-empty. For the same reason in case (a3) we have $f_i = f_1$ for all $i \in \{1, \dots, c\}$, while in case (a4) we have $f_i = f_1$ for all $i \in \{1, \dots, c-1\}$. We want to prove $F_1 = F_2 = \emptyset$, i.e. $\{Q_1, \dots, Q_a\} \cap J_{red} = \emptyset$. Since this is trivially true if $a = 0$, to prove it we may assume $a > 0$.

Claim. *Set $Z'' := (Z \backslash 2Q_a) \cup 3Q_a$. We claim that $h^1(Y, \mathcal{I}_{Z''}(C'+mH)) = 0$ for a general $(P_1, \dots, P_c, Q_1, \dots, Q_a\}) \in Y^{a+c}$.*

Proof of the Claim. To prove the claim it is sufficient to modify the proof of Lemma 5 in the following way. Recall that $h^0(T, \mathcal{O}_T(C'+mH)) = 12+m(s-2)$. Set $a_1' := \lfloor (12+m(s-2) - \sum_{i=1}^{c} m_i - 3)/2 \rfloor$ and $e_1' := 12+m(s-2) - \sum_{i=1}^{c} m_i - 3 - 2a_1'$. We have $a_1' > 0$, because $\sum_{i=1}^{c} m_i \le (s-2)(m-s+2)$. First assume $a \ge a_1' + e_1' + 1$. In the first step we specialize Z'' to a general union of $a - a_1' - e_1' - 1$ double points, c fat points of multiplicity $m_1, \dots, m_c$ supported by general points of T, a triple point of Y with support on T, a_1' double points of Y with support on T and a virtual scheme with length 3 obtained applying Remark 2 with respect to one point of T. By Remark 2 it is sufficient to prove the h^1-vanishing for the virtual residual scheme. From the next step on we do as in the proof of Lemma 5. If $a \le a_1' + e_1'$ at the first step we only insert $a - 1$

(not $a_1' + e_1'$) double points of Y whose support is a point of T. □

Set $Z_1 := Z\backslash 2Q_a$. By the claim $H^0(Y, \mathcal{I}_{Z_1}(C' + mH))$ generates $\mathcal{O}_Y(C' + mH)$ at Q_a up to jets of order 3. Hence the connected component of the scheme-theoretic base locus $\mathbb{B}$ of $\mathcal{I}_Z(C'+mH)$ supported by Q_a is the double point Q_a. Assume $f_a > 0$, i.e. assume $Q_a \in J_{red}$. Since $\mathbb{B}$ contains the third order jet of J_{red} at Q_a, we get a contradiction. Thus $F_1 = F_2 = \emptyset$, i.e. $Q_j \neq J$ for all j.

Now we use that we assumed (a5) or (a6) instead of (a3) or (a4). Fix $i \in \{1, \dots, c\}$ and call W_i the scheme obtained from the data of (a3) or (a4), except that we omit the point P_i. Since we have a smaller number of double points, we may modify the statement and the proof of Lemma 5 in the following way. In step (a) we add $(s-3)$ fat points of Y with multiplicity $(m-s+2)$ and support on T and one fat point of Y with multiplicity $(m-s+3)$ and support on T. The modify form of Lemma 5 gives that the linear system $|\mathcal{I}_{W_i}(C' + mH)|$ spans the jets of $\mathcal{O}_Y(C' + mH)$ up to multiplicity $m_i + 1$, not only up to multiplicity m_i. Hence as above we get $J \cap \{P_1, \dots, P_c\} = \emptyset$. Since W is smooth outside $\{P_1, \dots, P_c, Q_1, \dots, Q_a\}$, we get $W' \cap J + \emptyset$. Hence $h^0(W, \mathcal{O}_W) \geq 2$. Take any $M \in |\mathcal{O}_Y(C' + mH)|$. Since $h^1(Y, \mathcal{O}_Y) = 0$ (case $q = 0$ of [25], Lemma 2.2), the exact sequence

$$0 \to \mathcal{O}_Y(-C' - mH) \to \mathcal{O}_Y \to \mathcal{O}_M \to 0 \tag{11}$$

gives $h^0(M, \mathcal{O}_M) = 1 + h^1(Y, \mathcal{O}_Y(-C' - mH))$. Hence the integer $h^0(M, \mathcal{O}_M)$ is the same for all $M \in |\mathcal{O}_Y(C'+mH)|$. Since a general element of $|\mathcal{O}_Y(C'+mH)|$ is smooth and irreducible, while $h^0(W, \mathcal{O}_W) \geq 2$, we get a contradiction. □

Proof of Theorem 1. The existence of d', g', m, C' such that $m \geq s-3$ and a general element of $|\mathcal{O}_Y(C'+mH)|$ has arithmetic genus q is proved in [25]. The restriction $m < (d+s-6)/s$ and the choice of m comes from [25], p. 57 (before Proposition 2.6). Let X be the image in $\mathbb{P}^3$ of a general $W \in |\mathcal{I}_Z(C' + mH)|$. The curve W is an integral curve with arithmetic genus q and geometric genus g (Proposition 2). Hence to prove the theorem for the quadruple (d, s, g, q) it is sufficient to prove that the map $W \to X$ is an isomorphism. Obviously, this map is an isomorphism ouside $E \cap W$. Fix $P \in E$. Since Lemma 5 works in a larger range than Proposition 2, we may find Z with the additional property $h^1(Y, \mathcal{I}_{Z\cup\{P\}}(C' + mH)) = 0$. In this case $P \notin W$ for a general $W \in |\mathcal{I}_Z(C'+mH)|$. We may do this simultaneously at the finitely many points of E at which the finite map $\kappa : E \to L$ is not étale. As in the proof of Proposition 5 we see that we may assume that W is transversal to W. Hence $X \cong W$ if and only if $\kappa|W$ is injective. Set $\mathbb{B} := \{(P, Q) \in E \times E : P \neq Q, \kappa(P) = \kappa(Q)\}$. Fix any $(P, Q) \in \mathbb{B}$, such that κ is étale at P and at Q. For general Z we get $h^1(Y, \mathcal{I}_{Z\cup\{P,Q\}}(C' + mH)) = 0$ as in Lemma 5. Hence Q is not a base point of the linear system $|\mathcal{I}_{Z\cup\{P\}}(C'+mH)|$. Hence we may find W_P containing P and such that $\kappa^{-1}(\kappa(P))\cap W_P=\{P\}$. Hence if $\kappa|W$ is not injective, we may deform W so that a nearby curve W' has strictly less pairs of distinct points with the same image. Since $|\mathcal{I}_Z(C'+mH)|$ is irreducible, we get the injectivity of $\kappa|W$ for general W. □

Lemma 6. *Fix integers d,s,q such that $s\ge 5$, $d>s^2-3s+6$ and $(s-2)d-(s^3-7s^2+24s-30)<q\le(d^2+3(s-4)d-(\epsilon^2-\epsilon s+12s-36))/12s$, where ϵ is the unique integer such that $0\le\epsilon<s$ and $\epsilon\equiv d-6 \pmod s$. There is a smooth curve $C'\subset Y$ with associated integers d',g', and an integer m such that $s-3\le m<(d+s-6)/6$, $d=d'+rs$, $q=g'+(2d'+rs+3s-12)/2$, and a general member of $|C'+rH|$ is a smooth curve of genus q mapped isomorphically onto a smooth degree d curve of S. Set $\alpha:=(m-s+3)((12-2s+(m+1)m/2-(s-3)(s-4)/2$. Fix an integer g such that $q-\lfloor(\alpha-1)/3\rfloor+1\le g\le q$. Then there exists an integral and nodal curve $X\in|C'+rH|$ with geometric genus g which is mapped insomorphically into S and hence whose image in $\mathbb{P}^3$ is an integral and nodal degree d curve with geometric genus g and arithmetic genus q contained in a surface of degree s.*

Proof. The existence of C' comes from the proof of [25], Theorem 0.4. Recall that $H\cdot X=d$ and $p_a(X)=q$ for every $X\in|C'+rH|$. To get an integral curve with exactly $q-g$ ordinary node, just copy the case $c=0$ of Proposition 2, just quoting Lemma 3 instead of Lemma 5. □

Proof of Theorem 2. Copy the proof of Theorem 1, just quoting Lemmas 3 and 6 instead of Lemma 5 and Proposition 2. □

3. Proofs of Theorem 3 and Proposition 1

Remark 5. Fix integers d,s such that $d>s(s-1)\ge 5$. Let r be the only integer such that $0\le r<s$ and $d+r\equiv 0 \pmod s$. Let C be an integral space curve such that $\deg(C)=d$ and $h^0(\mathbb{P}^3,\mathcal{I}_C(s-1))=0$. We claim that $p_a(C)\le G_C(d,s)$ and that $p_a(C)=G_C(d,s)$ if and only if C is as described in [12], Theorem 3.1, i.e. it is linked to a plane curve of degree r by a complete intersection of a surface of degree s and a surface of degree $(d+r)/s$. Let $H\subset\mathbb{P}^3$ be a general plane. Let σ be the minimal degree of a plane curve containing $C\cap H$. Laudal's Generalized Trisecant Lemma was stated and proved for integral curves (see [23]). Hence $\sigma\ge s$ (see [23], Corollary 2.2). First assume $\sigma=s$. The statements of [12], Lemma 3.2 and Proposition 3.2, only require that C is integral. The inequality $p_a(C)\le G_C(d,s)$ is just the

case $s = \sigma$ of [12], Proposition 3.2, proving our claim if $\sigma = s$. If $\sigma > s$ note that the function $G_{C.M}(d, \sigma)$ of [12] is a decreasing function of σ. Note that we have $h^0(\mathbb{P}^3, \mathcal{I}_C(s)) \neq 0$. Hence in the range C we have $G'(d, s) = G(d, s) = G_C(d, s)$ even if we allow singular integral curves, taking as genus the arithmetic genus.

The next observation is just the case $m_i = 2$ of Remark 3.

Remark 6. Fix an integer $a > 0$, an integral projective curve C, $L \in \mathrm{Pic}(C)$ and any linear subspace $V \subseteq H^0(C, L)$. Let $S \subset C$ be a general union of a points. Since we are in characteristic zero, any non-constant rational map from C into a projective space is separable and in particular its differential is injective at a general point of C. Hence by induction on a the generality of S gives $\dim(V \cap H^0(C, \mathcal{I}_{2S} \otimes L)) = \max\{0, \dim(V) - 2a\}$.

In the proof of Lemma 7 we will use several times the following numerical observation.

Remark 7. Fix integers $y', m', u, v, y_1, u_1, v_1$ such that $m' > 0$, $y' \geq m'+1$, $u \geq 0$, $v \in \{0, 1\}$ and $y' + 1 = m' + 2u + v$. We have $y' \geq (m' - 1) + u + 2v$ if either $v = 0$ or $u > 0$. If $v = 1$ and $u = 0$, then $y' = m' + 1$. Hence we have $y' \geq (m' - 1) + u + 2v$ if $y' \geq m' + 2$. Now we handle the case $m' = 0$. Assume $u_1 \geq 0$, $v_1 \in \{0, 1\}$ and $y_1 + 1 = 2u_1 + v_1$. We have $y_1 \geq u_1 + 2v_1$ if either $v_1 = 0$ and $u_1 > 0$ or $u_1 \geq 2$. If $y_1 \geq 3$, then at least one of the previous inequalities is satisfied and hence $y_1 \geq u_1 + 2v_1$.

The first part of the next lemma is well-known (see, e.g. the case F_1 of [22]).

Lemma 7. *Fix integers y, m, a such that $y - 2 \geq m > 0$ and $0 \leq a \leq \lfloor ((y + 2)(y + 1)/2 - m(m - 1)/2)/3 \rfloor - 1$. Fix $P \in \mathbb{P}^2$ and a general $S \subset \mathbb{P}^2$ such that $\sharp(S) = a$. Set $Z := \{mP, \mathbb{P}^2\} \cup \{2S, \mathbb{P}^2\}$. Then $h^1(\mathbb{P}^2, \mathcal{I}_Z(y)) = 0$. If $a \leq \lfloor ((y + 2)(y + 1)/2 - m(m - 1)/2)/3 \rfloor - 2$, then a general $T \in |\mathcal{I}_Z(y)|$ is integral, with an ordinary point of multiplicity m at Q, an ordinary node at each point of S and no other singularity.*

Proof. It is sufficient to prove that $h^1(\mathbb{P}^2, \mathcal{I}_Z(y)) = 0$ for an integer $b \geq \lfloor ((y + 2)(y + 1)/2 - m(m - 1)/2 - 3)/3 \rfloor$. It is sufficient to find a virtual specialization of $2S$ for which the result is true. Fix a line R such that $P \in R$. Set $a_1 := \lfloor (y + 1 - m)/2 \rfloor$ and $e_1 := y + 1 - m - 2e_1$. Let Z_1 be the virtual specialization of Z which is a general union of $\{mP, \mathbb{P}^2\}$, $b - a_1 - e_1$ double points of $\mathbb{P}^2$, a_1 double points of $\mathbb{P}^2$ with support on R and (if $e_1 > 0$) the scheme obtained applying Remark 2 with $Y := \mathbb{P}^2$, $E := R$, and as P a general point of R. Hence $\mathrm{length}(Z_1) = y + 1$. Thus $h^i(R, \mathcal{I}_{Z_1 \cap R}(y)) = 0$, $i = 0, 1$. Hence it is sufficient to prove $h^1(\mathbb{P}^2, \mathcal{I}_{\mathrm{Res}_R(Z_1)}(y-1)) = 0$. The scheme $\mathrm{Res}_R(Z_1)$ is a general union of $\{(m-1)P, \mathbb{P}^2\}$, $b-a_1-e_1$ double points of $\mathbb{P}^2$, a_1 points of R and (if $e_1 > 0$) a double point of R. Set $w := \mathrm{length}(\mathrm{Res}_R(Z_1))$, $a_2 := \lfloor (y-w)/2 \rfloor$ and $e_2 := y - w - 2a_2$. Remark 7 gives $a_2 \geq 0$. We specialize $\mathrm{Res}_R(Z_1)$ to a general union Z_2 of $\{(m-1)P, \mathbb{P}^2\}$, $b - a_1 - e_1 - a_2 - e_2$ double points of $\mathbb{P}^2$, a_1 points of R, e_1 double points of R and (if $e_2 > 0$) the scheme obtained applying Remark 2 with $Y := \mathbb{P}^2$, $E := R$, and as P a general point of R. Hence $\mathrm{length}(Z_2 \cap R) = y$. Thus $h^i(R, \mathcal{I}_{Z'' \cap R}(y-1)) = 0$, $i = 0, 1$. Hence it is sufficient to prove $h^1(\mathbb{P}^2, \mathcal{I}_{\mathrm{Res}_R(Z_2)}(y-2)) = 0$. Then we continue in the following way. Fix an integer $t \leq y-3$ and assume given a scheme A_t for which we need to prove $h^1(\mathbb{P}^2, \mathcal{I}_{A_t}(y-t)) = 0$. A_t is the virtual residue $\mathrm{Res}_R(Z_t)$ of a zero-dimensional scheme Z_t. Since $h^0(R, \mathcal{O}_R(y-t-1)) = h^0(R, \mathcal{O}_R(y-t)) - 1$, to carry over the step $\mathcal{O}(y-t) \Longrightarrow \mathcal{O}(y-t-1)$ we only need to check that $\mathrm{length}(\mathrm{Res}_R(Z_t)) \leq y-t+1$. The scheme Z_t satisfies $\mathrm{length}(Z_t \cap R) = y-t+1$. Each connected component of the scheme Z_t gives a connected component with strictly smaller length, except at most one: if $e_{y-t-1} > 0$ then a length 1 connected component of Z_t has as virtual residue a double point of R. If $b < \sum_{i=1}^{t}(a_i + e_i)$ for some $t \leq y-2$, then we are done by Remark 7 and an application of Remark 2 with respect to $E := R$ without inserting any other double point with support on R. If $b \geq \sum_{i=1}^{y-2}(a_i + b_i)$, then we may continue until we are considering the linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$, because we may quote Remark 7 at each step. At the end for linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$ it only remains a virtual residual scheme $\Delta \subset R$. Set $\delta := \mathrm{length}(\Delta)$. Remark 7 gives $0 \leq \delta \leq 3$. Fix a general $O \in \mathbb{P}^2$ and a general $O_1 \in R$. First assume $\delta = 3$. Since $h^i(R, \mathcal{I}_\Delta(2)) = 0$, $i = 0, 1$, we get $h^i(\mathbb{P}^2, \mathcal{I}_{\Delta \cup \{2O, \mathbb{P}^2\}}(2)) = 0$, $i = 1, 2$. In this case $((y+2)(y+1)/2 - m(m-1)/2)/3$ is an integer and we win even for the integer $a' = ((y+2)(y+1)/2 - m(m-1)/2)/3$. If $\delta = 2$, then we take $\Delta \cup \{2O, \mathbb{P}^2\}$ and win for the integer $a' = ((y+2)(y+1)/2 - m(m-1)/2 - 1)/3$. Now assume $\delta = 1$. We take $\Delta \cup \{2O_1, \mathbb{P}^2\}$ and win for the integer $a' = ((y+2)(y+1)/2 - m(m-1)/2 - 3 + \delta)/3$. In all cases we get a scheme Z' union of $\{mP, \mathbb{P}^2\}$ and some double points such that $h^1(\mathbb{P}^2, \mathcal{I}_{Z'}(y)) = 0$ and $\mathrm{length}(Z') \geq (y+2)(y+1)/2 - 3$.

(a) Here and in the next two steps we prove the last sentence of the statement of the lemma. Hence here and in steps (b), (c) we assume $0 < a \leq \lfloor ((y+2)(y+1)/2 - m(m-1)/2)/3 \rfloor - 2$. Here and in steps (b) and (c) we fix a general $T \in |\mathcal{I}_Z(y)|$. By assumption a is not the maximal integer allowed to get $h^1 = 0$. We modify the first part of the proof in the following way. The aim is to get $h^1(\mathcal{I}_{mQ \cup 2S_1 \cup 3Q_a}(y)) = 0$, where $S = \{Q_1, \ldots, Q_a\}$ and $S_1 := S \backslash \{Q_a\}$.

At the first step we take $Q_a \in R$ and define the integers a_1', e_1' by the relations $y+1 = 3+2a_1'+e_1'$ (e.g. if $y = m+2$, then $a_1' = e_1' = 0$; if $y = m+3$, then $(a_1', e_1') = 0$; if $y = m+4$, then $(a_1', e_1') = (1,0)$). We insert a_1' double points of $\mathbb{P}^2$ with support on R and if $e_1' > 0$ we apply Remark 2 with respect to one general point of R. In the other steps we only use double points. At the end we get $h^1(\mathcal{I}_{mQ\cup 2S_1\cup 3Q_a}(y)) = 0$. Since $h^1(\mathcal{I}_{mQ\cup 2S_1\cup 3Q_a}(y)) = 0$, $H^0(\mathcal{I}_{mQ\cup 2S_1}(y))$ generates the jets of $\mathcal{O}_{\mathbb{P}^2}(y)$ at Q_a up to order 3. Hence a general element of $|\mathcal{I}_Z(y)|$ has an ordinary node at Q_a. Since the product of a copies of $\mathbb{P}^2$ is irreducible and S is general, T has an ordinary node at each point of S.

(b) Here we prove that T has an ordinary point with multiplicity m at P and it is smooth ouside $S \cup \{P\}$. Let $f : W \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at P. For each integer z set $\mathcal{O}_W(z) := f^*(\mathcal{O}_{\mathbb{P}^2}(z))$. Let $\Delta := f^{-1}(P)$ denote the exceptional divisor of f. Set $S' := f^{-1}(S)$. There is a natural isomorphim beteween $H^0(W, \mathcal{I}_{2S'}(y)(-m\Delta))$ and $H^0(\mathbb{P}^2, \mathcal{I}_{mP\cup 2S}(y))$. Let J be a general element of the linear system $|\mathcal{I}_{2S'}(y)(-m\Delta)|$. T has an ordinary point with multiplicity m at Q if and only if J intersects transversally Δ. Fix $O \in \Delta$. Recall that $\{2O, \Delta\}$ is the length 2 scheme with O as its reduction and contained in Δ. Recall that we could do the vanishing statement in the first part with the integer $a' := a+1$. Hence in the same way working on W we obtain $h^1(W, \mathcal{I}_{\{2O,\Delta\}\cup 2S'}(y)) = 0$, i.e. $|\mathcal{I}_{\{2O,\Delta\}\cup 2S'}(y)(-m\Delta)|$ has codimension 2 in $|\mathcal{I}_{2S'}(y)(-m\Delta)|$. Since $\dim(\Delta) = 1$ and J is general, J contains no scheme $\{2O, \Delta\}$ with $O \in \Delta$, i.e. it is transversal to Δ.

Now we prove that S is smooth ouside $S \cup \{P\}$. Our assumption on a and the h^1-vanishing part of the lemma gives $h^1(\mathbb{P}^2, \mathcal{I}_{mQ\cup 2S\cup 2U}(y)) = 0$ for a general $U \in \mathbb{P}^2$, i.e. $h^1(W, \mathcal{I}_{2S'\cup 2V}(y)(-m\Delta)) = 0$ for a general $V \in W$. By [5], Theorem 1.4, $\mathrm{Sing}(J) = S'$, i.e. T is smooth outside $S \cup \{P\}$.

(c) Here we check that T is integral and hence conclude the proof of the lemma. Our assumption on a gives $\dim(|\mathcal{I}_{Z\cup 2O}(y)|) = \dim(|\mathcal{I}_Z(y)|) - 3$ for a general $O \in \mathbb{P}^2$. Hence at a general $O \in \mathbb{P}^2$ the differential of the rational map induced by $|\mathcal{I}_Z(y)|$ has rank 2. Hence the image of the rational map induced by $|\mathcal{I}_Z(y)|$ is a surface. Hence if $|\mathcal{I}_Z(y)|$ has no base component, then a theorem of Bertini gives that a general member of $|\mathcal{I}_Z(y)|$ is irreducible (part (4) of [19], Theorem 6.3).

Now we check that $|\mathcal{I}_Z(y)|$ has no base component. Assume the contrary and call B the 1-dimensional part of its base locus. We have $T = A \cup B$ with A, B curves (not necessary reduced or irreducible). Set $u := \deg(A)$. Let m_1 be the multiplicity of A at P. We have $0 \le m_1 \le m$. The curve B has multiplicity $m - m_1$ at P, both A and B have P as an ordinary point with the prescribed multiplicity and their tangent cones at P have no common component. Hence A intersects B with multiplicity $m_1(m-m_1)$ at P. Since outside P the curve T has a ordinary nodes and no other singularity, outside P the curves A and B meets transversally at $u(y-u)$ points, each of them being an ordinary node of T. Let a_i, $i=1,2$, be the number of $j \in \{1,\dots,a\}$ such that B contains Q_j with multiplicity exactly i and call $S_i \subseteq \{Q_1,\dots,Q_a\}$, $i=1,2$, the corresponding subsets of $S := \{Q_1,\dots,Q_a\}$. The curve A contains a_1 points of S with multiplicity 1 and $a-a_1-a_2$ points of S with multiplicity 2. Bezout's Theorem gives

$$u(y-u) = m_1(m-m_1) + a_1. \tag{12}$$

Since S is general, moving $S \backslash S_1$ we see that for general S, T either $S_1 = S$ or $S_1 = \emptyset$. We have $m_1 \le u$ with strict inequality unless A is a union of m_1 distinct lines through P. We have $m - m_1 \le y - u$ with strict inequality unless B is a union of $y-u$ lines through P. Since $y \ge m+2$, (12) gives $a_1 > 0$. Hence $S_1 \ne \emptyset$. Hence $S_1 = S$, i.e. $a_1 = a$ and $a_2 = 0$. In step (b) we proved that $\mathcal{I}_{mP \cup 2S'}(y)$ generates the jets of $\mathcal{O}_{\mathbb{P}^2}(y)$ at Q_a up to order 3. However, since B is a fixed component of $|\mathcal{I}_Z(y)|$ and $Q_a \in B$, the vanishing of the order ≤ 2 jets force the vanishing at third order in the direction of the jet of order 3 of B at Q_a, contradiction. □

Remark 8. Let $Y \subset \mathbb{P}^3$ be a general degree s surface containing D. Y is smooth. If $r=0$, i.e. if $D=\emptyset$ we take as Y any smooth degree surface. We have $h^0(Y, \mathcal{O}_Y(n)) = \binom{n+3}{3}$ if $0 \le n \le s-1$ and $\binom{n+4}{3} - \binom{n-s+3}{3}$ if $n \ge s$. First assume $r > 0$, i.e. $D \ne \emptyset$. Fix a general hyperplane section C of Y. The curve C is isomorphic to a degree s smooth plane curve. Hence $p_a(C) = (s-1)(s-2)/2$, $h^0(C, \mathcal{O}_C(n)) = (n+2)(n+1)/2$ for all $0 \le n \le s-1$ and $h^0(C, \mathcal{O}_C(n)) = (n+2)(n+1)/2 - (n-s+2)(n-s+1)/2$ if $n \ge s$. Since $h^1(Y, \mathcal{O}_Y(n)) = 0$ for every $n \in \mathbb{Z}$, the restriction map $H^0(Y, \mathcal{O}_Y(n)) \to H^0(C, \mathcal{O}_C(n))$ is surjective for all n. Thus $h^0(Y, \mathcal{O}_Y(n)) = h^0(C, \mathcal{O}_C(n)) + h^0(Y, \mathcal{O}_Y(n-1))$ for all n.

(a) Notice that $h^0(C, \mathcal{O}_C(n)) > h^0(C, \mathcal{O}_C(n-1))$ for all $n > 0$ and that $h^0(C, \mathcal{O}_C(n)) \ge h^0(C, \mathcal{O}_C(n-1)) + 4$ for all integers $n \ge 2$.

(b) Here we check the inequality

$$h^0(C, \mathcal{O}_C(n)) + 1 \le 2 \cdot h^0(C, \mathcal{O}_C(n-1)) \tag{13}$$

for all integers $n \ge 3$. If $2 \le s-1$, then (13) is equivalent to the inequality $(n+2)(n+1)/2 + 1 \le 2((n+1)n/2)$, which is obviously true for all $n \ge 3$. If $n \ge s$, then (13) is equivalent to the inequality

$$(n+2)(n+1) - (n-s+1)(n-s) + 2 \le 2(n+1)n - 2(n-s)(n-s-1), \tag{14}$$

i.e. to the inequality

$$2 \le (n+1)(n-2) - (n-s)(n-s-1), \tag{15}$$

which is obviously true if $n \ge s \ge 5$. Now assume $r > 0$. Let E be the intersection with Y of a general plane through D; this plane is unique if $r \ge 2$. In the case $r > 0$ we only use that $\mathcal{O}_Y(1)(-D) = \mathcal{O}_Y(E)$ is effective. Hence for any zero-dimensional scheme $Z \subset Y$ such that $Z_{red} \cap E = \emptyset$ if Z imposes length(Z) independent conditions to $|\mathcal{O}_Y(n-1)|$, then it imposes length(Z) independent conditions to $|\mathcal{O}_Y(n)(-D)|$.

Lemma 8. *Fix integers $t \ge s \ge 5$, and a such that $0 \le a \le \lfloor (\binom{t+3}{3} - \binom{t-s+3}{3}) - 5)/3 \rfloor$. Let $Y \subset \mathbb{P}^3$ be any smooth degree s surface. Let $S \subset Y$ be a general subset such that $\sharp(S) = a$.*

(i) Then $h^1(Y, \mathcal{I}_{2S}(t)) = 0$, i.e. $h^0(Y, \mathcal{I}_{2S}(t)) = h^0(Y, \mathcal{O}_Y(t)) - 3a$.

(ii) Assume $a \le \lfloor (\binom{t+3}{3} - \binom{t-s+3}{3}) - 5)/3 \rfloor - 1$ and take a general $T \in |\mathcal{O}_{2S}(t)|$. Then T is integral Sing$(T) = S$ and T has an ordinary node at each point of S.

Proof. Let C be a smooth hyperplane section of Y. Hence C is isomorphic to a smooth degree s plane curve, for each integer n the restriction map $H^0(Y, \mathcal{O}_Y(n)) \to H^0(C, \mathcal{O}_C(n))$ is surjective,

$$h^0(Y, \mathcal{O}_Y)(t)) = \sum_{i=0}^{t} h^0(C, \mathcal{O}_C(i)),$$

$h^0(C, \mathcal{O}_C(n)) = (n+2)(n+1)/2$ if $0 \le n \le s-1$ and $h^0(C, \mathcal{O}_C(n)) = (n+2)(n+1)/2 - (n-s+2)(n-s+1)/2$ if $n \ge s$ (Remark 8).

(a) Here we check part (i). It is sufficient to find a virtual specialization of $2S$ for which the result is true. Set $a_1 := \lfloor ((t+2)(t+1)/2 - (t-s+2)(t-s+1)/2)/2 \rfloor$ and $e_1 := (t+2)(t+1)/2 - (t-s+2)(t-s+1)/2 - 2a_1$. Hence $e_1 \in \{0, 1\}$. First assume $a \ge a_1 + e_1$. Let Z_1 be the virtual specialization of $2S$ which is a general union of $a - a_1 - e_1$ double points of Y, a_1 double points of C and (if $e_1 > 0$) the scheme obtained applying Remark 2 with $E := C$ and as P a general point of C. Hence $Z_1 \cup C$ is a general union of e_1 points of C and a_1 double points of C. Remark 6 gives $h^0(C, \mathcal{I}_{Z_1 \cap C}(t)) = 0$. Hence it is sufficient to prove $h^1(Y, \mathcal{I}_{\mathrm{Res}_C(Z_1)}(t-1)) = 0$ (Remark 2). The scheme $\mathrm{Res}_C(Z_1)$ is a general union of $a - a_1 - e_1$ double points of Y, a_1 points of Y and e_1 double points of C. Hence length$(C \cap \mathrm{Res}_C(Z_1)) = a_1 + 2e_1$. Set $a_2 := \lfloor (h^0(C, \mathcal{O}_C(t-1)) - a_1 - 2e_1)/2 \rfloor$ and $e_2 := h^0(C, \mathcal{O}_C(t-1)) - a_1 - 2e_1 - a_2$. Hence $e_2 \in \{0, 1\}$. Part (b) of Remark 8 gives $a_2 \ge 0$. Assume $a \ge a_1 + e_1 + a_2 + e_2$. We specialize $\mathrm{Res}_C(Z_1)$ to the virtual scheme Z_2 which is a general union of $a - a_1 - e_1 - a_2 - e_2$ double points of Y, a_1 points of C, e_1 double points of C, a_2 double points with support on

C and (if $e_2 > 0$) the virtual scheme obtained applying Remark 2 at a general point of C. We have $\mathrm{length}(Z_2 \cap C) = h^0(C, \mathcal{O}_C(t-1))$. Hence Remark 6 gives $h^0(C, \mathcal{I}_{Z_1 \cap C}(t)) = 0$. Hence it is sufficient to prove $h^1(Y, \mathcal{I}_{\mathrm{Res}_{C(Z_1)}}(t-1)) = 0$ (Remark 2). And so on, defining integers a_i, e_i, $3 \le i \le t-2$, such that $e_i \in \{0, 1\}$ for all i. At each step we may do this inductive step by part (b) of Remark 8. We only stop if there is an integer k such that $1 \le k \le t-2$ and $a < \sum_{i=1}^{k}(a_i + e_i)$. This is not the case for maximal a, but anyway if $a < \sum_{i=1}^{k}(a_i + e_i)$ in the inductive step $\mathcal{O}_Y(t-k+1) \Longrightarrow \mathcal{O}_Y(t-k)$ we just specialize the remaining double points of Y to general double points with support on Y obtaining a scheme W such that $h^1(W, \mathcal{I}_{W \cap C}(t-k)) = 0$ and $\mathrm{Res}_C(W)$ is formed by at most a_k points of C. We have $h^1(C, \mathcal{I}_{\mathrm{Res}_{C(W)}}(t-k-1)) = 0$ by part (b) of Remark 8 and hence $h^1(W, \mathcal{I}_W(t-k)) = 0$ (Remark 1).

Part (b) of Remark 8 works for all $n \ge 3$. Hence we apply it until we arrive at a scheme $\Delta \subset C$ and we are looking at the linear system $|\mathcal{O}_Y(2)|$. Set $\delta := \mathrm{length}(\Delta)$. We have $h^0(Y, \mathcal{O}_Y(2)) = 10$, $h^0(C, \mathcal{O}_C(4)) = 15$, $h^0(C, \mathcal{O}_C(3)) = 10$, and $h^0(C, \mathcal{O}_C(2)) = 6$. Δ is the virtual residue of a virtual scheme Γ such that $\mathrm{length}(\Gamma \cap C) = 10$. Γ is a virtual residue of a virtual scheme Φ such that $\mathrm{length}(\Phi \cap C) = 15$. $\delta = 0$ if and only if $\Gamma \subset C$, i.e. if and only if Φ is either the union of 10 double points of Y with support on C or the union of 9 double points of Y with support on C and a virtual double point with support a general point of C. In both cases we have $\mathrm{length}(\Phi \cap C) \ge 19$, contradiction. Similarly, if $\delta = 1$, then Φ contains at least 8 double points of Y with support on C (here we use that $h^0(C, \mathcal{O}_C(3))$ is even) and hence $\mathrm{length}(\Phi \cap C) \ge 16$, contradiction. Thus $\delta \ge 2$. Since $\Delta \subset C$, we may always add a general double point O of Y, because $h^1(Y, \mathcal{I}_{2O}(1)) = 0$. In all cases we get $h^1(Y, \mathcal{I}_{2A}(t)) = 0$ with $\mathrm{length}(A) = 3 \cdot \sharp(A) \ge h^0(Y, \mathcal{O}_Y(t)) - 5$.

(b) Notice that we may apply (i) for $a' := a + 1$ general nodes. To check part (ii) of the lemma we may copy the proof of Lemma 7, except that here we have no multiple point of order $m \ge 3$. Hence it is sufficient to use twice the jets of order 3 at one of the prescribed nodes and then quote of [5], Theorem 1.4, and [19], part (4) of Theorem 6.3. □

Lemma 9. *Fix integers $t > s \ge 5$, $0 < r < s$, and a such that $0 \le a \le \lfloor (\binom{t+2}{3} - \binom{t-s+2}{3}) - 5)/3 \rfloor$. Let $Y \subset \mathbb{P}^3$ be any smooth degree s surface. Let $S \subset Y$ be a general subset such that $\sharp(S) = a$.*

(i) Then $h^1(Y, \mathcal{I}_{2S}(t)(-D)) = 0$, i.e.

$$h^0(Y, \mathcal{I}_{2S}(t)(-D)) = h^0(Y, \mathcal{O}_Y(t)(-D)) - 3a.$$

(ii) Assume $a \le \lfloor (\binom{t+2}{3} - \binom{t-s+2}{3}) - 5)/3 \rfloor - 1$ and take a general $T \in$

$|\mathcal{O}_{2S}(t)(-D)|$. Then T is integral, Sing$(T) = S$ and T has an ordinary node at each point of S.

Proof. Since $\mathcal{O}_Y(1)(-D)|$ is effective, we have $|\mathcal{O}_Y(n-1)| \subseteq |\mathcal{O}_Y(n)(-D)|$ for every integer $n \geq 2$. Hence part (i) follows from part (i) of Lemma 8 for the integer $t' = t-1$. Part (ii) follows from part (i) applied for the integer $a+1$, as in part (b) Lemma 8 (not from the statement of part (ii) of Lemma 8). □

Lemma 10. *Fix integers s, r, t such that $t \geq s \geq 5$ and $0 \leq r < s$. If $r = 0$, then set $t' := t$. If $r > 0$, then assume $t > s$ and set $t' := t-1$. Fix integers m_x, $1 \leq x \leq s-1$, and a_x, $1 \leq x \leq s-1$, such that $0 \leq m_x \leq t'-2-x$ and $0 \leq a_x \leq \lfloor (t'+3-x)(t'+2-x)/2 - m_x(m_x-1)/2-5)/3 \rfloor$ for all x. Set $u_0 := \lfloor (t'-s+3)(t'-s+2)/6 \rfloor - \epsilon$, where $\epsilon = 1$ if $t'-s+1 \in \{2,4\}$ and $\epsilon = 0$ otherwise. Fix an integer a_0 such that $0 \leq a_0 \leq u_0$ and set $a := \sum_{x=0}^{s-1} a_x$. Fix a plane $H \subset \mathbb{P}^3$ and a smooth degree r curve $D \subset H$, with the convention $D = \emptyset$ if $r = 0$.*

(i) Let $Y \subset \mathbb{P}^3$ be a general degree s surface containing D. Fix $s-1+a$ general points of Y, say P_x, $1 \leq x \leq s-1$, and Q_j, $1 \leq j \leq a$. Set $Z := \cup_{x=1}^{s-1}\{m_xP_x, Y\} \bigcup \cup_{j=1}^{a}\{2Q_j, Y\}$. Then $h^1(Y, \mathcal{I}_Z(t)(-D)) = 0$.

(ii) f $a \leq u_0 - 1 + \sum_{x=1}^{s-1} \lfloor (t'+3-x)(t'+2-x)/2 - m_x(m_x-1)/2-7)/3 \rfloor$, then a general $T \in |\mathcal{I}_Z(t)(-D))|$ is integral, with an ordinary point of multiplicity m_x at each P_x, $1 \leq x \leq t'-3$, an ordinary node at each Q_j, $1 \leq j \leq a$, and no other singularity.

Proof. The surface Y is smooth. By semicontinuity and the assumption that Y is general among the degree s surfaces containing D it is sufficient to prove the existence of a degree s surface W containing D and $s-1$ smooth points P_x, $1 \leq x \leq s-1$, Q_j, $1 \leq j \leq a$, of W for which the corresponding result is true, where $Z := \cup_{x=1}^{s-1}\{m_xP_x, W\} \bigcup \cup_{j=1}^{a}\{2Q_j, W\}$. We take as W the union of H (if $r = 0$) and $s-1$ general planes H_j, $j \leq j \leq s$. As explained in Remark 8 it is sufficient to find Z as above such that $h^1(\mathcal{I}_Z(t')) = 0$. Fix a general $S_i \subset H_i$, $1 \leq i \leq s-1$, such that $\sharp(S_i) = a_i$ and a general $O_i \in H_i$. Set $U := \cup_{i=1}^{s-1}\{2S_i, H_i\} \bigcup \cup_{x=1}^{s-1}\{m_xO_x, H_x\}$. We have $h^1(H_1, \mathcal{I}_{U\cap H_1}(t')) = 0$ (Lemma 7). And so on, using at each step Lemma 7 for a lower integer y and another plane $H_{t'+1-y}$. Then insert a_0 double points of H. Our definition of ϵ_1 is made to apply the case $n = 2$ of a theorem of Alexander-Hirschowitz on the postulation of general unions of double points of $\mathbb{P}^n$ (see [4]), i.e. u_0 is the maximal integer z such that $h^1(H, \mathcal{I}_{2A}(t'-s+1)) = 0$ for a general $B \subset H$ such that $\sharp(A) = b$.

Part (ii) is proved as in parts (a), (b) and (c) of Lemma 7 with the following modification (in that lemma we only had one point with multiplicity $m \geq 3$). Since $|\mathcal{I}_Z(t)(-D)|$ is a non-empty projective space, it is irreducible. Hence to check that T has an ordinary point of multiplicity m_x at each point P_x, $1 \leq x \leq s-1$, it is sufficient to fix an integer x such that $1 \leq x \leq s-1$ and find $T \in |\mathcal{I}_Z(t)(-D)|$ with an ordinary point with multiplicity m_x at P_x. This is the case considered in Lemma 7: blowing-up only the point P_x. □

Remark 9. For all integers y, m, a such that $y \geq m+2 \geq 4$ and $0 \leq a \leq \lfloor((y+2)(y+1)/2 - m(m-1)/2)/3\rfloor - 2$ set $g_{y,m,a} := (y-1)(y-2)/2 - m(m-1)/2 - a$. Let $Z(m,a) \subset \mathbb{P}^2$ be a general union of a fat point with multiplicity m and a double points. Lemma 10 gives that a general $C_{y,m,b} \in |\mathcal{I}_{Z(m,a)}(y)|$ is integral and with geometric genus $g_{y,m,a}$. Fixing y and m, while varying a between 0 and $\lfloor((y+2)(y+1)/2 - m(m-1)/2)/3\rfloor - 2$ we cover a connected interval of integers. Now assume $m > 0$. We have $g_{y,m,0} = g_{y,m-1,0} + m - 1$. We have $\lfloor((y+2)(y+1)/2 - m(m-1)/2)/3\rfloor - 2 \geq m-1$ if and only if

$$(y+2)(y+1) \geq m(m-1) + 6m + 6. \tag{16}$$

The inequality (16) is always satisfied, because $y \geq m+2 \geq 4$.

Proof of Theorem 3. For any integer $n \geq 2$ we have $\sum_{j=2}^{n} \binom{j}{2} = \binom{n+1}{3}$. Hence $\sum_{m=t'-s+4}^{t'-2} m(m-1)/2 = \binom{t'-1}{3} - \binom{t'-s+4}{3}$. Fix Z as in the statement of Lemma 10 with respect to all data m_x, a_x, $1 \leq x \leq s-1$, and a_0. Then use Remark 9. The lower bound for g is obtained taking $a_x = 0$ for all x, $m_x = t' - 2 - x$ and $a_0 = u_0 - 1$. Remark 9 gives that the set of all geometric general obtained in this way is connected. □

Proof of Proposition 1. Set $r := st - d$. Apply Lemma 8 if $d/s \in \mathbb{Z}$, i.e. $r = 0$. Apply Lemma 9 if $r > 0$. □

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