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CENTRAL NORMS AND CONTINUED FRACTIONS

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Abstract: We look at solutions to the norm-form equations $x^2 - Dy^2 = c$ in terms of the central norm being equal to c when the simple continued fraction expansion of $\sqrt{D}$ is odd. We essentially characterized all possible cases when the period length is even in earlier work including a generalization of a result of Lagrange for the case where D is prime. However, virtually nothing has been done in the odd case. We make inroads herein that characterize the central norms in the odd case for a wide range of cases that link to what is known in the literature for the even case.

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1. Introduction

The study of the solutions of norm form equations $x^2 - Dy^2 = c$ has a long and distinguished history, much of which may be found in the classic work by Dickson [1]. In particular, the study of the classic Pell equation ($c = \pm 1$) goes back to Archimedes (see [6], for instance). More recently, in the past century, pioneering work on the case $c = \pm 4N$ was done by Stolt in [11]-[13]. Also, in the latter part of the last century work was done for the case $c = 2^n$ by authors such as Tzanakis [14]. However, there has been relatively little done in this regard in the above works with the continued fraction connection. In previous

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work, [4]-[10], we looked at where c appeared as the central norm in the simple continued fraction expansion of of $\sqrt{D}$ when the period length thereof was even. In this case c necessarily divides $2D$. In particular, in the case where $c = 2$, we provided a rather palatable generalization of a result of Lagrange [10]. We now look at the road less travelled, the case where the period length is odd and $\gcd(c, D) = 1$.

2. Notation and Preliminaries

Herein, we will be concerned with the simple continued fraction expansions of $\sqrt{D}$, where D is an integer that is not a perfect square. We denote this expansion by,

$$\sqrt{D} = \langle q_0; \overline{q_1, q_2, \ldots, q_{\ell-1}, 2q_0} \rangle,$$

where $\ell = \ell(\sqrt{D})$ is the period length, $q_0 = \lfloor \sqrt{D} \rfloor$ (the *floor* of $\sqrt{D}$), and $q_1 q_2, \ldots, q_{\ell-1}$ is a palindrome.

The j-th *convergent* of α for $j \geq 0$ are given by,

$$\frac{A_j}{B_j} = \langle q_0; q_1, q_2, \ldots, q_j \rangle,$$

where

$$A_j = q_j A_{j-1} + A_{j-2}, \tag{2.1}$$

$$B_j = q_j B_{j-1} + B_{j-2}, \tag{2.2}$$

with $A_{-2} = 0$, $A_{-1} = 1$, $B_{-2} = 1$, $B_{-1} = 0$. The *complete quotients* are given by, $(P_j + \sqrt{D})/Q_j$, where $P_0 = 0$, $Q_0 = 1$, and for $j \geq 1$,

$$P_{j+1} = q_j Q_j - P_j, \tag{2.3}$$

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor,$$

and

$$D = P_{j+1}^2 + Q_j Q_{j+1}.$$

We will also need the following facts (which can be found in most introductory texts in number theory, such as [3], also, see [2] for a more advanced exposition).

$$A_j B_{j-1} - A_{j-1} B_j = (-1)^{j-1}. \tag{2.4}$$

Also,

$$A_{j-1} = P_j B_{j-1} + Q_j B_{j-2}, \tag{2.5}$$

$$DB_{j-1} = P_j A_{j-1} + Q_j A_{j-2}, \tag{2.6}$$

and

$$A_{j-1}^2 - B_{j-1}^2 D = (-1)^j Q_j. \tag{2.7}$$

In particular,

$$A_{\ell-1}^2 - B_{\ell-1}^2 D = (-1)^\ell, \tag{2.8}$$

and it follows that $(x_0, y_0) = (A_{\ell-1}, B_{\ell-1})$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = (-1)^\ell$.

When ℓ is even, $P_{\ell/2} = P_{\ell/2+1}$, so by equation (2.3),

$$Q_{\ell/2} \mid 2P_{\ell/2},$$

where $Q_{\ell/2}$ is called the *central norm* (via equation (2.7)), where

$$Q_{\ell/2} \mid 2D, \tag{2.9}$$

and

$$q_{\ell/2} = 2P_{\ell/2}/Q_{\ell/2}. \tag{2.10}$$

Similarly, when ℓ is odd, then $Q_{(\ell+1)/2} = Q_{(\ell-1)/2}$ is called the *central norm*.

In the following (which we need in the next section), and all subsequent results, the notation for the A_j, B_j, Q_j and so forth apply to the above-developed notation for the continued fraction expansion of $\sqrt{D}$.

Theorem 2.1. *Let D be a positive integer that is not a perfect square. Then the following are equivalent.*

1. $\ell = \ell(\sqrt{D})$ is even.

2. Either there exists a factorization $D = ab$ with $1 < a < b$ and

$$rx^2 - sy^2 = \pm 1, \tag{2.11}$$

has a solution, or there exists a factorization $D = ab$ with $1 \le a < b$ such that

$$rx^2 - sy^2 = \pm 2, \tag{2.12}$$

has a solution.

Furthermore, in the former case each of the following occurs, where

$$r\sqrt{a} + s\sqrt{b}$$

is the fundamental solution of equation (2.11):

(a) $Q_{\ell/2} = a$;

(b) $A_{\ell/2-1} = ra$ and $B_{\ell/2-1} = s$;

(c) $A_{\ell-1} = r^2a + s^2b$ and $B_{\ell-1} = 2rs$;

(d) $r^2a - s^2b = (-1)^{\ell/2}$;

and in the latter case, each of the following occurs, where

$$r\sqrt{a} + s\sqrt{b}$$

is the fundamental solution of equation (2.12):

(a) $Q_{\ell/2} = 2a$;

(b) $A_{\ell/2-1} = ra$ and $B_{\ell/2-1} = s$;

(c) $A_{\ell-1} = r^2a + s^2b$ and $B_{\ell-1} = 2rs$;

(d) $r^2a - s^2b = (-1)^{\ell/2}$.

Proof. All of this is proved in [6]. □

Lastly, we will need the following.

Theorem 2.2. *Suppose that $\ell = \ell(\sqrt{D})$ and k is any positive integer. Then if ℓ is even, all positive solutions of*

$$x^2 - y^2D = 1 \tag{2.13}$$

are given by

$$x = A_{k\ell-1} \text{ and } y = B_{kl-1},$$

whereas there are no solutions to

$$x^2 - y^2D = -1. \tag{2.14}$$

If ℓ is odd, then all positive solutions of equation (2.13) *are given by*

$$x = A_{2k\ell-1} \text{ and } y = B_{2k\ell-1},$$

whereas all positive solutions of equation (2.14) *are given by*

$$x = A_{(2k-1)\ell-1} \text{ and } y = B_{(2k-1)\ell-1}.$$

Proof. This appears in many introductory number theory texts possessing an in-depth section on continued fractions. For instance, see [3, Corollary 5.3.3, p. 249]. □

3. Central Norms

In what follows the notation from the previous section is in force.

The following generalizes [9, Theorem 5, p. 125], which only covered the case where $c = 2$. Also, it generalizes [8, Theorem 2, p. 275] which only covered the case where c is prime.

Theorem 3.1. *Suppose that for some $c, d \in \mathbb{N}$, $c > 1$, $D' = c^{2d}D$ is not a perfect square where $\gcd(c, D) = 1$, $\ell = \ell(\sqrt{D})$ is odd, and $D = f^2 + c^{4d} = P^2_{(\ell+1)/2} + Q^2_{(\ell+1)/2}$, and set $\ell' = \ell(\sqrt{D'})$. Then the following are equivalent where the notations A'_j, B'_j, P'_j, q'_j, Q'_j, occur in the simple continued fraction expansion of $\sqrt{D'}$ and A_j, B_j, etc. occur in the simple continued fraction expansion of $\sqrt{D}$ for any $j \in \mathbb{N}$.*

1. $A_{\ell'-1} = A^2_{\ell-1} + B^2_{\ell-1}D$.

2. ℓ' is even and,

$$\frac{A'_{\ell'/2-1}}{c^d} + B'_{\ell'/2-1}\sqrt{D} = A_{\ell-1} + B_{\ell-1}\sqrt{D}.$$

3. ℓ' is even, $\ell'/2$ is odd, and ${Q'}_{\ell'/2} = c^{2d} = Q_{(\ell+1)/2} = Q_{(\ell-1)/2}$, and in this case, each of the following holds:

$$c^{2d}{q'}_{\ell'/2} = 2{P'}_{\ell'/2}, \tag{3.1}$$

$$c^d A_{\ell-1} = {A'}_{\ell'/2-1} = c^{2d}({B'}_{\ell'/2} + {B'}_{\ell'/2-2})/2, \tag{3.2}$$

$$DB_{\ell-1} = D{B'}_{\ell'/2-1} = ({A'}_{\ell'/2} + {A'}_{\ell'/2-2})/2, \tag{3.3}$$

and

$$A_{(\ell-1)/2}A_{(\ell-3)/2} + B_{(\ell-1)/2}B_{(\ell-3)/2}D \tag{3.4}$$
$$+(A_{(\ell-1)/2}B_{(\ell-3)/2} + B_{(\ell-1)/2}A_{(\ell-3)/2})\sqrt{D} = c^{2d}(A_{\ell-1} + B_{\ell-1}\sqrt{D}).$$

Proof. Assume part 1 holds. First we show that ℓ' is even. If, on the contrary, ℓ' is odd, then $A^2_{\ell'-1} - B_{\ell'-1}D' = -1$ so since $(A_{\ell-}, B_{\ell-1})$ is the smallest solution of $x^2 - Dy^2 = -1$, then by Theorem 2.2,

$$(A_{\ell'-1}, B_{\ell'-1}c^d) = (A_{k\ell-1}, B_{k\ell-1}),$$

for some odd integer k. Therefore, , $A_{k\ell-1} = A_{\ell'-1} = A^2_{\ell-1} + B^2_{\ell-1}D = A_{2k-1}$, a contradiction. Hence, ℓ' is even, so we may invoke Theorem 2.1.

Suppose part 1 of Theorem 2.1 holds, with $D' = ab$ and

$$r^2a - s^2b = \pm 1, \tag{3.5}$$

with ${Q'}_{\ell'/2} = a$, then since

$${A'}_{\ell'-1} = ({A'}_{\ell'/2-1}/a)^2 a + b{B'}^2_{\ell/2-1} = A^2_{\ell-1} + DB^2_{\ell-1},$$

we must have $a = c^{2d}$ and $b = D$, so $a = c^{2d} = {Q'}_{\ell'/2}$. Thus,

$$A'^2_{\ell'/2-1} - B'^2_{\ell'/2-1}D' = -c^{2d},$$

and this is the smallest such solution of $x^2 - D'y^2 = -c^{2d}$, whence

$$(({A'}_{\ell'/2-1}/c^d), {B'}_{\ell'/2-1})$$

is the smallest solution of $x^2 - Dy^2 = -1$, and part 2 of this theorem follows. If part 2 of of Theorem 2.1 holds, then part 2 of this theorem follows similarly.

Assume part 2 holds. Then

$$(c^dA_{\ell-1})^2 - (c^dB_{\ell-1})^2D = {{A'}_{\ell'/2-1}}^2 - {B'}_{\ell'/2-1}D' = (-1)^{\ell'/2}{Q'}_{\ell'/2},$$

so

$$(-1)^{\ell'/2}{Q'}_{\ell'/2}/c^{2d} = A^2_{\ell-1} - B^2_{\ell-1}D = (-1)^{\ell} = -1,$$

from which we achieve ${Q'}_{\ell'/2} = c^{2d} = Q_{(\ell+1)/2} = Q_{(\ell-1)/2}$, and $\ell'/2$ is odd. Therefore, by equation (2.10), equation (3.1) holds. Now we show that equation (3.2) holds. By equations (2.5) and (3.1),

$$\begin{aligned}{A'}_{\ell'/2-1} = {P'}_{\ell'/2}{B'}_{\ell'/2-1} + {Q'}_{\ell'/2}{B'}_{\ell'/2-2} &= c^{2d}{q'}_{\ell'/2}{B'}_{\ell'/2-1} + c^{2d}{B'}_{\ell'/2-2}\\ &= c^{2d}({q'}_{\ell'/2}{B'}_{\ell'/2-1} + 2{B'}_{\ell'/2-2})/2 = c^{2d}({B'}_{\ell'/2} + {B'}_{\ell'/2-2})/2.\end{aligned}$$

Now we show that equation (3.3) holds. By equations (2.6) and (3.1),

$$\begin{aligned}2D'{B'}_{\ell'/2-1} = 2{P'}_{\ell'/2}{A'}_{\ell'/2-1} + {Q'}_{\ell'/2}{A'}_{\ell'/2-2} &= {q'}_{\ell'/2}c^{2d}{A'}_{\ell'/2-1} + 2c^{2d}{A'}_{\ell'/2-2}\\ &= c^{2d}({q'}_{\ell'/2}{B'}_{\ell'/2-1} + 2{A'}_{\ell'/2-2} = c^{2d}({A'}_{\ell'/2} + {A'}_{\ell'/2-2}),\end{aligned}$$

and dividing through by c^{2d} yields the result. Lastly, we show that Equation (3.4) holds. By [2, line: -4, p. 48], we have the following,

$$A_{(\ell-1)/2}A_{(\ell-3)/2} + B_{(\ell-1)/2}B_{(\ell-3)/2}D = A_{\ell-1}Q_{(\ell+1)/2},$$

and

$$A_{(\ell-1)/2}B_{(\ell-3)/2} + B_{(\ell-1)/2}A_{(\ell-3)/2} = B_{\ell-1}Q_{(\ell+1)/2},$$

from which the equation follows. This completes part 3.

If part 3 holds, then by Theorem 2.1,

$$A_{\ell'-1} = (A'_{\ell'/2-1}/c^d)^2 + {B'_{\ell'/2-1}}^2D = A^2_{\ell-1} + B^2_{\ell-1}D,$$

by the same argument as above about the minimality of $(A'_{\ell'/2-1}/c^d, B'_{\ell'/2-1})$ as a solution of $x^2 - Dy^2 = -1$, which gives us part 1, and secures the theorem. □

The following is a generalization, as noted previously, of the case where $c = 2$, namely of [9, Theorem 5, p. 125].

Corollary 3.1. *Let $D = c^{2d}D$ where $D > 1$ is not a perfect square, $d \in \mathbb{N}$, $\ell = \ell(\sqrt{D})$, and $\ell' = \ell(\sqrt{D'})$. Then if ℓ' is even, the following are equivalent.*

1. $Q'_{\ell'/2} = c^{2d}$.
2. $\frac{A'_{\ell'/2-1}}{c^d} + B'_{\ell'/2-1}\sqrt{D} = A_{\ell-1} + B_{\ell-1}\sqrt{D}$.

Example 3.1. Let $D = 629$, $c = 5$, $d = 1$, then $\ell(\sqrt{D}) = 5$ and $\ell(\sqrt{D'}) = \ell(\sqrt{25 \cdot 629}) = 10$, with $Q_3 = Q_{(\ell+1)/2} = Q_{(\ell-1)/2} = Q_2 = Q'_{\ell'/2} = Q'_5 = 25 = c^{2d}$. Also, $A'_{\ell'/2-1} = A'_4 = 39250 = c^d A_{\ell-1} = 5 \cdot 7850$, and $B'_{\ell'/2-1} = 313 = B_{\ell-1}$. Furthermore, $c^{2d}q'_{\ell'/2} = 25 \cdot 10 = 2P'_{\ell'/2} = 2 \cdot 125$,

$$c^d A_{\ell-1} = 5 \cdot 7850 = A'_{\ell'/2} = 391250 = c^{2d}(B'_{\ell'/2} + B'_{\ell'/2-2})/2 = 25(3135 + 5)/2,$$

$$DB_{\ell-1} = 629 \cdot 313 = DB'_{\ell'/2-1} = (A'_{\ell'/2} + A'_{\ell'/2-2})/2 = (393127 + 627)/2,$$

$$A_{(\ell-1)/2}A_{(\ell-3)/2} + B_{(\ell-1)/2}B_{(\ell-3)/2}D = 326 \cdot 301 + 13 \cdot 12 \cdot 629 = c^{2d}A_{\ell-1} = 25 \cdot 7850,$$

and

$$A_{(\ell-1)/2}B_{(\ell-3)/2} + B_{(\ell-1)/2}A_{(\ell-3)/2} = 326 \cdot 12 + 13 \cdot 301 = c^{2d}B_{\ell-1} = 25 \cdot 313.$$

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